Background for Lab 4

This laboratory concerns, in part, the phenomenon of beats. Recall that a mass m hanging from a spring and subject to a periodic force obeys the differential equation

$$my'' + \gamma y' + ky = F_0 \cos(\omega t)$$

where

- y is the position of the mass measured with respect to its equilibrium position.
- my'' is the mass times acceleration term in Newton's law.
- $\gamma y'$ is (minus) the frictional force acting on the mass.
- k is the spring constant and ky is (minus) the force exerted by the spring on the mass.
- $F_0 \cos(\omega t)$ is the periodic external force.

We have seen in class that, if $\gamma > 0$, this equation has general solution of the form

$$y(t) = R\cos(\omega t - \delta) + \begin{cases} Ae^{-\gamma t/2m}\cos(\mu t - \delta) & \text{underdamped} \\ C_1 e^{-\gamma t/2m} + C_2 t e^{-\gamma t/2m} & \text{critically damped} \\ C_1 e^{-r_1 t} + C_2 e^{-r_2 t} & \text{overdamped} \end{cases}$$

and that the $Ae^{-\gamma t/2m}\cos(\mu t - \delta)$ or $C_1 e^{-\gamma t/2m} + C_2 t e^{-\gamma t/2m}$ or $C_1 e^{-r_1 t} + C_2 e^{-r_2 t}$ part of the solution tends to zero exponentially as $t \to \infty$. This part of the solution is called the transient part of the solution. It damps to zero because of energy losses arising from the presence of the damping term $\gamma y'$ in the differential equation.

Beats arise when there is no damping (or at least the damping is small enough to be ignored), so that $\gamma = 0$. Then the differential equation simplifies to

$$my'' + ky = F_0 \cos(\omega t)$$

As e^{rt} obeys the homogeneous equation $my'' + ky = 0$ if and only if $r = \pm i\omega_0$ with $\omega_0 = \sqrt{\frac{k}{m}}$, the general solution of $my'' + ky = 0$ is $D_1 e^{i\omega_0 t} + D_2 e^{-i\omega_0 t}$, or equivalently $C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$ (with $C_1 = D_1 + D_2$ and $C_2 = iD_1 - iD_2$). Subbing $y = R\cos(\omega t)$ into $my'' + ky$ gives $F_0\cos(\omega t)$ if and only if $R = \frac{F_0}{k - m\omega^2} = \frac{F_0}{m(\omega_0^2 - \omega^2)}$. (We shall assume that $\omega_0 \neq \omega$.) Hence the general solution of $my'' + ky = F_0\cos(\omega t)$ is

$$y(t) = C_1\cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \tfrac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t)$$

If the mass starts out at rest in its equilibrium position then $y(0) = y'(0) = 0$. These initial conditions give $C_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)}$ and $C_2 = 0$ so that

$$y(t) = \tfrac{F_0}{m(\omega_0^2 - \omega^2)}\Big(\cos(\omega t) - \cos(\omega_0 t)\Big) = 2\tfrac{F_0}{m(\omega_0^2 - \omega^2)}\sin\left(\tfrac{\omega_0 - \omega}{2}t\right)\sin\left(\tfrac{\omega + \omega_0}{2}t\right)$$

by the trig identity $\cos a - \cos b = 2\sin\frac{a+b}{2}\sin\frac{b-a}{2}$. If the forcing frequency ω and the natural frequency ω_0 are reasonably close, $\sin\left(\frac{\omega+\omega_0}{2}t\right)$ oscillates at frequency roughly ω, while $\sin\left(\frac{\omega_0-\omega}{2}t\right)$ oscillates quite slowly. Hence we can think of $y(t)$ as a vibration with frequency $\frac{\omega+\omega_0}{2} \approx \omega$ and amplitude $2\frac{F_0}{m(\omega_0^2-\omega^2)}\sin\left(\frac{\omega_0-\omega}{2}t\right)$. The amplitude oscillates slowly in time, with frequency $\frac{\omega_0-\omega}{2}$. Guitar players exploit this phenomenon, called beats, to tune their guitars. They play two neighbouring strings at the same time, while placing a finger of their left hand on one string at the fret that should make the two strings have the same pitch. If the guitar is not quite in tune, the pitches of the two strings are not quite the same and the resulting sound gets alternately louder and softer with a frequency that is half the difference of the frequencies produced by the two vibrating strings.