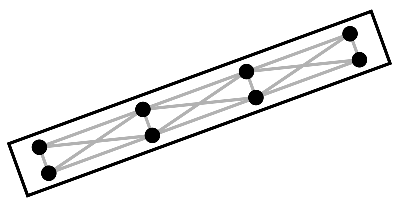
Torque

Newton's law of motion says that the position $\mathbf{r}(t)$ of a single particle moving under the influence of a force $\mathbf{F}$ obeys $m\mathbf{r}''(t) = \mathbf{F}$. Similarly, the positions $\mathbf{r}_i(t)$, $1 \le i \le n$, of a set of particles moving under the influence of forces $\mathbf{F}_i$ obey $m\mathbf{r}_i''(t) = \mathbf{F}_i$, $1 \le i \le n$. Often systems of interest consist of some small number of rigid bodies. Suppose that we are interested in the motion of a single rigid body, say a piece of wood. The piece of wood is made up of a huge number of atoms. So the system of equations determining the motion of all of the individual atoms in the piece of wood is huge. On the other hand, because the piece of wood is rigid, its configuration is completely determined by the position of, for example, its centre of mass and its orientation (I don't want to get into what is precisely meant by "orientation", but it is certainly determined by, for example, the positions of a few of the corners of the piece of wood). It is possible to extract from the huge system of equations that determine the motion of all of the individual atoms, a small system of equations that determine the motion of the centre of mass and the orientation.

So, imagine a piece of wood moving in $\mathbb{R}^3$. Furthermore, imagine that the piece of



wood consists of a huge number of particles joined by a huge number of weightless but very strong steel rods. The steel rod joining particle number one to particle number two just represents a force acting between particles number one and two. Suppose that

- there are n particles, with particle number i having mass m_i
- at time t, particle number i has position $\mathbf{r}_i(t)$
- at time t, the external force (gravity and the like) acting on particle number i is $\mathbf{F}_i(t)$.
- at time t, the force acting on particle number i, due to the steel rod joining particle number i to particle number j is $\mathbf{F}_{i,j}(t)$. If there is no steel rod joining particles number i and j, just set $\mathbf{F}_{i,j}(t) = 0$. In particular, $\mathbf{F}_{i,i}(t) = 0$.

The only assumptions that we shall make about the steel rod forces are

(A1) for each $i \ne j$, $\mathbf{F}_{i,j}(t) = -\mathbf{F}_{j,i}(t)$. In words, the steel rod joining particles i and j applies equal and opposite forces to particles i and j.

(A2) for each $i \ne j$, there is a function $M_{i,j}(t)$ such that $\mathbf{F}_{i,j}(t) = M_{i,j}(t)\big[\mathbf{r}_i(t) - \mathbf{r}_j(t)\big]$. In words, the force due to the rod joining particles i and j acts parallel to the line joining particles i and j. For (A1) to be true, we need $M_{i,j}(t) = M_{j,i}(t)$.

Newton's law of motion, applied to particle number i, now tells us that

$$m_i \mathbf{r}_i''(t) = \mathbf{F}_i(t) + \sum_{j=1}^{n} \mathbf{F}_{i,j}(t) \tag{1_i}$$

Adding up all of the equations (1_i), for $i = 1,\ 2,\ 3,\ \cdots,\ n$ gives

$$\sum_{i=1}^{n} m_i \mathbf{r}_i''(t) = \sum_{i=1}^{n} \mathbf{F}_i(t) + \sum_{1\le i,j\le n} \mathbf{F}_{i,j}(t) \tag{$\Sigma_i(1_i)$}$$

The sum $\sum_{1\le i,j\le n} \mathbf{F}_{i,j}(t)$ contains $\mathbf{F}_{1,2}(t)$ exactly once and it also contains $\mathbf{F}_{2,1}(t)$ exactly once and these two terms cancel exactly, by assumption (A1). In this way, all terms in $\sum_{1\le i,j\le n} \mathbf{F}_{i,j}(t)$ with $i \ne j$ exactly cancel. All terms with $i = j$ are assumed to be zero. So $\sum_{1\le i,j\le n} \mathbf{F}_{i,j}(t) = 0$ and the equation $\Sigma_i(1_i)$ simplifies to

$$\sum_{i=1}^{n} m_i \mathbf{r}_i''(t) = \sum_{i=1}^{n} \mathbf{F}_i(t) \tag{$\Sigma_i(1_i)$}$$

Denote by $M = \sum\limits_{i=1}^{n} m_i$ the total mass of the system, by $\mathbf{R}(t) = \frac{1}{M} \sum\limits_{i=1}^{n} m_i \mathbf{r}_i(t)$ the centre of mass of the system and by $\mathbf{F}(t) = \sum\limits_{i=1}^{n} \mathbf{F}_i(t)$ the total external force acting on the system. In this notation, equation $\Sigma_i(1_i)$ is

$$M\mathbf{R}''(t) = \mathbf{F}(t) \tag{2}$$

So the centre of mass of the system moves just like a single particle of mass M subject to the total external force.

Now take the cross product of $\mathbf{r}_i(t)$ and equation (1_i) and sum over i. This gives the equation $\sum_i \mathbf{r}_i(t) \times (1_i)$:

$$\sum_{i=1}^{n} m_i\ \mathbf{r}_i(t) \times \mathbf{r}_i''(t) = \sum_{i=1}^{n} \mathbf{r}_i(t) \times \mathbf{F}_i(t) + \sum_{1\le i,j\le n} \mathbf{r}_i(t) \times \mathbf{F}_{i,j}(t)$$

By the assumption (A2)

$$\begin{aligned}
\mathbf{r}_1(t) \times \mathbf{F}_{1,2}(t) &= M_{1,2}(t)\ \mathbf{r}_1(t) \times \big[\mathbf{r}_1(t) - \mathbf{r}_2(t)\big] \\
\mathbf{r}_2(t) \times \mathbf{F}_{2,1}(t) &= M_{2,1}(t)\ \mathbf{r}_2(t) \times \big[\mathbf{r}_2(t) - \mathbf{r}_1(t)\big] \\
&= -M_{1,2}(t)\ \mathbf{r}_2(t) \times \big[\mathbf{r}_1(t) - \mathbf{r}_2(t)\big] \\
\mathbf{r}_1(t) \times \mathbf{F}_{1,2}(t) + \mathbf{r}_2(t) \times \mathbf{F}_{2,1}(t) &= M_{1,2}(t)\ \big[\mathbf{r}_1(t) - \mathbf{r}_2(t)\big] \times \big[\mathbf{r}_1(t) - \mathbf{r}_2(t)\big] = 0
\end{aligned}$$

because the cross product of any two parallel vectors is zero. The last equation say that the $i=1$, $j=2$ term in $\sum\limits_{1\le i,j\le n} \mathbf{r}_i(t)\times\mathbf{F}_{i,j}(t)$ exactly cancels the $i=2$, $j=1$ term. In this way all of the terms in $\sum\limits_{1\le i,j\le n} \mathbf{r}_i(t)\times\mathbf{F}_{i,j}(t)$ with $i\ne j$ cancel. Each term with $i=j$ is exactly zero. So $\sum\limits_{1\le i,j\le n} \mathbf{r}_i(t)\times\mathbf{F}_{i,j}(t)=0$ and

$$\sum_{i=1}^n m_i\ \mathbf{r}_i(t)\times\mathbf{r}_i''(t)=\sum_{i=1}^n \mathbf{r}_i(t)\times\mathbf{F}_i(t) \tag{3}$$

Define

$$\mathbf{L}(t)=\sum_{i=1}^n m_i\ \mathbf{r}_i(t)\times\mathbf{r}_i'(t)$$
$$\mathbf{T}(t)=\sum_{i=1}^n \mathbf{r}_i(t)\times\mathbf{F}_i(t)$$

In this notation, (3) becomes

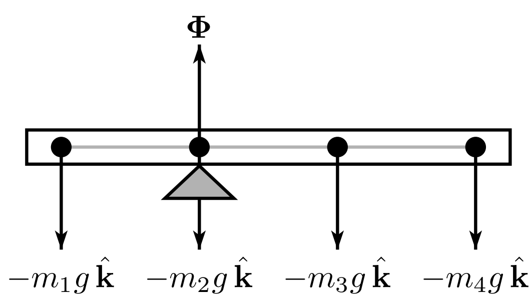
$$\tfrac{d}{dt}\mathbf{L}(t)=\mathbf{T}(t) \tag{4}$$

Equation (4) plays the rôle of Newton's law of motion for rotational motion. $\mathbf{T}(t)$ is called the torque and plays the rôle of "rotational force". $\mathbf{L}(t)$ is called the angular momentum (about the origin) and is a measure of the rate at which the piece of wood is rotating. For example, if a particle of mass m is traveling in a circle of radius ρ in the xy–plane at ω radians per unit time, then $\mathbf{r}(t)=\rho\cos(\omega t)\hat{\boldsymbol{\imath}}+\rho\sin(\omega t)\hat{\boldsymbol{\jmath}}$ and

$$m\mathbf{r}(t)\times\mathbf{r}'(t)=m\big[\rho\cos(\omega t)\hat{\boldsymbol{\imath}}+\rho\sin(\omega t)\hat{\boldsymbol{\jmath}}\big]\times\big[-\omega\rho\sin(\omega t)\hat{\boldsymbol{\imath}}+\omega\rho\cos(\omega t)\hat{\boldsymbol{\jmath}}\big]=m\rho^2\ \omega\ \hat{\mathbf{k}}$$

is proportional to ω, which is the rate of rotation about the origin and is in the direction $\hat{\mathbf{k}}$, which is normal to the plane containing the circle. In any event, in order for the piece of wood to remain stationary, equations (2) and (4) force $\mathbf{F}(t)=\mathbf{T}(t)=0$.

Now suppose that the piece of wood is a seesaw that is supported on a fulcrum at $\mathbf{p}$. The forces consist of gravity, $-m_ig\hat{\mathbf{k}}$, acting on particle number i, for each $1\le i\le n$ and the force $\boldsymbol{\Phi}$ imposed by the fulcrum that is pushing up on the particle at $\mathbf{p}$. The total



external force is $\mathbf{F} = \mathbf{\Phi} - \sum_{i=1}^{n} m_i g\hat{\mathbf{k}} = \mathbf{\Phi} - Mg\hat{\mathbf{k}}$. If the seesaw is to remain stationary, this must be zero so that $\mathbf{\Phi} = Mg\hat{\mathbf{k}}$. The total torque (about the origin) is

$$\mathbf{T} = \mathbf{p} \times \mathbf{\Phi} - \sum_{i=1}^{n} m_i g \mathbf{r}_i \times \hat{\mathbf{k}} = g\Big(M\mathbf{p} - \sum_{i=1}^{n} m_i \mathbf{r}_i\Big) \times \hat{\mathbf{k}}$$

If the seesaw is to remain stationary, this must also be zero. This will be the case if the fulcrum is placed at

$$\mathbf{p} = \tfrac{1}{M} \sum_{i=1}^{n} m_i \mathbf{r}_i$$

which is the centre of mass of the piece of wood.

More generally, suppose that the external forces acting on the piece of wood consist of $\mathbf{F}_i$, acting on particle number i, for each $1 \le i \le n$, and a "fulcrum force" $\mathbf{\Phi}$ acting on a particle at $\mathbf{p}$. The total external force is $\mathbf{F} = \mathbf{\Phi} + \sum_{i=1}^{n} \mathbf{F}_i$. If the piece of wood is to remain stationary, this must be zero so that $\mathbf{\Phi} = -\sum_{i=1}^{n} \mathbf{F}_i$. The total torque (about the origin) is

$$\mathbf{T} = \mathbf{p} \times \mathbf{\Phi} + \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i = \sum_{i=1}^{n} (\mathbf{r}_i - \mathbf{p}) \times \mathbf{F}_i$$

If the piece of wood is to remain stationary, this must also be zero. That is, the torque about point $\mathbf{p}$ due to all of the forces $\mathbf{F}_i$, $1 \le i \le n$, must be zero.