Periodic Extensions

If a function $f(x)$ is only defined for $0 < x < \ell$ we can get many Fourier expansions for f by using the following

Main Idea If $F(x)$ is periodic of period 2ℓ (and hence has a Fourier series expansion) and if $f(x) = F(x)$ for $0 < x < \ell$ then, for all x between 0 and ℓ,

$$\begin{aligned} f(x) &= F(x) \\ &= \text{Fourier series for } F(x) \end{aligned}$$

Motivated by this observation we define $F(x)$ to be a **periodic extension** of $f(x)$ if

$$i)\ F(x) = f(x) \ \text{ for } \ 0 < x < \ell$$

$$ii)\ F(x) \text{ is periodic of period } 2\ell$$

There are many periodic extensions of $f(x)$. Most of them are pretty useless. For example define $g(x) = 1$ for $0 < x < \pi$. We are not defining $g(x)$ for $x \ge \pi$ or $x \le 0$. Then

$$G(x) = \begin{cases} 1 & \text{if } \ 2n\pi \le x < (2n+1)\pi \ \text{ for some integer } n \\ -x + (2n+2)\pi & \text{if } \ (2n+1)\pi \le x < (2n+2)\pi \ \text{ for some integer } n \end{cases}$$

is a periodic extension of g. Its graph is drawn in the figure at the end of this handout.

There are two periodic extensions for $f(x)$ that are very useful. Given $f(x)$ defined for $0 < x < \ell$ we define its **even periodic extension** $F^e(x)$ by

a) $F^e(x) = f(x)$ for $0 < x < \ell$

b) $F^e(x)$ is even (This fixes $F^e(x)$ for $-\ell < x < 0$.)

c) $F^e(x)$ has period 2ℓ (This fixes $F^e(x)$ for all remaining x's, except $x = n\pi$.)

and we define its **odd periodic extension** $F^o(x)$ by

a) $F^o(x) = f(x)$ for $0 < x < \ell$

b) $F^o(x)$ is odd

c) $F^o(x)$ has period 2ℓ

By the Main Idea we have, for all $0 < x < \ell$

$$\begin{aligned} f(x) = F^e(x) &= \tfrac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\tfrac{k\pi x}{\ell}\right) \\ &= F^o(x) = \sum_{k=1}^{\infty} b_k \sin\left(\tfrac{k\pi x}{\ell}\right) \end{aligned}$$

where

$$a_k = \tfrac{2}{\ell}\int_0^\ell F^e(x)\cos\left(\tfrac{k\pi x}{\ell}\right)dx = \tfrac{2}{\ell}\int_0^\ell f(x)\cos\left(\tfrac{k\pi x}{\ell}\right)dx$$
$$b_k = \tfrac{2}{\ell}\int_0^\ell F^o(x)\sin\left(\tfrac{k\pi x}{\ell}\right)dx = \tfrac{2}{\ell}\int_0^\ell f(x)\sin\left(\tfrac{k\pi x}{\ell}\right)dx$$

For example consider, again, the function $g(x)$ which is only defined for $0 < x < \pi$ and takes the value $g(x) = 1$ for all $0 < x < \pi$. The even and odd periodic extensions, $G^e(x)$ and $G^o(x)$ of this function are graphed on the next page. Both $G^e(x)$ and $G^o(x)$ take the value 1 for all $0 < x < \pi$. Both $G^e(x)$ and $G^o(x)$ have period 2π. But $G^e(x)$ is an even function while $G^o(x)$ is an odd function. Because it is an even periodic function, $G^e(x)$ has the Fourier series expansion

$$G^e(x) = \tfrac{a_0}{2} + \sum_{k=1}^\infty a_k\cos\left(\tfrac{k\pi x}{\ell}\right)$$

with

$$a_k = \tfrac{2}{\pi}\int_0^\pi 1\cos\left(\tfrac{k\pi x}{\ell}\right)dx = \begin{cases} 2 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

That is,

$$G^e(x) = \tfrac{a_0}{2} + \sum_{k=1}^\infty a_k\cos\left(\tfrac{k\pi x}{\ell}\right) = 1 \qquad\qquad \text{(surprise!)}$$

Because it is an odd periodic function, $G^o(x)$ has the Fourier series expansion

$$G^o(x) = \sum_{k=1}^\infty b_k\sin\left(\tfrac{k\pi x}{\ell}\right)$$

with

$$b_k = \tfrac{2}{\pi}\int_0^\pi 1\sin\left(\tfrac{k\pi x}{\ell}\right)dx = \begin{cases} 0 & \text{if } k \text{ is even} \\ \tfrac{4}{k\pi} & \text{if } k \text{ is odd} \end{cases}$$

That is

$$G^o(x) = \sum_{\substack{k=1 \\ k \text{ odd}}}^\infty \tfrac{4}{k\pi}\sin\left(\tfrac{k\pi x}{\ell}\right)$$

For all $0 < x < \pi$ we have both

$$g(x) = G^e(x) = \tfrac{a_0}{2} + \sum_{k=1}^\infty a_k\cos\left(\tfrac{k\pi x}{\ell}\right) = 1$$
$$g(x) = G^o(x) = \sum_{k=1}^\infty b_k\sin\left(\tfrac{k\pi x}{\ell}\right) = \sum_{\substack{k=1 \\ k \text{ odd}}}^\infty \tfrac{4}{k\pi}\sin\left(\tfrac{k\pi x}{\ell}\right)$$

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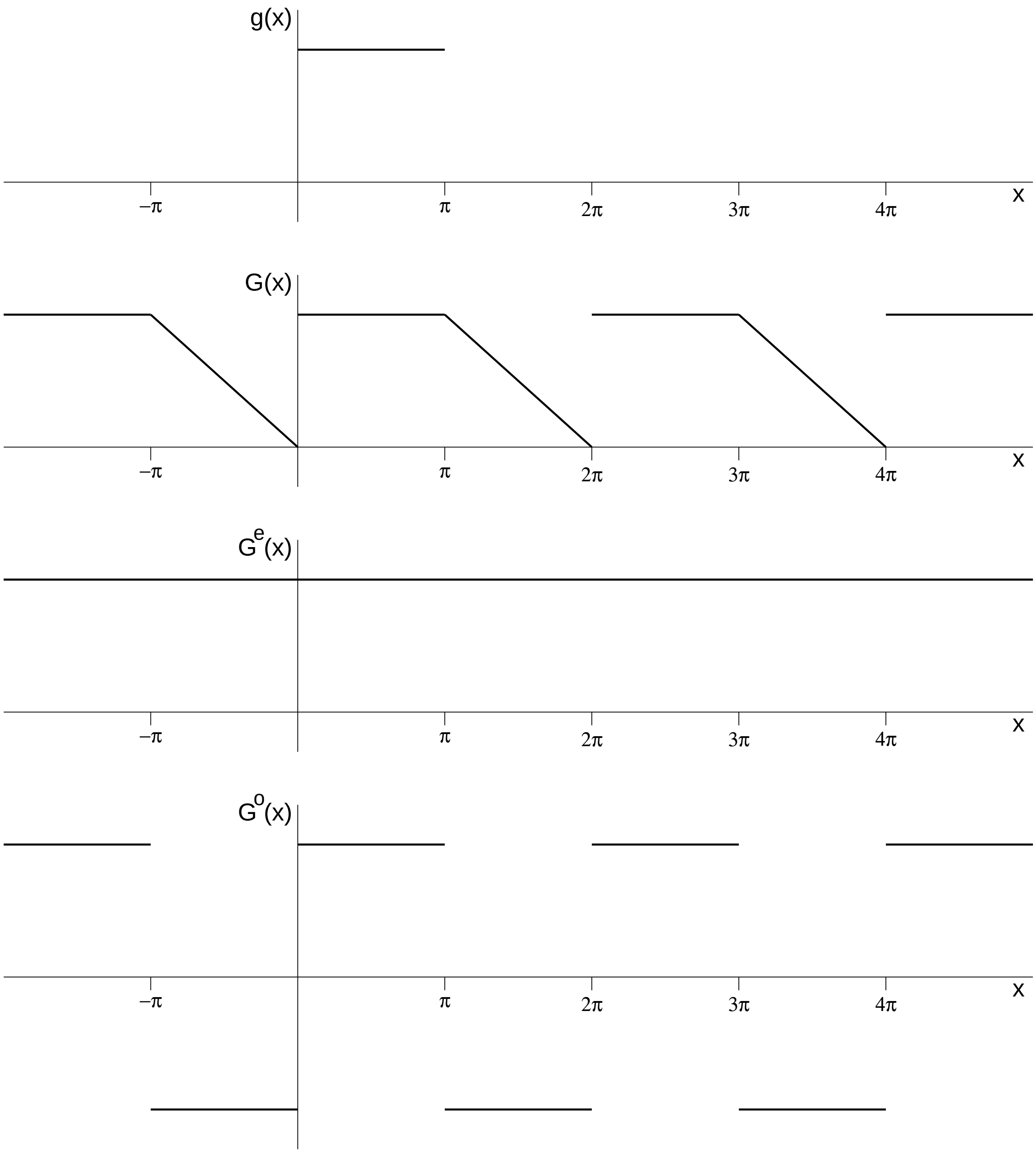
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