Euler Wobble

Consider a rigid body with inertia tensor

$$\mathcal{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \qquad \text{with } I_1 = 1,\ I_2 = \tfrac{1}{2},\ I_3 = \tfrac{1}{3}$$

Under free motion, the angular momentum $\vec{M}$, expressed in body coordinates, obeys Euler's equations (see "A Summary of Rigid Body Formulae")

$$\dot{M}_1 = M_2 M_3 \qquad \dot{M}_2 = -2M_1M_3 \qquad \dot{M}_3 = M_1M_2$$

These equations imply that both $\vec{M}^2 = M_1^2 + M_2^2 + M_3^2$ and the energy $E = M_1^2 + 2M_2^2 + 3M_3^2$ are conserved:

$$\tfrac{d}{dt}\big(M_1^2 + M_2^2 + M_3^2\big) = 2M_1\dot{M}_1 + 2M_2\dot{M}_2 + 2M_3\dot{M}_3 = 2M_1M_2M_3 - 4M_1M_2M_3 + 2M_1M_2M_3 = 0$$
$$\tfrac{d}{dt}\big(M_1^2 + 2M_2^2 + 3M_3^2\big) = 2M_1\dot{M}_1 + 4M_2\dot{M}_2 + 6M_3\dot{M}_3 = 2M_1M_2M_3 - 8M_1M_2M_3 + 6M_1M_2M_3 = 0$$

Assuming that $\vec{M}^2 \neq 0$, we may choose units of time so that

$$M_1^2 + M_2^2 + M_3^2 = 1 \qquad M_1^2 + 2M_2^2 + 3M_3^2 = E \tag{$*$}$$

Observe that the condition $M_1^2 + M_2^2 + M_3^2 = 1$ forces

$$1 = M_1^2 + M_2^2 + M_3^2 \le M_1^2 + 2M_2^2 + 3M_3^2 \le 3M_1^2 + 3M_2^2 + 3M_3^2 = 3 \qquad \text{so that } 1 \le E \le 3$$

Here are sketchs of the curve $(*)$ for various values of E.

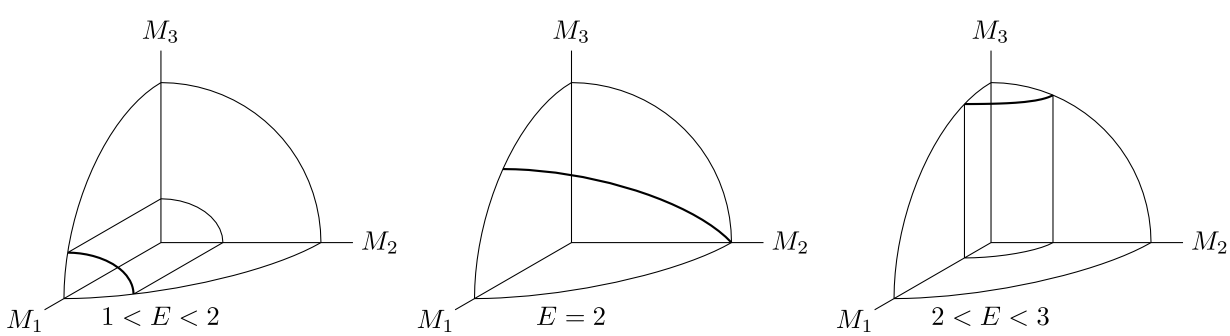
Case 1: $E = 1$. In this case, $(*)$ forces $\vec{M} = \pm(1, 0, 0)$.

Case 2: $1 < E < 2$. The equations $(*)$ are equivalent to $M_1^2 + M_2^2 + M_3^2 = 1$, $M_2^2 + 2M_3^2 = E - 1$. So $(*)$ is the intersection of the unit sphere with an elliptical cylinder centred on the x–axis.

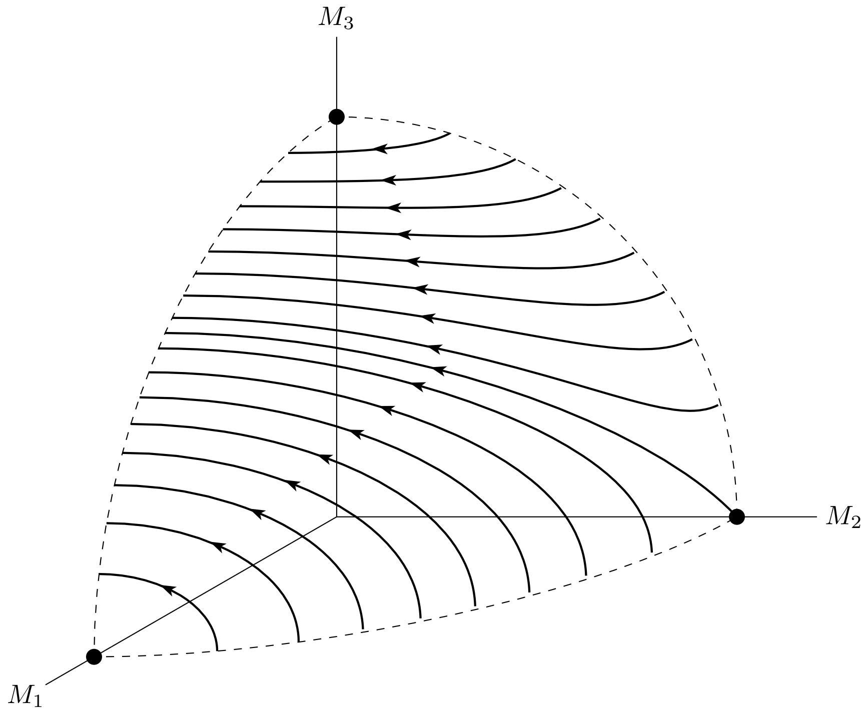
Case 3: $E = 2$. The equations $(*)$ are equivalent to $M_1^2 + M_2^2 + M_3^2 = 1$, $M_1^2 - M_3^2 = 0$. So $(*)$ is the intersection of the unit sphere with the planes $M_1 = \pm M_3$. Each sign gives a great circle.

Case 4: $2 < E < 3$. The equations $(*)$ are equivalent to $M_1^2 + M_2^2 + M_3^2 = 1$, $2M_1^2 + M_2^2 = 3 - E$. So $(*)$ is the intersection of the unit sphere with an elliptical cylinder centred on the z–axis.

Case 5: $E = 3$. In this case, $(*)$ forces $\vec{M} = \pm(0, 0, 1)$.



The following figure provides a sketch of some representative trajectories in the first octant.



We conclude that

- ◦ Trajectories that start exactly at $(\pm 1, 0, 0)$ or $(0, \pm 1, 0)$ or $(0, 0, \pm 1)$ (i.e. rotations of the rigid body **exactly** about one of its principal axes) do not move at all. That is, $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$ are critical points.
- ◦ Trajectories that start near, but not exactly at $(\pm 1, 0, 0)$ or $(0, 0, \pm 1)$ (i.e. rotations of the rigid body almost about the longest or shortest principal axes) just circle about their starting points for all time. That is, $(\pm 1, 0, 0)$ and $(0, 0, \pm 1)$ are stable critical points.
- ◦ Trajectories that start near, but not exactly at $(0, 1, 0)$ (i.e. rotations of the rigid body almost about the middle principal axis) move to near $(0, -1, 0)$ and then back to near $(0, 1, 0)$ and so on. There is one exceptional trajectory (called the separatrix) that starts near $(0, 1, 0)$ and tries to go to $(0, -1, 0)$ exactly, but slows as it gets closer and closer to $(0, -1, 0)$ and never actually gets to $(0, -1, 0)$. So $(0, \pm 1, 0)$ are unstable critical points.