

Approximating Functions Near a Specified Point

Suppose that you are interested in the values of some function $f(x)$ for x near some fixed point x_0 . The function is too complicated to work with directly. So you wish to work instead with some other function $F(x)$ that is both simple and a good approximation to $f(x)$ for x near x_0 . We'll consider an example of this scenario later. First, we use the Fundamental Theorem of Calculus and integration by parts to develop several different approximations.

The Fundamental Theorem of Calculus says that $\int_{x_0}^x f'(t) dt = f(x) - f(x_0)$, or equivalently,

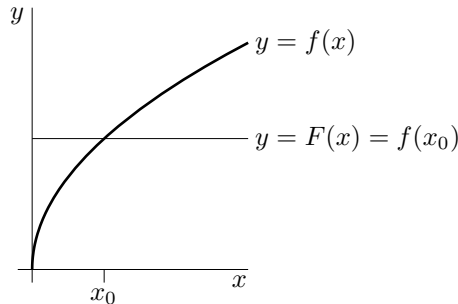
$$f(x) = f(x_0) + \int_{x_0}^x f'(t) dt \quad (1')$$

First approximation

The simplest functions are those that are constants. Our first, and crudest, approximation is the constant function $f(x_0)$.

$$f(x) \approx f(x_0) \quad (1)$$

Here is a figure showing the graphs of a typical $f(x)$ and approximating function $F(x)$. At $x = x_0$, $f(x)$ and



$F(x)$ take the same value. For x very near x_0 , the values of $f(x)$ and $F(x)$ remain close together. But the quality of the approximation deteriorates fairly quickly as x moves away from x_0 . Equation (1') says that the error you make when you approximate $f(x)$ by $f(x_0)$ (namely $|f(x) - f(x_0)|$) is exactly $\left| \int_{x_0}^x f'(t) dt \right|$. Usually this integral is too complicated to evaluate in a useful way (or you wouldn't be approximating $f(x)$ in this first place). But if you can find a constant M_1 such that $|f'(t)| \leq M_1$ for all t between x_0 and x (we'll see examples of this later), then

$$\text{error} = \left| \int_{x_0}^x f'(t) dt \right| \leq \left| \int_{x_0}^x M_1 dt \right| = M_1 |x - x_0|$$

So the error grows at most linearly as x moves away from x_0 .

Second approximation – the tangent line, or linear, approximation

We now develop a better approximation by applying integration by parts to the integral in (1'). Note that the integration variable is t . As far as the integration is concerned, x is a constant. We use $u(t) = f'(t)$ and $v(t) = x - t$. Since x is a constant, $dv = v'(t)dt = -dt$.

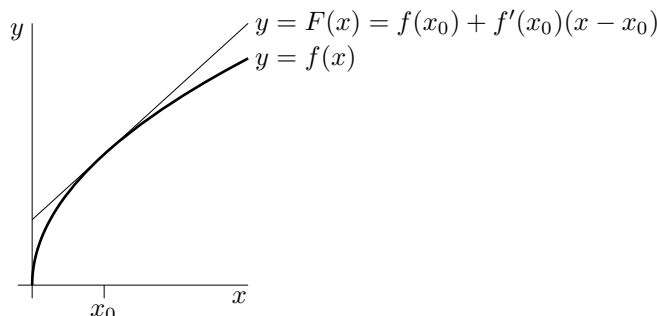
$$\begin{aligned} f(x) &= f(x_0) + \int_{x_0}^x f'(t) dt = f(x_0) - \int_{x_0}^x u(t)v'(t) dt = f(x_0) - u(x)v(x) + u(x_0)v(x_0) + \int_{x_0}^x v(t)u'(t) dt \\ &= f(x_0) + f'(x_0)(x - x_0) + \int_{x_0}^x (x - t)f''(t) dt \end{aligned} \quad (2')$$

Our second approximation is

$$\boxed{f(x) \approx f(x_0) + f'(x_0)(x - x_0)} \quad (2)$$

It is exactly the tangent line approximation to $f(x)$ at x_0 . The original function $f(x)$ and the approximating function $F(x) = f(x_0) + f'(x_0)(x - x_0)$ have the same value and the same slope at $x = x_0$. That is, $f(x_0) = F(x_0)$ and $f'(x_0) = F'(x_0)$. Here is a figure showing the graphs of a typical $f(x)$ and approximating function $F(x)$.

Observe that the graph of $f(x_0) + f'(x_0)(x - x_0)$ remains close to the graph of $f(x)$ for a much larger range



of x than did the graph of $f(x_0)$. Equation (2') says that the error you make when you approximate $f(x)$ by $f(x_0) + f'(x_0)(x - x_0)$ is exactly $\left| \int_{x_0}^x (x - t)f''(t) dt \right|$. Again, this integral is usually too complicated to evaluate in a useful way. But if you can find a constant M_2 such that $|f''(t)| \leq M_2$ for all t between x_0 and x , then

$$\text{error} = \left| \int_{x_0}^x (x - t)f''(t) dt \right| \leq \left| \int_{x_0}^x (x - t)M_2 dt \right| = \left| -M_2 \frac{(x-t)^2}{2} \Big|_{t=x_0}^{t=x} \right| = \frac{M_2}{2}(x - x_0)^2$$

So the error grows at most quadratically as x moves away from x_0 .

Third approximation – the quadratic approximation

We develop a still better approximation by applying integration by parts a second time – this time to the integral in (2'). We use $u(t) = f''(t)$ and $v(t) = \frac{1}{2}(x - t)^2$, so that $v'(t) = -(x - t)$.

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \int_{x_0}^x (x - t)f''(t) dt = f(x_0) + f'(x_0)(x - x_0) - \int_{x_0}^x u(t)v'(t) dt \\ &= f(x_0) + f'(x_0)(x - x_0) - u(x)v(x) + u(x_0)v(x_0) + \int_{x_0}^x v(t)u'(t) dt \\ &= f(x_0) + f'(x_0)(x - x_0) - \frac{1}{2}f''(x)(x - x_0)^2 + \frac{1}{2}f''(x_0)(x - x_0)^2 + \frac{1}{2} \int_{x_0}^x (x - t)^2 f^{(3)}(t) dt \\ &= f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \frac{1}{2} \int_{x_0}^x (x - t)^2 f^{(3)}(t) dt \end{aligned} \quad (3')$$

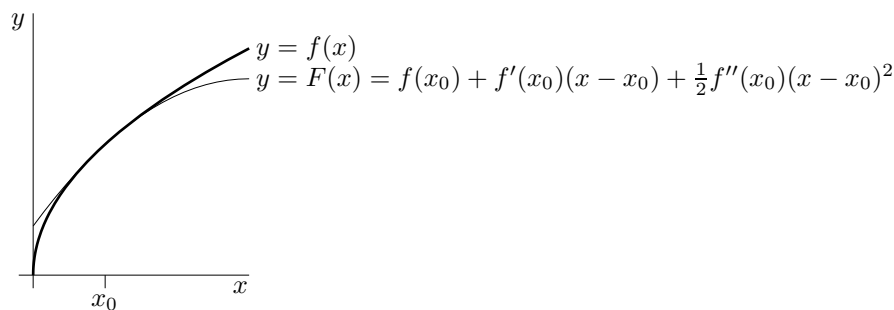
Our third approximation is

$$\boxed{f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2} \quad (3)$$

It is called the quadratic approximation. Equation (3') says that the error you make when you approximate $f(x)$ by $f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2$ is exactly $\left| \frac{1}{2} \int_{x_0}^x (x - t)^2 f^{(3)}(t) dt \right|$. If you can find a constant M_3 such that $|f^{(3)}(t)| \leq M_3$ for all t between x_0 and x , then

$$\text{error} = \left| \frac{1}{2} \int_{x_0}^x (x - t)^2 f^{(3)}(t) dt \right| \leq \left| \frac{1}{2} \int_{x_0}^x (x - t)^2 M_3 dt \right| = \left| -M_3 \frac{(x-t)^3}{2 \times 3} \Big|_{t=x_0}^{t=x} \right| = \frac{M_3}{3!} |x - x_0|^3$$

where $3!$ (read “three factorial”) means $1 \times 2 \times 3$. Here is a figure showing the graphs of a typical $f(x)$ and quadratic approximating function $F(x)$. The third approximation looks better than both the first and second.



Still better approximations – Taylor polynomials

We can use the same strategy to generate still better approximations by polynomials of any degree we like. Integrating by parts repeatedly gives, for any natural number n ,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \frac{1}{3!}f^{(3)}(x_0)(x - x_0)^3 + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n + \frac{1}{n!} \int_{x_0}^x (x - t)^n f^{(n+1)}(t) dt \quad (4')$$

If you can find a constant M_{n+1} such that $|f^{(n+1)}(t)| \leq M_{n+1}$ for all t between x_0 and x , then the error in the approximation

$$\boxed{f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \frac{1}{3!}f^{(3)}(x_0)(x - x_0)^3 + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n} \quad (4)$$

is at most

$$\text{error} = \left| \frac{1}{n!} \int_{x_0}^x (x - t)^n f^{(n+1)}(t) dt \right| \leq \left| \frac{1}{n!} \int_{x_0}^x (x - t)^n M_{n+1} dt \right| = \left| -M_{n+1} \frac{(x-t)^{n+1}}{n! \times (n+1)} \Big|_{t=x_0}^{t=x} \right| = \frac{M_{n+1}}{(n+1)!} |x - x_0|^{n+1}$$

The right hand side of (4) is called the Taylor polynomial of degree n for f .

There is a second formula for the error that we can derive easily from $E_n = \frac{1}{n!} \int_{x_0}^x (x - t)^n f^{(n+1)}(t) dt$. The only assumption that we shall make is that $f^{(n+1)}(t)$ is continuous in t for t running between x_0 and x . Let m and M denote the largest and smallest values, respectively, that $f^{(n+1)}(t)$ takes for t between x_0 and x . Then for each t between x_0 and x , the integrand $(x - t)^n f^{(n+1)}(t)$ must lie between $(x - t)^n m$ and $(x - t)^n M$. So E_n must lie between

$$\frac{1}{n!} \int_{x_0}^x (x - t)^n m dt = \frac{m}{(n+1)!} (x - x_0)^{n+1} \quad \text{and} \quad \frac{1}{n!} \int_{x_0}^x (x - t)^n M dt = \frac{M}{(n+1)!} (x - x_0)^{n+1}$$

That is, $\frac{E_n}{\frac{1}{(n+1)!} (x - x_0)^{n+1}}$ must lie between m and M . Since $f^{(n+1)}(t)$ is continuous and takes the values m and M for some t 's between x_0 and x , $f^{(n+1)}(t)$ must take all values between m and M as t runs from x_0 to x . In particular there must exist a t^* between x_0 and x such that

$$f^{(n+1)}(t^*) = \frac{E_n}{\frac{1}{(n+1)!} (x - x_0)^{n+1}} \Rightarrow E_n = \frac{1}{(n+1)!} f^{(n+1)}(t^*) (x - x_0)^{n+1}$$

In conclusion, we have that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \frac{1}{3!}f^{(3)}(x_0)(x - x_0)^3 + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n + E_n$$

with

$$E_n = \frac{1}{n!} \int_{x_0}^x (x - t)^n f^{(n+1)}(t) dt = \frac{1}{(n+1)!} f^{(n+1)}(t^*) (x - x_0)^{n+1}$$

for some t between x_0 and x .

Example

Let's find all Taylor polynomial for $\sin x$ and $\cos x$ at $x_0 = 0$. To do so, we merely need compute all derivatives of $\sin x$ and $\cos x$ at $x_0 = 0$. First, compute all derivatives at general x .

$$\begin{aligned} f(x) = \sin x & \quad f'(x) = \cos x & \quad f''(x) = -\sin x & \quad f^{(3)}(x) = -\cos x & \quad f^{(4)}(x) = \sin x & \quad \dots \\ g(x) = \cos x & \quad g'(x) = -\sin x & \quad g''(x) = -\cos x & \quad g^{(3)}(x) = \sin x & \quad g^{(4)}(x) = \cos x & \quad \dots \end{aligned}$$

The pattern starts over again with the fourth derivative being the same as the original function. Now set $x = x_0 = 0$.

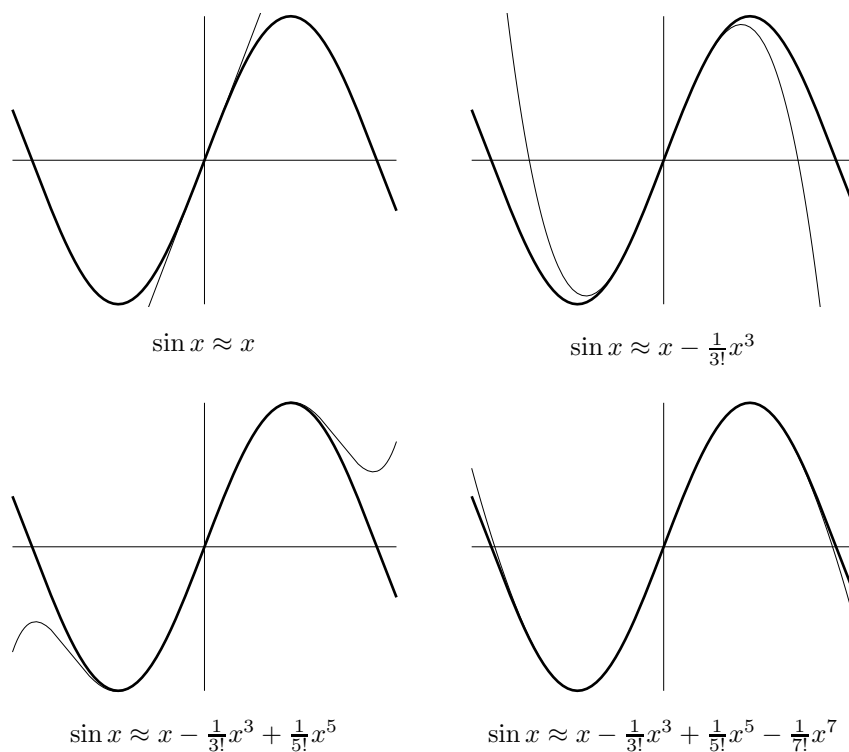
$$\begin{aligned} f(x) = \sin x & \quad f(0) = 0 & \quad f'(0) = 1 & \quad f''(0) = 0 & \quad f^{(3)}(0) = -1 & \quad f^{(4)}(0) = 0 & \quad \dots \\ g(x) = \cos x & \quad g(0) = 1 & \quad g'(0) = 0 & \quad g''(0) = -1 & \quad g^{(3)}(0) = 0 & \quad g^{(4)}(0) = 1 & \quad \dots \end{aligned}$$

For $\sin x$, all even numbered derivatives are zero. The odd numbered derivatives alternate between 1 and -1 . For $\cos x$, all odd numbered derivatives are zero. The even numbered derivatives alternate between 1 and -1 . So, the Taylor polynomials that best approximate $\sin x$ and $\cos x$ near $x = x_0 = 0$ are

$$\sin x \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots \quad \cos x \approx 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots$$

Note that all of our derivative formulae for trig functions were developed under the assumption that angles are measured in radians. When applying these approximation formulae, which were developed using trig function derivatives, we are obliged to express x in radians.

Here are graphs of $\sin x$ and its Taylor polynomials (about $x_0 = 0$) up to degree seven.



Visually, we cannot distinguish between $\sin x$ and the approximation $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$ for $|x|$ running from 0 almost to π .

Now imagine that we are designing a scientific pocket calculator. To have \sin and \cos buttons on the calculator we need algorithms for computing $\sin x$ and $\cos x$ accurate to nine decimal places. Because \sin and

\cos have period 2π , it suffices to consider $-\pi \leq x \leq \pi$. Because \sin is odd and \cos is even, it suffices to consider $0 \leq x \leq \pi$. Because $\sin(\pi - x) = \sin x$ and $\cos(\pi - x) = -\cos x$, it suffices to consider $0 \leq x \leq \frac{\pi}{2}$. Because $\sin(\frac{\pi}{2} - x) = \cos x$ and $\cos(\frac{\pi}{2} - x) = \sin x$, it suffices to consider $0 \leq x \leq \frac{\pi}{4}$. Using double angle formulae, it is possible to cut down the range of x 's even further, but let's stop here and design algorithms for computing $\sin x$ and $\cos x$ accurate to nine decimal places when $0 \leq x \leq \frac{\pi}{4}$. We have already seen that every derivative of $f(x) = \sin x$ and $g(x) = \cos x$ is either $\sin x$ or $\cos x$ or $-\sin x$ or $-\cos x$. Thus every derivative of $f(x) = \sin x$ and $g(x) = \cos x$ never has magnitude bigger than one. So applying the approximation (4) introduces an error no large than $\frac{M_{n+1}}{(n+1)!}|x - x_0|^{n+1}$ with $M_{n+1} = 1$. When $x_0 = 0$ and $0 \leq x \leq \frac{\pi}{4}$, the error is at most $\frac{1}{(n+1)!}(\frac{\pi}{4})^{n+1}$. Here is a table giving values of $\frac{1}{(n+1)!}(\frac{\pi}{4})^{n+1}$ (accurate to two significant digits) for various values of n .

n	5	6	7	8	9	10	11
$\frac{1}{(n+1)!}(\frac{\pi}{4})^{n+1}$	3.3×10^{-4}	3.7×10^{-5}	3.6×10^{-6}	3.1×10^{-7}	2.5×10^{-8}	1.8×10^{-9}	1.2×10^{-10}

So applying the approximation (4) with $n = 11$ gives

$$\sin x \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} \quad \cos x \approx 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10}$$

with errors that must be smaller than 1.2×10^{-10} when $0 \leq x \leq \frac{\pi}{4}$.