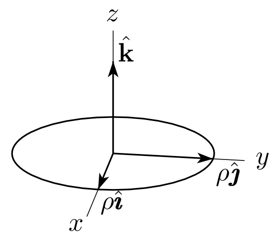
Parametrizing Circles

These notes discuss a simple strategy for parametrizing circles in three dimensions. We start with the circle in the xy–plane that has radius ρ and is centred on the origin. This is easy to parametrize:



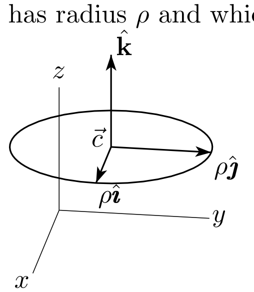
$$\vec{r}(t) = \rho\cos t\,\hat{\boldsymbol{\imath}} + \rho\sin t\,\hat{\boldsymbol{\jmath}} \qquad 0 \le t \le 2\pi$$

Note that we can check that $\vec{r}(t)$ lies on the desired circle by checking, firstly, that $\vec{r}(t)$ lies in the correct plane (in this case, the xy–plane) and, secondly, that the distance from $\vec{r}(t)$ to the centre of the circle is ρ:

$$\left|\vec{r}(t) - \vec{0}\right| = \left|\rho\cos t\,\hat{\boldsymbol{\imath}} + \rho\sin t\,\hat{\boldsymbol{\jmath}}\right| = \sqrt{(\rho\cos t)^2 + (\rho\sin t)^2} = \rho$$

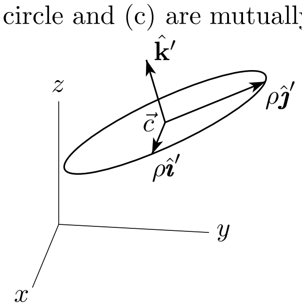
since $\sin^2 t + \cos^2 t = 1$.

Now let's move the circle so that its centre is at some general point $\vec{c}$. To parametrize this new circle, which still has radius ρ and which is still parallel to the xy–plane, we just translate by $\vec{c}$:



$$\vec{r}(t) = \vec{c} + \rho\cos t\,\hat{\boldsymbol{\imath}} + \rho\sin t\,\hat{\boldsymbol{\jmath}} \qquad 0 \le t \le 2\pi$$

Finally, let's consider a circle in general position. The secret to parametrizing a general circle is to replace $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ by two new vectors $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$ which (a) are unit vectors, (b) are parallel to the plane of the desired circle and (c) are mutually perpendicular.



$$\vec{r}(t) = \vec{c} + \rho\cos t\,\hat{\boldsymbol{\imath}}' + \rho\sin t\,\hat{\boldsymbol{\jmath}}' \qquad 0 \le t \le 2\pi$$

To check that this is correct, observe that

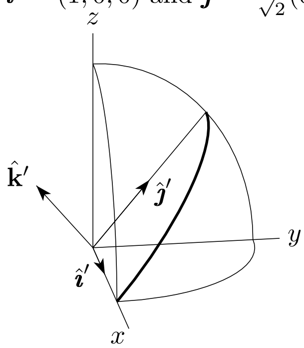
- $\vec{r}(t) - \vec{c}$ is parallel to the plane of the desired circle because $\vec{r}(t) - \vec{c} = \rho\cos t\,\hat{\boldsymbol{\imath}}' + \rho\sin t\,\hat{\boldsymbol{\jmath}}'$ and both $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$ are parallel to the plane of the desired circle
- $\vec{r}(t) - \vec{c}$ is of length ρ for all t because

$$\begin{aligned} |\vec{r}(t) - \vec{c}|^2 &= (\vec{r}(t) - \vec{c}) \cdot (\vec{r}(t) - \vec{c}) \\ &= (\rho\cos t\,\hat{\boldsymbol{\imath}}' + \rho\sin t\,\hat{\boldsymbol{\jmath}}') \cdot (\rho\cos t\,\hat{\boldsymbol{\imath}}' + \rho\sin t\,\hat{\boldsymbol{\jmath}}') \\ &= \rho^2\cos^2 t\;\hat{\boldsymbol{\imath}}' \cdot \hat{\boldsymbol{\imath}}' + \rho^2\sin^2 t\;\hat{\boldsymbol{\jmath}}' \cdot \hat{\boldsymbol{\jmath}}' + 2\rho\cos t\sin t\;\hat{\boldsymbol{\imath}}' \cdot \hat{\boldsymbol{\jmath}}' \\ &= \rho^2(\cos^2 t + \sin^2 t) = \rho^2 \end{aligned}$$

since $\hat{\boldsymbol{\imath}}' \cdot \hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{\jmath}}' \cdot \hat{\boldsymbol{\jmath}}' = 1$ ($\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$ are both unit vectors) and $\hat{\boldsymbol{\imath}}' \cdot \hat{\boldsymbol{\jmath}}' = 0$ ($\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$ are perpendicular).

To find such a parametrization in practice, we need to find the centre $\vec{c}$ of the circle, the radius ρ of the circle and two mutually perpendicular unit vectors, $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$, in the plane of the circle. It is often easy to find at least one point $\vec{p}$ on the circle. Then we can take $\hat{\boldsymbol{\imath}}' = \frac{\vec{p}-\vec{c}}{|\vec{p}-\vec{c}|}$. It is also often easy to find a unit vector, $\hat{\mathbf{k}}'$, that is normal to the plane of the circle. Then we can choose $\hat{\boldsymbol{\jmath}}' = \hat{\mathbf{k}}' \times \hat{\boldsymbol{\imath}}'$.

Example 1 Let C be the intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane $z = y$. The intersection of any plane with any sphere is a circle. The plane in question passes through the centre of the sphere, so C has the same centre and same radius as the sphere. So C has radius 2 and centre $(0, 0, 0)$. The point $(2, 0, 0)$ satisfies both $x^2 + y^2 + z^2 = 4$ and $z = y$ and so is on C. We may choose $\hat{\boldsymbol{\imath}}'$ to be the unit vector in the direction from the centre $(0, 0, 0)$ of the circle towards $(2, 0, 0)$. Namely $\hat{\boldsymbol{\imath}}' = (1, 0, 0)$. Since the plane of the circle is $z - y = 0$, the vector $\vec{\nabla}(z - y) = (0, -1, 1)$ is perpendicular to the plane of C. So we may take $\hat{\mathbf{k}}' = \frac{1}{\sqrt{2}}(0, -1, 1)$. Then $\hat{\boldsymbol{\jmath}}' = \hat{\mathbf{k}}' \times \hat{\boldsymbol{\imath}}' = \frac{1}{\sqrt{2}}(0, -1, 1) \times (1, 0, 0) = \frac{1}{\sqrt{2}}(0, 1, 1)$. Subbing in $\vec{c} = (0, 0, 0)$, $\rho = 2$, $\hat{\boldsymbol{\imath}}' = (1, 0, 0)$ and $\hat{\boldsymbol{\jmath}}' = \frac{1}{\sqrt{2}}(0, 1, 1)$ gives



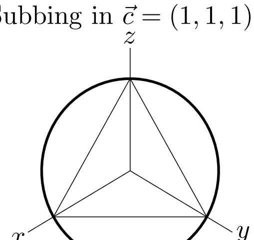
$$\vec{r}(t) = 2\cos t\,(1, 0, 0) + 2\sin t\,\tfrac{1}{\sqrt{2}}(0, 1, 1) = 2\big(\cos t, \tfrac{\sin t}{\sqrt{2}}, \tfrac{\sin t}{\sqrt{2}}\big) \qquad 0 \le t \le 2\pi$$

To check this, note that $x = 2\cos t$, $y = \sqrt{2}\sin t$, $z = \sqrt{2}\sin t$ satisfies both $x^2 + y^2 + z^2 = 4$ and $z = y$.

Example 2 Let C be the circle that passes through the three points $(3, 0, 0)$, $(0, 3, 0)$ and $(0, 0, 3)$. All three points obey $x + y + z = 3$. So the circle lies in the plane $x + y + z = 3$. We guess, by symmetry, or by looking at the figure below, that the centre of the circle is at the centre of mass of the three points, which is $\frac{1}{3}[(3, 0, 0) + (0, 3, 0) + (0, 0, 3)] = (1, 1, 1)$. We can check this by checking that $(1, 1, 1)$ is equidistant from the three points:

$$\begin{aligned}
\big|(3, 0, 0) - (1, 1, 1)\big| &= \big|(2, -1, -1)\big| = \sqrt{6}\\
\big|(0, 3, 0) - (1, 1, 1)\big| &= \big|(-1, 2, -1)\big| = \sqrt{6}\\
\big|(0, 0, 3) - (1, 1, 1)\big| &= \big|(-1, -1, 2)\big| = \sqrt{6}
\end{aligned}$$

This tells us both that $(1, 1, 1)$ is indeed the centre and that the radius of C is $\sqrt{6}$. We may choose $\hat{\boldsymbol{\imath}}'$ to be the unit vector in the direction from the centre $(1, 1, 1)$ of the circle towards $(3, 0, 0)$. Namely $\hat{\boldsymbol{\imath}}' = \frac{1}{\sqrt{6}}(2, -1, -1)$. Since the plane of the circle is $x+y+z = 3$, the vector $\vec{\nabla}(x+y+z) = (1, 1, 1)$ is perpendicular to the plane of C. So we may take $\hat{\mathbf{k}}' = \frac{1}{\sqrt{3}}(1, 1, 1)$. Then $\hat{\boldsymbol{\jmath}}' = \hat{\mathbf{k}}' \times \hat{\boldsymbol{\imath}}' = \frac{1}{\sqrt{18}}(1, 1, 1) \times (2, -1, -1) = \frac{1}{\sqrt{18}}(0, 3, -3) = \frac{1}{\sqrt{2}}(0, 1, -1)$. Subbing in $\vec{c} = (1, 1, 1)$, $\rho = \sqrt{6}$, $\hat{\boldsymbol{\imath}}' = \frac{1}{\sqrt{6}}(2, -1, -1)$ and $\hat{\boldsymbol{\jmath}}' = \frac{1}{\sqrt{2}}(0, 1, -1)$ gives



$$\begin{aligned}
\vec{r}(t) &= (1, 1, 1) + \sqrt{6}\cos t\,\tfrac{1}{\sqrt{6}}(2, -1, -1) + \sqrt{6}\sin t\,\tfrac{1}{\sqrt{2}}(0, 1, -1)\\
&= \big(1 + 2\cos t, 1 - \cos t + \sqrt{3}\sin t, 1 - \cos t - \sqrt{3}\sin t\big) \qquad 0 \le t \le 2\pi
\end{aligned}$$

To check this, note that $\vec{r}(0) = (3, 0, 0)$, $\vec{r}\big(\frac{2\pi}{3}\big) = (0, 3, 0)$ and $\vec{r}\big(\frac{4\pi}{3}\big) = (0, 0, 3)$ since $\cos\frac{2\pi}{3} = \cos\frac{4\pi}{3} = -\frac{1}{2}$, $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$ and $\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$.