A Lightning Fast Review of Eigenvalues and Eigenvectors

Let A be an $n\times n$ matrix. Then, by **definition**, $\vec{v}$ is an eigenvector of A with eigenvalue λ if and only if

(i) $\vec{v} \neq \vec{0}$

(ii) $A\vec{v} = \lambda\vec{v}$

Fix λ for a moment. Then $(A-\lambda I)\vec{v} = \vec{0}$ has a nonzero solution $\vec{v}$ if and only if the matrix $A-\lambda I$ fails to have an inverse. This, in turn, is the case if and only if the matrix has determinant zero. Hence the eigenvalues of A are determined by

$$\det(A-\lambda I) = 0$$

This determinant is a polynomial in λ of degree n. Consequently, it has precisely n roots, counting multiplicity. Given an eigenvalue λ of A the corresponding eigenvectors are found by solving

$$(A-\lambda I)\vec{v} = \vec{0}$$

Each distinct eigenvector has at least one eigenvector. If λ has multiplicity m then λ may have form 1 to m linearly independent eigenvectors.

Example 1 $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

$$0 = \det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1)$$

The eigenvalues of A are $\lambda = -1, 2$.

$$\text{For } \lambda = -1, \quad \begin{pmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{pmatrix}\vec{v} = 0 \quad\Longrightarrow\quad \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}\vec{v} = \vec{0} \quad\Longrightarrow\quad \vec{v} = \text{const}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\text{For } \lambda = 2, \quad \begin{pmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{pmatrix}\vec{v} = 0 \quad\Longrightarrow\quad \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}\vec{v} = \vec{0} \quad\Longrightarrow\quad \vec{v} = \text{const}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Example 2 $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$0 = \det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^3$$

The eigenvalues of A are $\lambda = 1, 1, 1$. For $\lambda = 1$

$$\begin{pmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix}\vec{v} = 0 \quad\Longrightarrow\quad \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{0} \quad\Longrightarrow\quad v_2 + v_3 = 0$$

which has general solution $v_1 = \alpha$, arbitrary, $v_3 = \beta$, arbitrary, and $v_2 = -\beta$ or

$$\vec{v} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

There are two linearly independent eigenvectors of eigenvalue 1, which can be chosen to be $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

Remarks

1) If A is 2×2, then $\det(A - \lambda I) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A$.
   If A is 3×3, then $\det(A - \lambda I) = -\lambda^3 + (\operatorname{tr} A)\lambda^2 - \left(\sum_{i=1}^3 \det M_i\right)\lambda + \det A$.
   Here $\operatorname{tr} A = \sum_{i=1}^3 A_{ii}$ is the trace of A and M_i is the 2×2 matrix gotten by deleting the $i^{\rm th}$ row and column from A.
2) If A is triangular (i.e. all entries on one side of the diagonal of A are zero) then the eigenvalues of A are just the diagonal entries of A.
3) If $A_{ij} = \overline{A_{ji}}$ for all $1 \le i, j \le n$, (such matrices are called symmetric, or Hermitian or self-adjoint) then all of the eigenvalues of A are real and A has n linearly independent eigenvectors (even if A has multiple eigenvalues).
4) Suppose that the $n \times n$ matrix A has n linearly independent eigenvectors $\vec{v}_i$ with corresponding eigenvalues λ_i. Define $V = (\vec{v}_1, \cdots, \vec{v}_n)$ (i.e. V is the matrix whose $j^{\rm th}$ column is $\vec{v}_j$) and define Λ to be the diagonal matrix whose (j, j) entry is λ_j. Then

   $$A\vec{v}_j = \lambda_j \vec{v}_j, \ 1 \le j \le n$$

   is equivalent to

   $$AV = A(\vec{v}_1, \cdots, \vec{v}_n) = (A\vec{v}_1, \cdots, A\vec{v}_n) = (\lambda_1\vec{v}_1, \cdots, \lambda_n\vec{v}_n) = V\Lambda$$

   by the definition of matrix multiplication. Hence

   $$A = V\Lambda V^{-1} \qquad V^{-1}AV = \Lambda$$