Complex Numbers and Exponentials

A complex number is nothing more than a point in the xy–plane. The sum and product of two complex numbers (x_1, y_1) and (x_2, y_2) is defined by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$(x_1, y_1)\,(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$$

respectively. It is conventional to use the notation $x + iy$ (or in electrical engineering country $x + jy$) to stand for the complex number (x, y). In other words, it is conventional to write x in place of $(x, 0)$ and i in place of $(0, 1)$. In this notation, The sum and product of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is given by

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$
$$z_1z_2 = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$$

Addition and multiplication of complex numbers obey the familiar algebraic rules

$$\begin{array}{cc} z_1 + z_2 = z_2 + z_1 & z_1z_2 = z_2z_1 \\ z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 & z_1(z_2z_3) = (z_1z_2)z_3 \\ 0 + z_1 = z_1 & 1z_1 = z_1 \\ z_1(z_2 + z_3) = z_1z_2 + z_1z_3 & (z_1 + z_2)z_3 = z_1z_3 + z_2z_3 \end{array}$$

The negative of any complex number $z = x + iy$ is defined by $-z = -x + (-y)i$, and obeys $z + (-z) = 0$. The inverse of any complex number $z = x + iy$, other than 0, is defined by $\frac{1}{z} = \frac{x}{x^2+y^2} + \frac{-y}{x^2+y^2}i$ and obeys $\frac{1}{z}z = 1$. The complex number i has the special property

$$i^2 = (0 + 1i)(0 + 1i) = (0 \times 0 - 1 \times 1) + i(0 \times 1 + 1 \times 0) = -1$$

The absolute value, or modulus, $|z|$ of $z = x + iy$ is given by

$$|z| \;=\; \sqrt{x^2 + y^2} \;=\; z\bar{z}$$

where $\bar{z} = x - iy$ is called the complex conjugate of z . It is just the distance between z and

© Joel Feldman. 2000. All rights reserved.

the origin. We have

$$
\begin{aligned}
|z_1 z_2| &= \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2} \\
&= \sqrt{x_1^2x_2^2 - 2x_1x_2y_1y_2 + y_1^2y_2^2 + x_1^2y_2^2 + 2x_1y_2x_2y_1 + x_2^2y_1^2} \\
&= \sqrt{x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2} \\
&= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \\
&= |z_1||z_2|
\end{aligned}
$$

and

$$z^{-1} = \frac{z^*}{|z|^2}$$

for all complex numbers z_1, z_2 and $z \neq 0$.

The Complex Exponential

Definition and Basic Properties. For any complex number $z = x + iy$ the exponential e^z, is defined by

$$e^{x+iy} = e^x \cos y + ie^x \sin y$$

For any two complex numbers z_1 and z_2

$$
\begin{aligned}
e^{z_1}e^{z_2} &= e^{x_1}(\cos y_1 + i\sin y_1)e^{x_2}(\cos y_2 + i\sin y_2) \\
&= e^{x_1+x_2}(\cos y_1 + i\sin y_1)(\cos y_2 + i\sin y_2) \\
&= e^{x_1+x_2}\{(\cos y_1\cos y_2 - \sin y_1\sin y_2) + i(\cos y_1\sin y_2 + \cos y_2\sin y_1)\} \\
&= e^{x_1+x_2}\{\cos(y_1+y_2) + i\sin(y_1+y_2)\} \\
&= e^{(x_1+x_2)+i(y_1+y_2)} \\
&= e^{z_1+z_2}
\end{aligned}
$$

so that the familiar multiplication formula also applies to complex exponentials. For any complex number $a = \alpha + i\beta$ and real number t

$$e^{at} = e^{\alpha t + i\beta t} = e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)]$$

© Joel Feldman. 2000. All rights reserved.

so that the derivative with respect to t

$$\begin{aligned}\tfrac{d}{dt}e^{at} &= \alpha e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)] + e^{\alpha t}[-\beta\sin(\beta t) + i\beta\cos(\beta t)]\\ &= (\alpha + i\beta)e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)]\\ &= ae^{at}\end{aligned}$$

is also the familiar one.

Relationship with $\sin$ **and** $\cos$**.** When θ is a real number

$$\begin{aligned}e^{i\theta} &= \cos\theta + i\sin\theta\\ e^{-i\theta} &= \cos\theta - i\sin\theta\end{aligned}$$

are complex numbers of modulus one. Solving for $\cos\theta$ and $\sin\theta$ (by adding and subtracting the two equations)

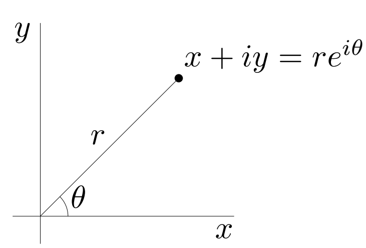
$$\begin{aligned}\cos\theta &= \tfrac{1}{2}(e^{i\theta} + e^{-i\theta})\\ \sin\theta &= \tfrac{1}{2i}(e^{i\theta} - e^{-i\theta})\end{aligned}$$

These formulae make it easy derive trig identities. For example

$$\begin{aligned}\cos\theta\cos\phi &= \tfrac{1}{4}(e^{i\theta} + e^{-i\theta})(e^{i\phi} + e^{-i\phi})\\ &= \tfrac{1}{4}(e^{i(\theta+\phi)} + e^{i(\theta-\phi)} + e^{i(-\theta+\phi)} + e^{-i(\theta+\phi)})\\ &= \tfrac{1}{4}(e^{i(\theta+\phi)} + e^{-i(\theta+\phi)} + e^{i(\theta-\phi)} + e^{i(-\theta+\phi)})\\ &= \tfrac{1}{2}\big(\cos(\theta+\phi) + \cos(\theta-\phi)\big)\end{aligned}$$

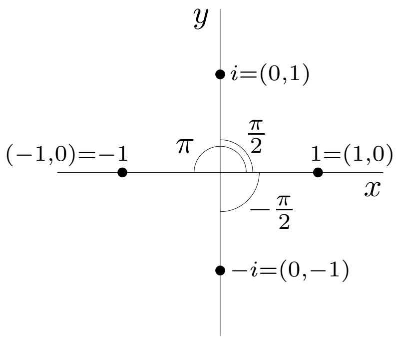
Polar Coordinates. Let $z = x + iy$ be any complex number. Writing x and y in polar coordinates in the usual way gives

$$x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}$$



© Joel Feldman. 2000. All rights reserved.

In particular



$$\begin{aligned} 1 &= e^{i0} = e^{2\pi i} = e^{2k\pi i} && \text{for } k = 0, \pm 1, \pm 2, \cdots \\ -1 &= e^{i\pi} = e^{3\pi i} = e^{(1+2k)\pi i} && \text{for } k = 0, \pm 1, \pm 2, \cdots \\ i &= e^{i\pi/2} = e^{\frac{5}{2}\pi i} = e^{(\frac{1}{2}+2k)\pi i} && \text{for } k = 0, \pm 1, \pm 2, \cdots \\ -i &= e^{-i\pi/2} = e^{\frac{3}{2}\pi i} = e^{(-\frac{1}{2}+2k)\pi i} && \text{for } k = 0, \pm 1, \pm 2, \cdots \end{aligned}$$

The polar coordinate representation makes it easy to find square roots, third roots and so on. Fix any positive integer n. The $n^{\rm th}$ roots of unity are, by definition, all solutions z of

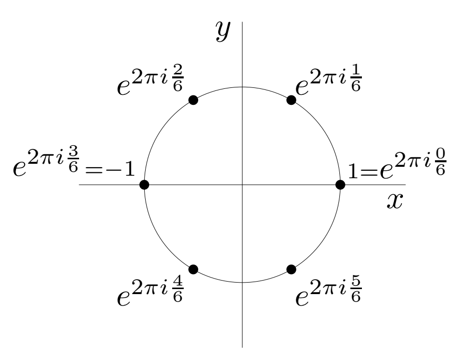
$$z^n = 1$$

Writing $z = re^{i\theta}$

$$r^n e^{n\theta i} = 1e^{0i}$$

The polar coordinates (r, θ) and (r', θ') represent the same point in the xy–plane if and only if $r = r'$ and $\theta = \theta' + 2k\pi$ for some integer k. So $z^n = 1$ if and only if $r^n = 1$, i.e. $r = 1$, and $n\theta = 2k\pi$ for some integer k. The $n^{\rm th}$ roots of unity are all complex numbers $e^{2\pi i\frac{k}{n}}$ with k integer. There are precisely n distinct $n^{\rm th}$ roots of unity because $e^{2\pi i\frac{k}{n}} = e^{2\pi i\frac{k'}{n}}$ if and only if $2\pi\frac{k}{n} - 2\pi i\frac{k'}{n} = 2\pi\frac{k-k'}{n}$ is an integer multiple of 2π. That is, if and only if $k - k'$ is an integer multiple of n. The are n distinct nth roots of unity are

$$1 \ , \ e^{2\pi i\frac{1}{n}} \ , \ e^{2\pi i\frac{2}{n}} \ , \ e^{2\pi i\frac{3}{n}} \ , \ \cdots \ , \ e^{2\pi i\frac{n-1}{n}}$$



© Joel Feldman. 2000. All rights reserved.