A Lightning Fast Review of Determinants

Let A be an $n\times n$ matrix. Then, $\det A$ is the number given by

$$\begin{aligned}\det A &= \sum_{j=1}^{n} A_{ij}(-1)^{i+j}\det M_{ij}\\ &= \sum_{i=1}^{n} A_{ij}(-1)^{i+j}\det M_{ij}\end{aligned}$$

In the upper formula $1\le i\le n$ is any fixed row. The upper formula is called expansion along the $i^{\rm th}$ row. In the lower formula $1\le j\le n$ is any fixed column. The lower formula is called expansion along the $j^{\rm th}$ column. The sign $(-1)^{i+j}$ is given by the checkerboard pattern

$$\begin{pmatrix} + & - & + & - & \cdots\\ - & + & - & + & \cdots\\ + & - & + & - & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}$$

In both formulae M_{ij} is the $(n-1)\times(n-1)$ matrix formed by deleting the $i^{\rm th}$ row and $j^{\rm th}$ column from A. So the above formulae express $n\times n$ determinants in terms of $(n-1)\times(n-1)$ determinants. Applying them $n-1$ times reduces the problem of computing an $n\times n$ determinant to the problem of computing many 1×1 determinants. Of course $\det\left(\, a_{11}\,\right) = a_{11}$.

2×2 **case:** Expanding along the first column

$$\begin{aligned}\det\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} &= a_{11}\det\left(\, a_{22}\,\right) - a_{21}\det\left(\, a_{12}\,\right)\\ &= a_{11}a_{22} - a_{12}a_{21}\end{aligned}$$

3×3 **case:** Expanding along the first row

$$\begin{aligned}&\det\begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}\\ &= a_{11}\det\begin{pmatrix} a_{22} & a_{23}\\ a_{32} & a_{33}\end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21} & a_{23}\\ a_{31} & a_{33}\end{pmatrix} + a_{13}\det\begin{pmatrix} a_{21} & a_{22}\\ a_{31} & a_{32}\end{pmatrix}\\ &= a_{11}\left(a_{22}a_{33} - a_{32}a_{23}\right) - a_{12}\left(a_{21}a_{33} - a_{31}a_{23}\right) + a_{13}\left(a_{21}a_{32} - a_{31}a_{22}\right)\\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}\end{aligned}$$

Properties

1.

$$\det\left(\, \vec a_1 + \alpha\vec b_1, \vec a_2, \cdots, \vec a_n\,\right) = \det\left(\, \vec a_1, \vec a_2, \cdots, \vec a_n\,\right) + \alpha\det\left(\, \vec b_1, \vec a_2, \cdots, \vec a_n\,\right)$$

and similarly for other rows and columns. That is, the determinant is linear in each row and column. Linearity in the first row is an immediate consequence of the formula for expansion along the first row.

2. If A has a zero row or column, then $\det A = 0$. To see this, just expand along the zero row or column.
3. If any row of A is a multiple of another row of A then $\det A = 0$. Similarly for columns. To see this use linearity, i.e. property 1, to replace one row (or column) by zero and then apply property 2.
4. $$\det\left(\cdots,\vec a_i,\cdots,\vec a_j,\cdots\right) = \det\left(\cdots,\vec a_i,\cdots,\vec a_j+\alpha\vec a_i,\cdots\right)$$
   and similarly for rows. This is immediate from property 1 followed by property 3.
5. Exchanging two rows or columns changes the sign of a determinant. This may be seen as follows:

$$\begin{aligned}
\det\left(\cdots,\vec a_i,\cdots,\vec a_j,\cdots\right) &= \det\left(\cdots,\vec a_i,\cdots,\vec a_j-\vec a_i,\cdots\right) && \text{(property 4)}\\
&= \det\left(\cdots,\vec a_i+\vec a_j-\vec a_i,\cdots,\vec a_j-\vec a_i,\cdots\right) && \text{(property 4)}\\
&= \det\left(\cdots,\vec a_j,\cdots,\vec a_j-\vec a_i,\cdots\right)\\
&= \det\left(\cdots,\vec a_j,\cdots,\vec a_j,\cdots\right) - \det\left(\cdots,\vec a_j,\cdots,\vec a_i,\cdots\right)\\
& && \text{(property 1)}\\
&= -\det\left(\cdots,\vec a_j,\cdots,\vec a_i,\cdots\right) && \text{(property 3)}
\end{aligned}$$

6. The determinant of any triangular matrix is the product of its diagonal entries.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots\\ 0 & a_{22} & a_{23} & \cdots\\ 0 & 0 & a_{33} & \cdots\\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = a_{11}a_{22}a_{33}\cdots$$

   To see this, repeatedly expand along the first column.
7. $$\det AB = \det A \det B$$
8. $$\begin{aligned}\det A = 0 \qquad &\iff \qquad A^{-1} \text{ does not exist}\\ &\iff \qquad A\vec x = \vec 0 \text{ has a nonzero solution}\end{aligned}$$

Example

$$\det\begin{pmatrix} 0 & 1 & -2 & 3 \\ -2 & 0 & 2 & 4 \\ 2 & -1 & 0 & 1 \\ -3 & -2 & -1 & 0 \end{pmatrix} = -2\det\begin{pmatrix} 0 & 1 & -2 & 3 \\ 1 & 0 & -1 & -2 \\ 2 & -1 & 0 & 1 \\ -3 & -2 & -1 & 0 \end{pmatrix} \qquad \text{(property 1)}$$

$$= -2\det\begin{pmatrix} 0 & 1 & -2 & 3 \\ 1 & 0 & -1 & -2 \\ 0 & -1 & 2 & 5 \\ 0 & -2 & -4 & -6 \end{pmatrix} \begin{matrix} \\ \\ \text{row 3 } -2\times\text{row 2} \\ \text{row 4 } +3\times\text{row 2} \end{matrix} \qquad \text{(ppty 4)}$$

$$= 2\det\begin{pmatrix} 1 & -2 & 3 \\ -1 & 2 & 5 \\ -2 & -4 & -6 \end{pmatrix} \qquad \text{(expand on column 1)}$$

$$= 4\det\begin{pmatrix} 1 & -2 & 3 \\ -1 & 2 & 5 \\ -1 & -2 & -3 \end{pmatrix} \qquad \text{(property 1)}$$

$$= 4\det\begin{pmatrix} 1 & -2 & 3 \\ 0 & 0 & 8 \\ -1 & -2 & -3 \end{pmatrix} \text{ row 2 } +\text{row 1} \qquad \text{(property 4)}$$

$$= -32\det\begin{pmatrix} 1 & -2 \\ -1 & -2 \end{pmatrix} \qquad \text{(expand on row 2)}$$

$$= -32\det\begin{pmatrix} 1 & -2 \\ 0 & -4 \end{pmatrix} \begin{matrix} \\ \text{row 2 } +\text{row 1} \end{matrix} \qquad \text{(property 4)}$$

$$= (-32)\times 1\times(-4) \qquad \text{(property 6)}$$

$$= 128$$

Applications

1. Testing for invertibility. In the eigenvalue problem one must find λ so that $A\vec{\xi} = \lambda\vec{\xi}$ has a nonzero solution $\vec{\xi}$, i.e. so that $(A-\lambda I)\vec{\xi}$ has a nonzero solution. By property 8, this is the case if and only if $\det(A-\lambda I) = 0$. This gives a polynomial equation for the eigenvalue λ.
2. Concise formulae.

   a) $$\left(A^{-1}\right)_{ij} = (-1)^{i+j}\frac{\det M_{ij}}{\det A}$$

   b) $$\text{If } \det A \neq 0, \text{ then } A\vec{x} = \vec{b} \iff x_i = \frac{\det(A \text{ with column } i \text{ replaced by } \vec{b})}{\det A}$$

   c) $$\vec{a}\times\vec{b} = \det\begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

   In the formula for the inverse, M_{ij} is the matrix constructed from A by deleting the i^{th} row and j^{th} column. Formulae a) and b) are good for getting properties of inverses and solutions of linear equations. But they are horrendously inefficient for computing with.
4. The volume of a parallelepiped with edges $\vec{a}_1, \cdots, \vec{a}_n$ is $|\det(\vec{a}_1 \ \cdots \ \vec{a}_n)|$. This is why, when you make the change of variables $\vec{x} = A\vec{x}'$ in an integral, you replace $dx_1\cdots dx_n$ by $|\det A|\, dx_1'\cdots dx_n'$.