Roots of Polynomials

Here are some tricks for finding roots of polynomials that work well on exams and homework assignments, where polynomials tend to have integer coefficients and lots of integer, or at least rational roots.

Trick # 1

If r or $-r$ is an integer root of a polynomial $a_nx^n + \cdots + a_1x + a_0$ with integer coefficients, then r is a factor of the constant term a_0.

To see that this is true, just observe that for any root $\pm r$

$$a_n(\pm r)^n + \cdots + a_1(\pm r) + a_0 = 0 \quad \Longrightarrow \quad a_0 = -\big[a_n(\pm r)^n + \cdots + a_1(\pm r)\big]$$

Every term on the right hand side is an integer times a strictly positive power of r. So the right hand side, and hence the left hand side, is some integer times r.

Example. $P(\lambda) = \lambda^3 - \lambda^2 + 2$.

The constant term in this polynomial is $2 = 1 \times 2$. So the only candidates for integer roots are $\pm 1,\ \pm 2$. Trying each in turn

$$P(1) = 2 \qquad P(-1) = 0 \qquad P(2) = 6 \qquad P(-2) = -10$$

so the only integer root is -1.

Trick # 2

If b/d or $-b/d$ is a rational root in lowest terms (i.e. b and d are integers with no common factors) of a polynomial $a_nx^n + \cdots + a_1x + a_0$ with integer coefficients, then the numerator b is a factor of the constant term a_0 and the denominator d is a factor of a_n.

For any root $\pm b/d$

$$a_n(\pm b/d)^n + \cdots + a_1(\pm b/d) + a_0 = 0$$

Multiply through by d^n

$$a_0d^n = -\big[a_n(\pm b)^n + a_{n-1}d(\pm b)^{n-1} + \cdots + a_1d^{n-1}(\pm b)\big]$$

Every term on the right hand side is an integer times a strictly positive power of b. So the right hand side is some integer times b. The left hand side is d^na_0 and d does not contain any factor that is a factor of b. So a_0 must be some integer times b. Similarly, every term on the right hand side of

$$a_n(\pm b)^n = -\big[a_{n-1}d(\pm b)^{n-1} \cdots + a_1d^{n-1}(\pm b) + a_0d^n\big]$$

is an integer times a strictly positive power of d. So the right hand side is some integer times d. The left hand side is $a_n(\pm b)^n$ and b does not contain any factor that is a factor of d. So a_n must be some integer times d.

Example. $P(\lambda) = 2\lambda^2 - \lambda - 3$.

The constant term in this polynomial is $3 = 1 \times 3$ and the coefficient of the highest power of λ is $2 = 1 \times 2$. So the only candidates for integer roots are $\pm 1,\ \pm 3$ and the only candidates for fractional roots are $\pm\frac{1}{2},\ \pm\frac{3}{2}$.

$$P(1) = -2 \quad P(-1) = 0 \quad P\big(\pm 3\big) = 18 \mp 3 - 3 \neq 0 \quad P\big(\pm \tfrac{1}{2}\big) = \tfrac{1}{2} \mp \tfrac{1}{2} - 3 \neq 0 \quad P\big(\pm \tfrac{3}{2}\big) = \tfrac{9}{2} \mp \tfrac{3}{2} - 3$$

so the roots are -1 and $\frac{3}{2}$.

Trick # 3

Once you have found one root r of a polynomial, you can divide by $\lambda - r$, using the long division algorithm you learned in public school, but with 10 replaced by λ, to reduce the degree of the polynomial by one.

Example. $P(\lambda) = \lambda^3 - \lambda^2 + 2$.

We have already determined that -1 is a root of this polynomial. So we divide $\frac{\lambda^3-\lambda^2+2}{\lambda+1}$.

$$
\begin{array}{r|l}
 & \lambda^2 - 2\lambda + 2 \\
\cline{2-2}
\lambda + 1 & \lambda^3 - \lambda^2 + \qquad 2 \\
 & \underline{\lambda^3 + \lambda^2} \\
 & \quad -2\lambda^2 \\
 & \quad \underline{-2\lambda^2 - 2\lambda} \\
 & \qquad\qquad 2\lambda + 2 \\
 & \qquad\qquad \underline{2\lambda + 2} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

The first term, λ^2, in the quotient is chosen so that when you multiply it by the denominator, $\lambda^2(\lambda+1) = \lambda^3+\lambda^2$, the leading term, λ^3, matches the leading term in the numerator, $\lambda^3 - \lambda^2 + 2$, exactly. When you subtract $\lambda^2(\lambda+1) = \lambda^3 + \lambda^2$ from the numerator $\lambda^3 - \lambda^2 + 2$ you get the remainder $-2\lambda^2 + 2$. Just like in public school, the 2 is not normally "brought down" until it is actually needed. The next term, -2λ, in the quotient is chosen so that when you multiply it by the denominator, $-2\lambda(\lambda+1) = -2\lambda^2 - 2\lambda$, the leading term $-2\lambda^2$ matches the leading term in the remainder exactly. And so on. Note that we finally end up with a remainder 0. Since -1 is a root of the numerator, $\lambda^3 - \lambda^2 + 2$, the denominator $\lambda - (-1)$ must divide the numerator exactly.

Here is an alternative to long division that involves more writing. In the previous example, we know that $\frac{\lambda^3-\lambda^2+2}{\lambda+1}$ must be a polynomial (since -1 is a root of the numerator) of degree 2. So

$$\frac{\lambda^3 - \lambda^2 + 2}{\lambda + 1} = a\lambda^2 + b\lambda + c$$

for some, as yet unknown, coefficients a, b and c. Cross multiplying and simplifying

$$\begin{aligned}
\lambda^3 - \lambda^2 + 2 &= (a\lambda^2 + b\lambda + c)(\lambda + 1) \\
&= a\lambda^3 + (a+b)\lambda^2 + (b+c)\lambda + c
\end{aligned}$$

Matching coefficients of the various powers of λ on the left and right hand sides

$$a = 1 \qquad a + b = -1 \qquad b + c = 0 \qquad c = 2$$

forces

$$a = 1 \qquad b = -2 \qquad c = 2$$