The Fourier Transform

As we have seen, any (sufficiently smooth) function $f(t)$ that is periodic can be built out of sin's and cos's. We have also seen that complex exponentials may be used in place of sin's and cos's. We shall now use complex exponentials because they lead to less writing and simpler computations, but yet can easily be converted into sin's and cos's. If $f(t)$ has period 2ℓ, its (complex) Fourier series expansion is

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\frac{\pi}{\ell}t} \qquad \text{with} \qquad c_k = \tfrac{1}{2\ell}\int_{-\ell}^{\ell} f(t)e^{-ik\frac{\pi}{\ell}t}\,dt \tag{1}$$

Not surprisingly, each term $c_k e^{ik\frac{\pi}{\ell}t}$ in this expansion also has period 2ℓ, because $c_k e^{ik\frac{\pi}{\ell}(t+2\ell)} = c_k e^{ik\frac{\pi}{\ell}t}e^{i2k\pi} = c_k e^{ik\frac{\pi}{\ell}t}$. We now develop an expansion for non-periodic functions, by allowing complex exponentials (or equivalently sin's and cos's) of all possible periods, not just 2ℓ, for some fixed ℓ. So, from now on, do not assume that $f(t)$ is periodic.

For simplicity we'll only develop the expansions for functions that are zero for all sufficiently large $|t|$. With a little more work, one can show that our conclusions apply to a much broader class of functions. Let $L > 0$ be sufficiently large that $f(t) = 0$ for all $|t| \ge L$. We can get a Fourier series expansion for the part of $f(t)$ with $-L < t < L$ by using the periodic extension trick. Define $F_L(t)$ to be the unique function determined by the requirements that

$$\begin{gathered} i)\ F_L(t) = f(t) \ \text{ for } \ -L < t \le L \\ ii)\ F_L(t) \text{ is periodic of period } 2L \end{gathered}$$

Then, for $-L < t < L$,

$$f(t) = F_L(t) = \sum_{k=-\infty}^{\infty} c_k(L) e^{ik\frac{\pi}{L}t} \qquad \text{where} \qquad c_k(L) = \tfrac{1}{2L}\int_{-L}^{L} f(t)e^{-ik\frac{\pi}{L}t}\,dt \tag{2}$$

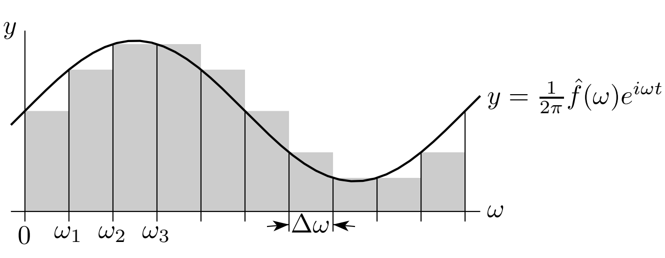
If we can somehow take the limit $L \to \infty$, we will get a representation of f that is is valid for all t's , not just those in some finite interval $-L < t < L$. This is exactly what we shall do, by the simple expedient of interpreting the sum in (2) as a Riemann sum approximation to a certain integral. For each integer k, define the k^{th} frequency to be $\omega_k = k\frac{\pi}{L}$ and denote by $\Delta\omega = \frac{\pi}{L}$ the spacing, $\omega_{k+1} - \omega_k$, between any two successive frequencies. Also define $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$. Since $f(t) = 0$ for all $|t| \ge L$

$$c_k(L) = \tfrac{1}{2L}\int_{-L}^{L} f(t)e^{-ik\frac{\pi}{L}t}\,dt = \tfrac{1}{2L}\int_{-\infty}^{\infty} f(t)e^{-i(k\frac{\pi}{L})t}\,dt = \tfrac{1}{2L}\int_{-\infty}^{\infty} f(t)e^{-i\omega_k t}\,dt = \tfrac{1}{2L}\hat{f}(\omega_k) = \tfrac{1}{2\pi}\hat{f}(\omega_k)\,\Delta\omega$$

In this notation,

$$f(t) = F_L(t) = \tfrac{1}{2\pi}\sum_{k=-\infty}^{\infty} \hat{f}(\omega_k)e^{i\omega_k t}\Delta\omega \tag{3}$$

for any $-L < t < L$. As we let $L \to \infty$, the restriction $-L < t < L$ disappears, and the right hand side converges exactly to the integral $\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}\,d\omega$. To see this, cut the domain of integration into



small slices of width $\Delta\omega$ and approximate, as in the above figure, the area under the part of the graph of $\frac{1}{2\pi}\hat{f}(\omega)e^{i\omega t}$ with ω between ω_k and $\omega_k + \Delta\omega$ by the area of a rectangle of base $\Delta\omega$ and height $\frac{1}{2\pi}\hat{f}(\omega_k)e^{i\omega_k t}$. This approximates the integral $\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega t}\,d\omega$ by the sum $\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\hat{f}(\omega_k)e^{i\omega_k t}\Delta\omega$. As $L \to \infty$ the approximation gets better and better so that the sum approaches the integral. So taking the limit of (3) as $L \to \infty$ gives

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega t}\,d\omega \qquad \text{where} \qquad \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt \tag{4}$$

The function $\hat{f}$ is called the Fourier transform of f. It is to be thought of as the frequency profile of the signal $f(t)$.

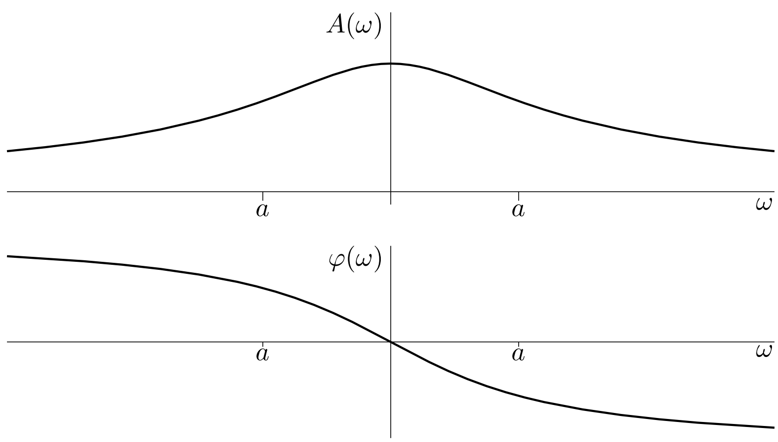
Example 1 Suppose that a signal gets turned on at $t = 0$ and then decays exponentially, so that

$$f(t) = \begin{cases} e^{-at} & \text{if } t \ge 0 \\ 0 & \text{if } t < 0 \end{cases}$$

for some $a > 0$. The Fourier transform of this signal is

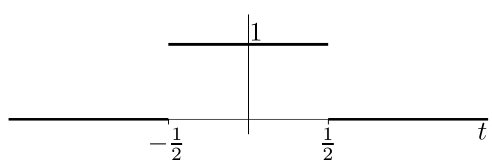
$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt = \int_0^{\infty} e^{-at}e^{-i\omega t}\,dt = \int_0^{\infty} e^{-t(a+i\omega)}\,dt = \frac{e^{-t(a+i\omega)}}{-(a+i\omega)}\bigg|_0^{\infty} = \frac{1}{a+i\omega}$$

Since $a + i\omega$ has modulus $\sqrt{a^2+\omega^2}$ and phase $\tan^{-1}\frac{\omega}{a}$, we have that $\hat{f}(\omega) = A(\omega)e^{i\phi(\omega)}$ with $A(\omega) = \frac{1}{|a+i\omega|} = \frac{1}{\sqrt{a^2+\omega^2}}$ and $\phi(\omega) = -\tan^{-1}\frac{\omega}{a}$. The amplitude $A(\omega)$ and phase $\phi(\omega)$ are plotted below.



Example 2 Suppose that a signal consists of a single rectangular pulse of width 1 and height 1. Let's say that it gets turned on at $t = -\frac{1}{2}$ and turned off at $t = \frac{1}{2}$. The standard name for this "normalized" rectangular pulse is

$$\mathrm{rect}(t) = \begin{cases} 1 & \text{if } -\frac{1}{2} < t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$



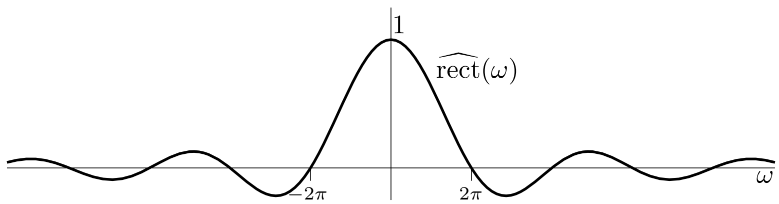
It is also called a normalized boxcar function. The Fourier transform of this signal is

$$\widehat{\mathrm{rect}}(\omega) = \int_{-\infty}^{\infty}\mathrm{rect}(t)\;e^{-i\omega t}\,dt = \int_{-1/2}^{1/2} e^{-i\omega t}\,dt = \frac{e^{-i\omega t}}{-i\omega}\bigg|_{-1/2}^{1/2} = \frac{e^{i\omega/2} - e^{-i\omega/2}}{i\omega} = \frac{2}{\omega}\sin\frac{\omega}{2}$$

when $\omega \neq 0$. When $\omega = 0$, $\widehat{\mathrm{rect}}(0) = \int_{-1/2}^{1/2} dt = 1$. By l'Hôpital's rule

$$\lim_{\omega\to 0} \widehat{\mathrm{rect}}(\omega) = \lim_{\omega\to 0} 2\frac{\sin\frac{\omega}{2}}{\omega} = \lim_{\omega\to 0} 2\frac{\frac{1}{2}\cos\frac{\omega}{2}}{1} = 1 = \widehat{\mathrm{rect}}(0)$$

so $\widehat{\mathrm{rect}}(\omega)$ is continuous at $\omega = 0$. There is a standard function called "sinc" that is defined[1] by $\mathrm{sinc}\,\omega = \frac{\sin\omega}{\omega}$. In this notation $\widehat{\mathrm{rect}}(\omega) = \mathrm{sinc}\,\frac{\omega}{2}$. Here is a graph of $\widehat{\mathrm{rect}}(\omega)$.



Properties of the Fourier Transform

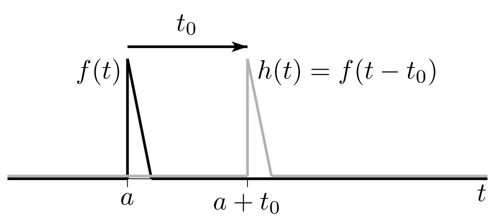
Linearity

If α and β are any constants and we build a new function $h(t) = \alpha f(t) + \beta g(t)$ as a linear combination of two old functions $f(t)$ and $g(t)$, then the Fourier transform of h is

$$\begin{aligned} \hat{h}(\omega) &= \int_{-\infty}^{\infty} h(t)e^{-i\omega t}\, dt = \int_{-\infty}^{\infty} \big[\alpha f(t) + \beta g(t)\big] e^{-i\omega t}\, dt = \alpha \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt + \beta \int_{-\infty}^{\infty} g(t)e^{-i\omega t}\, dt \\ &= \alpha \hat{f}(\omega) + \beta \hat{g}(\omega) \end{aligned} \tag{L}$$

Time Shifting

Suppose that we build a new function $h(t) = f(t - t_0)$ by time shifting a function $f(t)$ by t_0. The easy way to check the direction of the shift is to note that if the original signal $f(t)$ has a jump when its argument $t = a$, then the new signal $h(t) = f(t - t_0)$ has a jump when $t - t_0 = a$, which is at $t = a + t_0$, t_0 units to the right of the original jump.



The Fourier transform of h is

$$\begin{aligned} \hat{h}(\omega) &= \int_{-\infty}^{\infty} h(t)e^{-i\omega t}\, dt = \int_{-\infty}^{\infty} f(t - t_0)e^{-i\omega t}\, dt = \int_{-\infty}^{\infty} f(t')e^{-i\omega(t'+t_0)}\, dt' \text{ where } t = t' + t_0,\ dt = dt' \\ &= e^{-i\omega t_0} \int_{-\infty}^{\infty} f(t')e^{-i\omega t'}\, dt' = e^{-i\omega t_0}\hat{f}(\omega) \end{aligned} \tag{T}$$

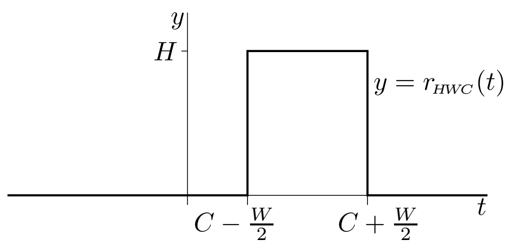
(1) There are actually two different commonly used definitions. The first, which we shall use, is $\mathrm{sinc}\,\omega = \frac{\sin\omega}{\omega}$. The second is $\mathrm{sinc}\,\omega = \frac{\sin\pi\omega}{\pi\omega}$. It is sometimes called the normalized sinc function.

Scaling

If we build a new function $h(t) = f\left(\frac{t}{\alpha}\right)$ by scaling time by a factor of $\alpha > 0$, then the Fourier transform of h is

$$\begin{aligned}\hat{h}(\omega) &= \int_{-\infty}^{\infty} h(t)e^{-i\omega t}\ dt = \int_{-\infty}^{\infty} f\left(\tfrac{t}{\alpha}\right)e^{-i\omega t}\ dt = \alpha\int_{-\infty}^{\infty} f(t')e^{-i\omega\alpha t'}\ dt' \text{ where } t = \alpha t',\ dt = \alpha dt' \\ &= \alpha\hat{f}(\alpha\omega)\end{aligned} \tag{S}$$

Example 3 Now consider a signal that consists of a single rectangular pulse of height H, width W and centre C.



The function $r_{HWC}(t) = H\,\mathrm{rect}\left(\frac{t-C}{W}\right)$ (with rect the normalized rectangular pulse of Example 2) has height H and jumps when $\frac{t-C}{W} = \pm\frac{1}{2}$, i.e. $t = C \pm \frac{1}{2}W$ and so is the specified signal. By combining properties (L), (T) and (S), we can determine the Fourier transform of $r_{HWC}(t) = H\,\mathrm{rect}\left(\frac{t-C}{W}\right)$ for any H, C and W. We build it up in three steps.

- First we consider the special case in which $H = 1$ and $C = 0$. Then we have $R_1(t) = \mathrm{rect}\left(\frac{t}{W}\right)$. So, according to the scaling property (S) with

  $$f(t) = \mathrm{rect}(t), \quad h(t) = R_1(t) = \mathrm{rect}\left(\tfrac{t}{W}\right) = f\left(\tfrac{t}{W}\right) = f\left(\tfrac{t}{\alpha}\right) \quad \text{with } \alpha = W$$

  we have

  $$\hat{R}_1(\omega) = \hat{h}(\omega) = \alpha\hat{f}(\alpha\omega) = W\widehat{\mathrm{rect}}(W\omega) = W\tfrac{2}{W\omega}\sin\tfrac{W\omega}{2} = \tfrac{2}{\omega}\sin\tfrac{W\omega}{2}$$

- Next we allow a nonzero C too and consider $R_2(t) = \mathrm{rect}\left(\frac{t-C}{W}\right) = R_1(t-C)$. By (T), with

  $$f(t) = R_1(t),\ h(t) = R_2(t) = R_1(t-C) = f(t-t_0),\ \text{with } t_0 = C$$

  the Fourier transform is

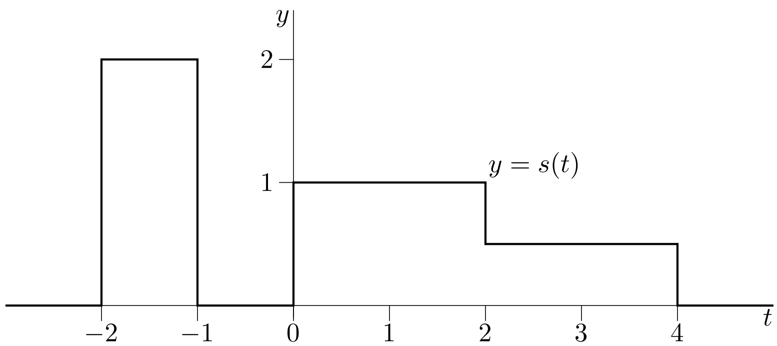
  $$\hat{R}_2(\omega) = \hat{h}(\omega) = e^{-i\omega t_0}\hat{f}(\omega) = e^{-i\omega C}\hat{R}_1(\omega) = We^{-i\omega C}\widehat{\mathrm{rect}}\big(W\omega\big)$$

- Finally, by (L), with $\alpha = H$ and $\beta = 0$, the Fourier transform of $r_{HWC} = H\,\mathrm{rect}\left(\frac{t-C}{W}\right) = HR_2(t)$ is

  $$\hat{r}_{HWC}(\omega) = H\hat{R}_2(\omega) = HWe^{-i\omega C}\widehat{\mathrm{rect}}(W\omega) = HWe^{-i\omega C}\tfrac{2}{W\omega}\sin\tfrac{W\omega}{2} = e^{-i\omega C}\tfrac{2H}{\omega}\sin\tfrac{W\omega}{2}$$

Example 4

Now suppose that we have a signal that consists of a number of rectangular pulses of various heights and widths. Here is an example



We can write this signal as a sum of rectangular pulses. As we saw in the last example, $r_{HWC}(t) = H\,\mathrm{rect}\big(\frac{t-C}{W}\big)$ is a single rectangular signal with height H, centre C and width W. So

$$s(t) = s_1(t) + s_2(t) + s_3(t) \quad \text{where} \quad s_n(t) = r_{HWC}(t) \text{ with } \begin{cases} H = 2, \;\; W = 1, \; C = -1.5 & \text{for } n = 1 \\ H = 1, \;\; W = 2, \; C = 1 & \text{for } n = 2 \\ H = 0.5, \; W = 2, \; C = 3 & \text{for } n = 3 \end{cases}$$

So, using (L) and $\hat{r}_{HWC}(\omega) = e^{-i\omega C}\frac{2H}{\omega}\sin\frac{W\omega}{2}$, which we derived in Example 3,

$$\hat{s}(\omega) = \hat{s}_1(\omega) + \hat{s}_2(\omega) + \hat{s}_3(\omega) = \tfrac{4}{\omega}e^{i\frac{3}{2}\omega}\sin\tfrac{\omega}{2} + \tfrac{2}{\omega}e^{-i\omega}\sin\omega + \tfrac{1}{\omega}e^{-i3\omega}\sin\omega$$

Differentiation

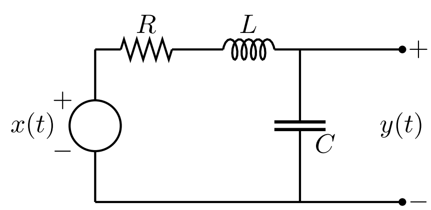
If we build a new function $h(t) = f'(t)$ by differentiating an old function $f(t)$, then the Fourier transform of h is

$$\hat{h}(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}\, dt = \int_{-\infty}^{\infty} f'(t)e^{-i\omega t}\, dt$$

Now integrate by parts with $u = e^{-i\omega t}$ and $dv = f'(t)\,dt$ so that $du = -i\omega e^{-i\omega t}\,dt$ and $v = f(t)$. Assuming that $f(\pm\infty) = 0$, this gives

$$\hat{h}(\omega) = \int_{-\infty}^{\infty} u\, dv = uv\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v\, du = -\int_{-\infty}^{\infty} f(t)\,(-i\omega)e^{-i\omega t}\, dt = i\omega\hat{f}(\omega) \tag{D}$$

Example 5 The differentiation property is going to be useful when we use the Fourier transform to solve differential equations. As an example, let's take another look at the RLC circuit



thinking of the voltage $x(t)$ as an input signal and the voltage $y(t)$ as an output signal. If we feed any input signal $x(t)$ into an RLC circuit, we get an output $y(t)$ which obeys the differential equation

$$LCy''(t) + RCy'(t) + y(t) = x(t) \tag{5}$$

You should be able to quickly derive this equation, which is also (16) in the notes "Fourier Series", on your own. Take the Fourier transform of this whole equation and use that

- the Fourier transform of $y'(t)$ is $i\omega\hat{y}(\omega)$ and
- the Fourier transform of $y''(t)$ is $i\omega$ times the Fourier transform of $y'(t)$ and so is $-\omega^2\hat{y}(\omega)$

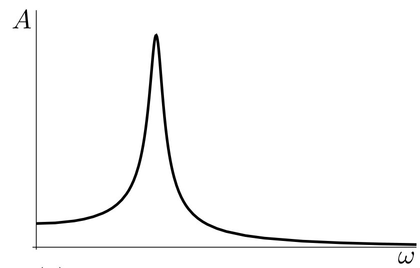
So the Fourier transform of (5) is

$$-LC\omega^2\hat{y}(\omega) + iRC\omega\hat{y}(\omega) + \hat{y}(\omega) = \hat{x}(\omega)$$

This is trivially solved for

$$\hat{y}(\omega) = \tfrac{\hat{x}(\omega)}{-LC\omega^2+iRC\omega+1} \tag{6}$$

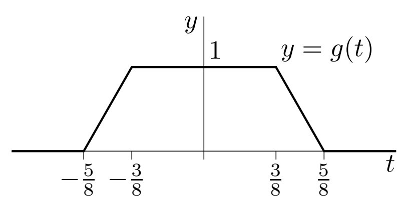
which exhibits classic resonant behaviour. The circuit acts independently on each frequency ω component of the signal. The amplitude $|\hat{y}(\omega)|$ of the frequency ω part of the output signal is the amplitude $|\hat{x}(\omega)|$ of the frequency ω part of the input signal multiplied by $A(\omega) = \frac{1}{|-LC\omega^2+iRC\omega+1|} = \frac{1}{\sqrt{(1-LC\omega^2)^2+R^2C^2\omega^2}}$.



We shall shortly learn how to convert (6) into an equation that expresses the time domain output signal $y(t)$ in terms of the time domain input signal $x(t)$.

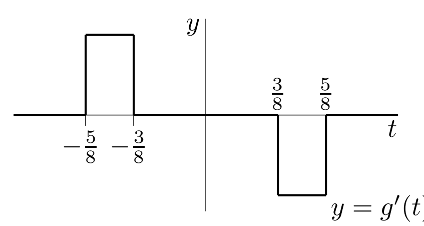
Example 6 The Fourier transform, $\widehat{\mathrm{rect}}(\omega)$, of the rectangular pulse function of Example 2 decays rather slowly, like $\frac{1}{\omega}$ for large ω. We can try suppressing large frequencies by eliminating the discontinuities at $t = \pm\frac{1}{2}$ in $\mathrm{rect}(t)$. For example

$$g(t) = \begin{cases} 0 & \text{if } t \le -\frac{5}{8} \\ 4(t+\frac{5}{8}) & \text{if } -\frac{5}{8} \le t \le -\frac{3}{8} \\ 1 & \text{if } -\frac{3}{8} \le t \le \frac{3}{8} \\ 4(\frac{5}{8}-t) & \text{if } \frac{3}{8} \le t \le \frac{5}{8} \\ 0 & \text{if } t \ge \frac{5}{8} \end{cases}$$



It is not very difficult to evaluate the Fourier transform $\hat{g}(\omega)$ directly. But it easier to use properties (L)–(D), since

$$g'(t) = \left\{\begin{array}{ll} 0 & \text{if } t \le -\frac{5}{8} \\ 4 & \text{if } -\frac{5}{8} \le t \le -\frac{3}{8} \\ 0 & \text{if } -\frac{3}{8} \le t \le \frac{3}{8} \\ -4 & \text{if } \frac{3}{8} \le t \le \frac{5}{8} \\ 0 & \text{if } t \ge \frac{5}{8} \end{array}\right\} = s_4(t) + s_5(t)$$



$$\text{where} \quad s_n(t) = r_{HWC}(t) \text{ with } \begin{cases} H = 4, \quad W = \frac{1}{4}, \ C = -\frac{1}{2} & \text{for } n = 4 \\ H = -4, \ W = \frac{1}{4}, \ C = \frac{1}{2} & \text{for } n = 5 \end{cases}$$

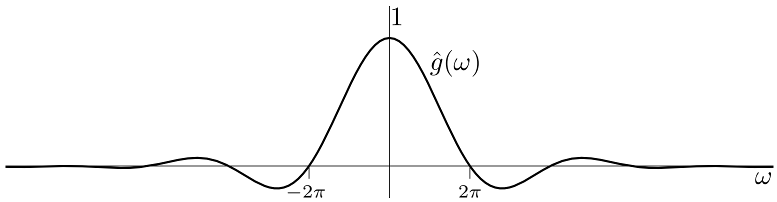
By Example 3, the Fourier transform of

$$\hat{s}_4(\omega) + \hat{s}_5(\omega) = \tfrac{8}{\omega}e^{i\omega/2}\sin\tfrac{\omega}{8} - \tfrac{8}{\omega}e^{-i\omega/2}\sin\tfrac{\omega}{8} = 2i\tfrac{8}{\omega}\sin\tfrac{\omega}{2}\sin\tfrac{\omega}{8}$$

By Property (D), the Fourier transform of $g'(t)$ is $i\omega$ times $\hat{g}(\omega)$. So the Fourier transform of $g(t)$ is $\frac{1}{i\omega}$ times the Fourier transform of $g'(t)$:

$$\hat{g}(\omega) = \tfrac{1}{i\omega}\left(2i\tfrac{8}{\omega}\sin\tfrac{\omega}{2}\sin\tfrac{\omega}{8}\right) = \tfrac{16}{\omega^2}\sin\tfrac{\omega}{2}\sin\tfrac{\omega}{8}$$

This looks somewhat like the the Fourier transform in Example 2 but exhibits faster decay for large ω.



Parseval's Relation

The energy carried by a signal $f(t)$ is defined(2) to be

$$\int_{-\infty}^{\infty} |f(t)|^2 \ dt = \int_{-\infty}^{\infty} f(t)\overline{f(t)} \ dt$$

We would like to express this energy in terms of the Fourier transform $\hat{f}(\omega)$. To do so, substitute in that

$$\overline{f(t)} = \overline{\tfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t} \ d\omega} = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{f}(\omega)} \ \overline{e^{i\omega t}} \ d\omega = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{f}(\omega)}e^{-i\omega t} \ d\omega$$

This gives

$$\begin{aligned}\int_{-\infty}^{\infty} |f(t)|^2 \ dt &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\overline{\hat{f}(\omega)}e^{-i\omega t} \ d\omega \ dt = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{f}(\omega)} \left[\int_{-\infty}^{\infty} f(t)e^{-i\omega t} \ dt\right] d\omega \\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{f}(\omega)} \ \hat{f}(\omega) \ d\omega \qquad\qquad = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \ d\omega\end{aligned}$$

This formula, $\int_{-\infty}^{\infty} |f(t)|^2 \ dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \ d\omega$, is called Parseval's relation.

Duality

The duality property says that if we build a new time–domain function $g(t) = \hat{f}(t)$ by exchanging the roles of time and frequency, then the Fourier transform of g is

$$\hat{g}(\omega) = 2\pi f(-\omega) \tag{Du}$$

To verify this, just write down just the definition of $\hat{g}(\omega)$ and the Fourier inversion formula (4) for $f(t)$ and, in both integrals, make a change of variables so that the integration variable is s:

$$\begin{aligned}\hat{g}(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t} \ dt = \int_{-\infty}^{\infty} \hat{f}(t)e^{-i\omega t} \ dt \quad &\overset{t=s}{=} \int_{-\infty}^{\infty} \hat{f}(s) \ e^{-i\omega s} \ ds \\ f(t) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t} \ d\omega &\overset{\omega=s}{=} \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(s) \ e^{ist} \ ds\end{aligned}$$

So $\hat{g}(\omega)$, which is given by the last integral of the first line is exactly 2π times the last integral of the second line with t replaced by $-\omega$, which is $2\pi f(-\omega)$.

(2) By way of motivation for this definition, imagine that $f(t)$ is the voltage $v(t)$ at time t across a resistor of resistance R. Then the power (i.e. energy per unit time) dissipated by the resistor is $v(t)i(t) = \frac{1}{R}v(t)^2$. So the total energy dissipated by the resistor will be $\frac{1}{R}\int v(t)^2 \ dt$, which, up to a constant, agrees with our definition.

Example 7 In this example, we shall compute the Fourier transform of $\mathrm{sinc}(t) = \frac{\sin t}{t}$. Our starting point is Example 2, where we saw that the Fourier transform of the rectangular pulse $\mathrm{rect}(t)$ of height one and width one is $\widehat{\mathrm{rect}}(\omega) = \frac{2}{\omega}\sin\frac{\omega}{2} = \mathrm{sinc}\,\frac{\omega}{2}$. So, by duality (Du), with $f(t) = \mathrm{rect}(t)$ and $g(t) = \hat{f}(t) = \widehat{\mathrm{rect}}(t)$, the Fourier transform of $g(t) = \mathrm{sinc}\,\frac{t}{2}$ is $\hat{g}(\omega) = 2\pi f(-\omega) = 2\pi\,\mathrm{rect}(-\omega) = 2\pi\,\mathrm{rect}(\omega)$, since $\mathrm{rect}(t)$ is even. So we now know that the Fourier transform of $\mathrm{sinc}\,\frac{t}{2}$ is $2\pi\,\mathrm{rect}(t)$. To find the Fourier transform of $\mathrm{sinc}(t)$, we just need to scale the $\frac{1}{2}$ out of $\frac{t}{2}$. So we apply the scaling property (S) with $f(t) = \mathrm{sinc}\,\frac{t}{2}$ and $h(t) = \mathrm{sinc}(t) = f(2t) = f\left(\frac{t}{\alpha}\right)$ where $\alpha = \frac{1}{2}$. By (S), the Fourier transform of $h(t) = \mathrm{sinc}(t)$ is $\alpha\hat{f}(\alpha\omega) = \alpha\; 2\pi\,\mathrm{rect}(\alpha\omega) = \pi\,\mathrm{rect}\left(\frac{\omega}{2}\right)$.

Multiplication and Convolution

A very common operation in signal processing is that of filtering. It is used to eliminate high frequency noise and also to eliminate sixty cycle hum, arising from the ordinary household AC current. It is also used to extract the signal from any one desired radio or TV station. In general, the filter is described by a function $\hat{H}(\omega)$ of frequency. For example you might have $\hat{H}(\omega) = 1$ for desired frequencies and $\hat{H}(\omega) = 0$ for undesirable frequencies. When a signal $f(t)$ is fed into the filter, an output signal $g(t)$, whose Fourier transform is $\hat{f}(\omega)\hat{H}(\omega)$ is produced. For example, the RLC circuit of Example 5 is a filter with $\hat{H}(\omega) = \frac{1}{-LC\omega^2 + iRC\omega + 1}$.

The question "what is $g(t)$?" remains. Of course it is the inverse Fourier transform

$$g(t) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\hat{H}(\omega)e^{i\omega t}\; d\omega$$

of $\hat{f}(\omega)\hat{H}(\omega)$, but we would like to express it more directly in terms of the original time–domain signal $f(t)$. So let's substitute $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau}\; d\tau$ (which expresses $\hat{f}(\omega)$ in terms $f(t)$) and see if we can simplify the result.

$$\begin{aligned} g(t) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau}\hat{H}(\omega)e^{i\omega t}\; d\tau d\omega \\ &= \int_{-\infty}^{\infty} f(\tau)\left[\tfrac{1}{2\pi}\int_{-\infty}^{\infty}\hat{H}(\omega)e^{i\omega(t-\tau)}\; d\omega\right] d\tau \\ &= \int_{-\infty}^{\infty} f(\tau)H(t-\tau)\; d\tau \end{aligned}$$

The last integral is called a convolution integral and is denoted

$$(f * H)(t) = \int_{-\infty}^{\infty} f(\tau)H(t-\tau)\; d\tau$$

If we make the change of variables $\tau = t - \tau'$, $d\tau = -d\tau'$ we see that we can also express

$$(f * H)(t) = \int_{\infty}^{-\infty} f(t-\tau')H(\tau')\;(-d\tau') = \int_{-\infty}^{\infty} f(t-\tau')H(\tau')\; d\tau'$$

We conclude that

$$\hat{g}(\omega) = \hat{f}(\omega)\hat{H}(\omega) \Longrightarrow g(t) = (f * H)(t) \tag{C}$$

Example 8 The convolution integral gives a sort of moving weighted average, with the weighting determined by H. Its value at time t involves the values of f for times near t. I say "sort of" because there is no requirement that $\int_{-\infty}^{\infty} H(t)\; dt = 1$ or even that $H(t) \geq 0$.

Suppose for example, that we use the filter with

$$\hat{H}(\omega) = \tfrac{2}{\omega} \sin \tfrac{\omega}{2}$$

The graph of this function was given in Example 2. It is a “low pass filter” in the sense that it lets through most small frequencies ω and suppresses high frequencies. It is not really a practical filter, but I have chosen it anyway because it is relatively easy to see the effect of this filter in t–space. From Example 2 we know that

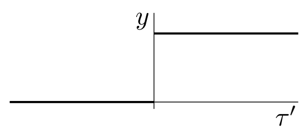
$$H(t) = \begin{cases} 1 & \text{if } -\frac{1}{2} < t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

So when this filter is applied to a signal $f(t)$, the output at time t is

$$\begin{aligned}(f * H)(t) &= \int_{-\infty}^{\infty} f(t-\tau)H(\tau)\ d\tau = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t-\tau)\ d\tau = -\int_{t+\frac{1}{2}}^{t-\frac{1}{2}} f(\tau')\ d\tau' \quad \text{where } \tau' = t-\tau,\ d\tau' = -d\tau \\ &= \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} f(\tau')\ d\tau'\end{aligned}$$

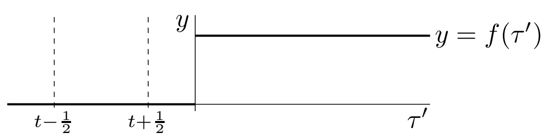
which is the area under the part of the graph of $f(\tau')$ from $\tau' = t - \frac{1}{2}$ to $\tau' = t + \frac{1}{2}$. To be concrete, suppose that the input signal is

$$f(\tau') = \begin{cases} 1 & \text{if } \tau' \geq 0 \\ 0 & \text{if } \tau' < 0 \end{cases}$$

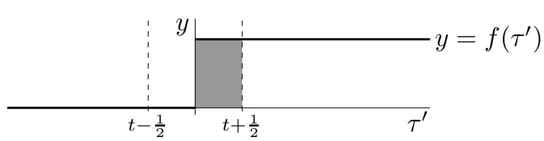


Then the output signal at time t is the area under the part of this graph from $\tau' = t - \frac{1}{2}$ to $\tau' = t + \frac{1}{2}$.

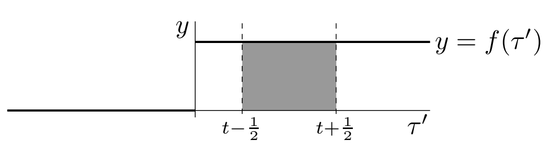
- if $t + \frac{1}{2} < 0$, that is $t < -\frac{1}{2}$, then $f(\tau') = 0$ for all $t - \frac{1}{2} < \tau' < t + \frac{1}{2}$ (see the figure below) so that $(f * H)(t) = 0$



- if $t + \frac{1}{2} \geq 0$ but $t - \frac{1}{2} \leq 0$, that is $-\frac{1}{2} \leq t \leq \frac{1}{2}$, then $f(\tau') = 0$ for $t - \frac{1}{2} < \tau' < 0$ and $f(\tau') = 1$ for $0 < \tau' < t + \frac{1}{2}$ (see the figure below) so that $(f * H)(t) = \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} f(\tau')\ d\tau' = \int_0^{t+\frac{1}{2}} 1\ d\tau' = t + \frac{1}{2}$

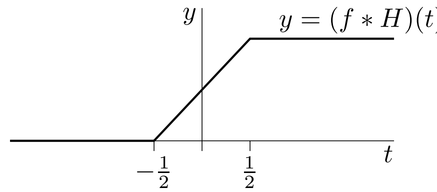


- if $t - \frac{1}{2} > 0$, that is $t > \frac{1}{2}$, then $f(\tau') = 1$ for all $t - \frac{1}{2} < \tau' < t + \frac{1}{2}$ and $(f * H)(t) = \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} f(\tau')\ d\tau' = \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} 1\ d\tau' = 1$.



All together, the output signal is

$$(f*H)(t) = \begin{cases} 0 & \text{if } t < -\frac{1}{2} \\ t+\frac{1}{2} & \text{if } -\frac{1}{2} \le t \le \frac{1}{2} \\ 1 & \text{if } t > \frac{1}{2} \end{cases}$$

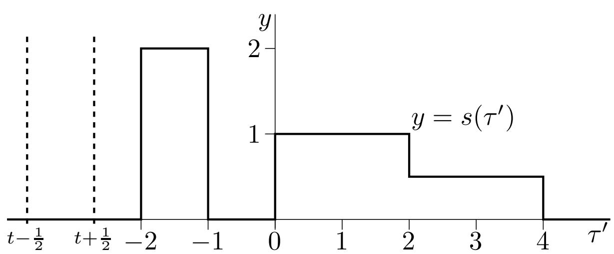


Example 9

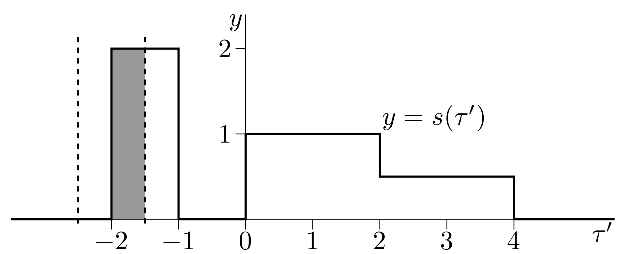
For a similar, but more complicated example, we can through the procedure of Example 8 with $H(t)$ still being $\mathrm{rect}(t)$ but with $f(t)$ replaced by the more complicated signal $s(t)$ of Example 4. This time the output signal at time t is

$$(s*H)(t) = \int_{-\infty}^{\infty} s(t-\tau)H(\tau)\ d\tau = \int_{-\frac{1}{2}}^{\frac{1}{2}} s(t-\tau)\ d\tau = \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} s(\tau')\ d\tau' \quad \text{where } \tau' = t-\tau$$

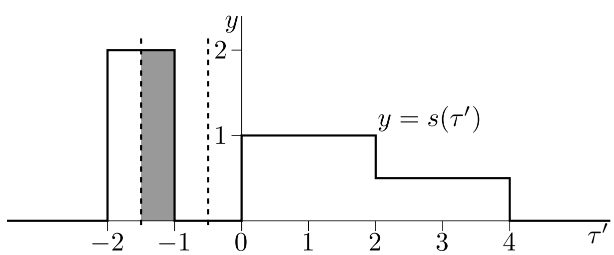
which is the area under the part of the graph of $s(\tau')$ from $\tau' = t-\frac{1}{2}$ to $\tau' = t+\frac{1}{2}$. We can read off this area from the figures below. In each figure, the left hand vertical dotted line is $\tau' = t-\frac{1}{2}$ and the right hand vertical dotted line is $\tau' = t+\frac{1}{2}$. The first figure has t so small that the entire interval $t-\frac{1}{2} < \tau' < t+\frac{1}{2}$ is to the left of $\tau' = -2$, where the graph of $s(\tau')$ "starts". Then, in each successive figure, we increase t, moving the interval to the right. In the final figure t is large enough that the entire interval $t-\frac{1}{2} < \tau' < t+\frac{1}{2}$ is to the right of $\tau' = 4$, where the graph of $s(\tau')$ "ends".



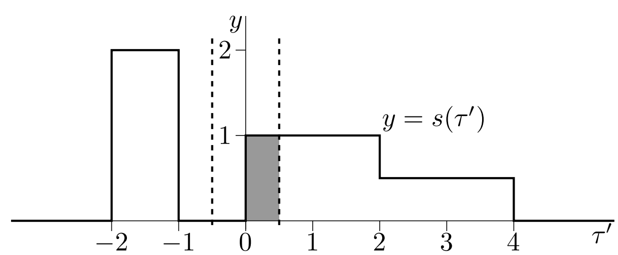
Case: $t+\frac{1}{2} < -2$, that is $t < -\frac{5}{2}$
area $= 0$



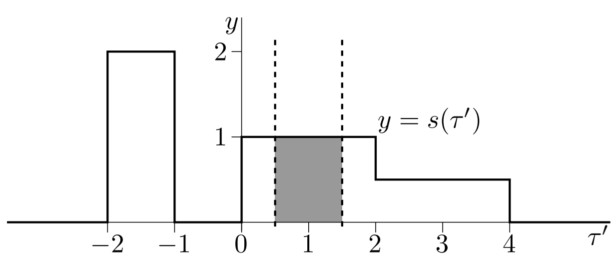
Case: $-2 \le t+\frac{1}{2} \le -1$, that is $-\frac{5}{2} \le t \le -\frac{3}{2}$
area $= 2((t+\frac{1}{2}) - (-2)) = 2t+5$



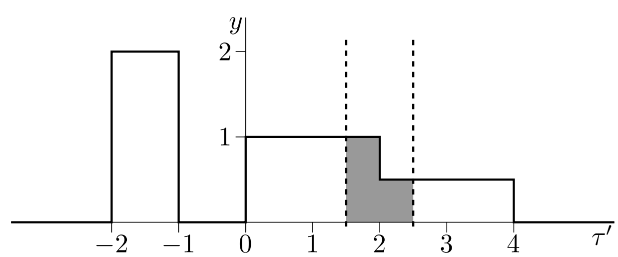
Case: $-1 \le t+\frac{1}{2} \le 0$, that is $-\frac{3}{2} \le t \le -\frac{1}{2}$
area $= 2((-1) - (t-\frac{1}{2})) = -1-2t$



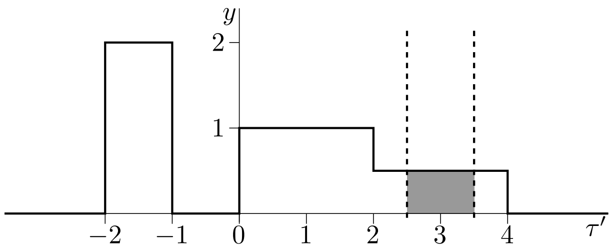
Case: $0 \le t+\frac{1}{2} \le 1$, that is $-\frac{1}{2} \le t \le \frac{1}{2}$
area $= t+\frac{1}{2}$



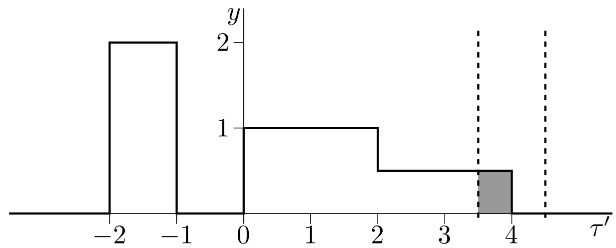
Case: $1 \le t+\frac{1}{2} \le 2$, that is $\frac{1}{2} \le t \le \frac{3}{2}$
area $= 1$



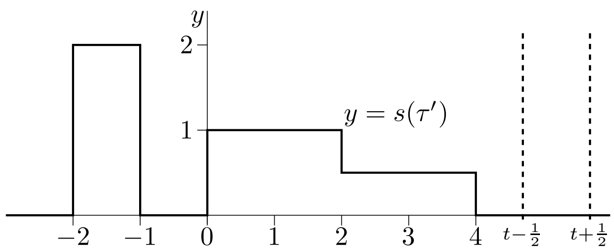
Case: $2 \le t+\frac{1}{2} \le 3$, that is $\frac{3}{2} \le t \le \frac{5}{2}$
area $= [2-(t-\frac{1}{2}) + \frac{1}{2}[(t+\frac{1}{2})-2] = \frac{7}{4} - \frac{1}{2}t$



Case: $3 \leq t + \frac{1}{2} \leq 4$, that is $\frac{5}{2} \leq t \leq \frac{7}{2}$
area $= \frac{1}{2}$

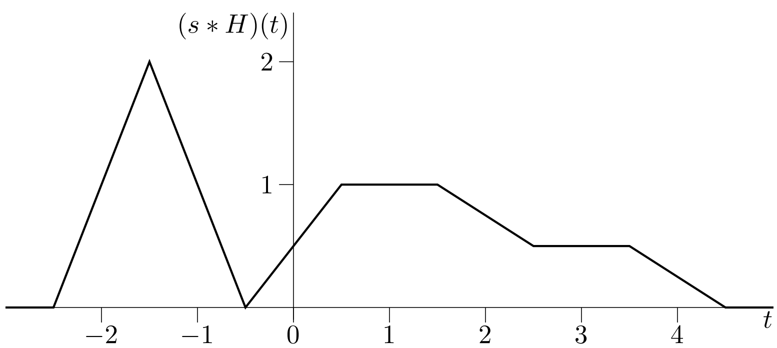


Case: $4 \leq t + \frac{1}{2} \leq 5$, that is $\frac{7}{2} \leq t \leq \frac{9}{2}$
area $= \frac{1}{2}[4 - (t - \frac{1}{2})] = \frac{9}{4} - \frac{1}{2}t$



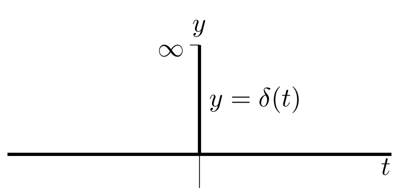
Case: $t - \frac{1}{2} > 4$, that is $t > \frac{9}{2}$
area $= 0$

All together, the graph of the output signal is



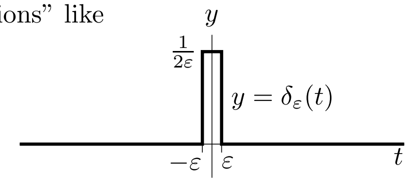
Impulses

The inverse Fourier transform $H(t)$ of $\hat{H}(\omega)$ is called the impulse response function of the filter, because it is the output generated when the input is an impulse at time 0. An impulse, usually denoted $\delta(t)$ (and called a "delta function") takes the value 0 for all times $t \neq 0$ and the value ∞ at time $t = 0$. In fact it is so infinite at time 0 that the area under its graph is exactly 1. Of course there isn't any such function, in



the usual sense of the word. But it is possible to generalize the concept of a function (to something called a distribution or a generalized function) so as to accommodate delta functions. One generalization involves taking the limit as $\varepsilon \to 0$ of "approximate delta functions" like

$$\delta_\varepsilon(t) = \begin{cases} \frac{1}{2\varepsilon} & \text{if } -\varepsilon < t < \varepsilon \\ 0 & \text{otherwise} \end{cases}$$



A treatment of these generalization procedures is well beyond the scope of this course. Fortunately, in practice it suffices to be able to compute the value of the integral $\int_{-\infty}^{\infty} \delta(t)f(t)\ dt$ for any continuous function $f(t)$ and that is easy. Because $\delta(t) = 0$ for all $t \neq 0$, $\delta(t)f(t)$ is the same as $\delta(t)f(0)$. (Both are zero for $t \neq 0$.) Because $f(0)$ is a constant, the area under $\delta(t)f(0)$ is $f(0)$ times the area under $\delta(t)$, which we already said is 1. So

$$\int_{-\infty}^{\infty} \delta(t)f(t)\ dt = f(0)$$

In particular, choosing $f(t) = e^{-i\omega t}$ gives the Fourier transform of $\delta(t)$:

$$\hat{\delta}(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-i\omega t}\, dt = e^{-i\omega t}\big|_{t=0} = 1$$

By the time shifting property (T), the Fourier transform of $\delta(t - t_0)$ (where t_0 is a constant) is $e^{-i\omega t_0}$. We may come to the same conclusion by first making the change of variables $\tau = t - t_0$ to give

$$\int_{-\infty}^{\infty} \delta(t - t_0)e^{-i\omega t}\, dt \stackrel{\tau=t-t_0}{=} \int_{-\infty}^{\infty} \delta(\tau)e^{-i\omega(\tau+t_0)}\, d\tau$$

and then applying $\int_{-\infty}^{\infty} \delta(\tau)f(\tau)\ d\tau = f(0)$ with $f(\tau) = e^{-i\omega(\tau+t_0)}$.

Returning to the impulse response function, we can now verify that the output generated by a filter $\hat{H}(\omega)$ in response to the impulse input signal $\delta(t)$ is indeed

$$(\delta * H)(t) = \int_{-\infty}^{\infty} \delta(\tau)H(t - \tau)\ d\tau = H(t)$$

where we have applied $\int_{-\infty}^{\infty} \delta(\tau)f(\tau)\ d\tau = f(0)$ with $f(\tau) = H(t - \tau)$.

Example 10 We saw in (6) that for an RLC circuit

$$\hat{H}(\omega) = \tfrac{1}{-LC\omega^2 + iRC\omega + 1}$$

To make the numbers work out cleanly, let's choose $R = 1$, $L = 6$ and $C = 5$. Then

$$\hat{H}(\omega) = \tfrac{1}{-6\omega^2 + i5\omega + 1} = \tfrac{1}{(3i\omega+1)(2i\omega+1)}$$

Recall from Example 1 that

$$f(t) = \begin{cases} e^{-at} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \qquad \Rightarrow \qquad \hat{f}(\omega) = \tfrac{1}{a+i\omega} \tag{7}$$

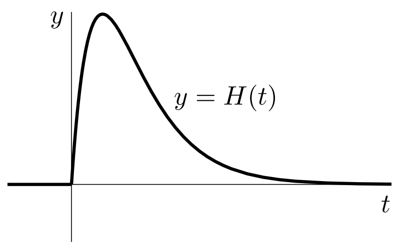
So we can determine the impulse response function $H(t)$ for the RLC filter just by using partial fractions to write $H(\omega)$ as a linear combination of $\frac{1}{a+i\omega}$'s. Since

$$\hat{H}(\omega) = \frac{1}{(3i\omega + 1)(2i\omega + 1)} = \frac{a}{3i\omega + 1} + \frac{b}{2i\omega + 1} = \frac{a(2i\omega + 1) + b(3i\omega + 1)}{(3i\omega + 1)(2i\omega + 1)} = \frac{(2a + 3b)i\omega + (a + b)}{(3i\omega + 1)(2i\omega + 1)}$$
$$\iff 2a + 3b = 0,\ a + b = 1 \iff b = 1 - a,\ 2a + 3(1 - a) = 0 \iff a = 3,\ b = -2$$

we have, using (7) with $a = \frac{1}{3}$ and $a = \frac{1}{2}$,

$$\hat{H}(\omega) = \frac{3}{3i\omega + 1} - \frac{2}{2i\omega + 1} = \frac{1}{\frac{1}{3} + i\omega} - \frac{1}{\frac{1}{2} + i\omega} \qquad \Rightarrow \qquad H(t) = \begin{cases} e^{-t/3} - e^{-t/2} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

which has graph



Example 10 (again) Here is a sneakier way to do the partial fraction expansion of Example 10. We know that $\hat{H}(\omega)$ has a partial fraction expansion of the form

$$\hat{H}(\omega) = \frac{1}{(3i\omega + 1)(2i\omega + 1)} = \frac{a}{3i\omega + 1} + \frac{b}{2i\omega + 1}$$

The fast way to determine a is to multiply both sides of this equation by $(3i\omega + 1)$

$$\frac{1}{2i\omega + 1} = a + \frac{b(3i\omega + 1)}{2i\omega + 1}$$

and then evaluate both sides at $3i\omega + 1 = 0$. Then the b term becomes zero, because of the factor $(3i\omega + 1)$ in the numerator and we get

$$a = \frac{1}{2i\omega + 1}\bigg|_{i\omega = -\frac{1}{3}} = \frac{1}{\frac{1}{3}} = 3$$

Similarly, multiplying by $(2i\omega + 1)$ rather than $(3i\omega + 1)$,

$$b = \frac{1}{3i\omega + 1}\bigg|_{2i\omega + 1 = 0} = \frac{1}{3i\omega + 1}\bigg|_{i\omega = -\frac{1}{2}} = \frac{1}{-\frac{1}{2}} = -2$$

Properties of the Fourier Transform

Property	*Signal*	*Fourier Transform*
	$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{x}(\omega)e^{i\omega t}\,d\omega$	$\hat{x}(\omega)=\int_{-\infty}^{\infty}x(t)e^{-i\omega t}\,dt$
	$y(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{y}(\omega)e^{i\omega t}\,d\omega$	$\hat{y}(\omega)=\int_{-\infty}^{\infty}y(t)e^{-i\omega t}\,dt$
Linearity	$Ax(t)+By(t)$	$A\hat{x}(\omega)+B\hat{y}(\omega)$
Time shifting	$x(t-t_0)$	$e^{-i\omega t_0}\hat{x}(\omega)$
Frequency shifting	$e^{i\omega_0 t}x(t)$	$\hat{x}(\omega-\omega_0)$
Scaling	$x\big(\frac{t}{\alpha}\big)$	$\lvert\alpha\rvert\hat{x}(\alpha\omega)$
Time shift & scaling	$x\big(\frac{t-t_0}{\alpha}\big)$	$\lvert\alpha\rvert e^{-i\omega t_0}\hat{x}(\alpha\omega)$
Frequency shift & scaling	$\lvert\alpha\rvert e^{i\omega_0 t}x(\alpha t)$	$\hat{x}\big(\frac{\omega-\omega_0}{\alpha}\big)$
Conjugation	$\overline{x(t)}$	$\overline{\hat{x}(-\omega)}$
Time reversal	$x(-t)$	$\hat{x}(-\omega)$
t–Differentiation	$x'(t)$	$i\omega\hat{x}(\omega)$
	$x^{(n)}(t)$	$(i\omega)^n\hat{x}(\omega)$
ω–Differentiation	$tx(t)$	$i\frac{d}{d\omega}\hat{x}(\omega)$
	$t^n x(t)$	$\big(i\frac{d}{d\omega}\big)^n\hat{x}(\omega)$
Convolution	$\int_{-\infty}^{\infty}x(\tau)y(t-\tau)\,d\tau$	$\hat{x}(\omega)\hat{y}(\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{x}(\theta)\hat{y}(\omega-\theta)\,d\theta$
Duality	$\hat{x}(t)$	$2\pi x(-\omega)$
Parseval	$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert\hat{x}(\omega)\rvert^2\,d\omega$	
	$e^{-at}u(t)=\begin{cases}0 & \text{if } t<0,\\ e^{-at} & \text{if } t>0\end{cases}$	$\dfrac{1}{a+i\omega}$ (a constant, $\mathrm{Re}\,a>0$)
	$e^{-at}u(-t)=\begin{cases}e^{-at} & \text{if } t<0,\\ 0 & \text{if } t>0\end{cases}$	$-\dfrac{1}{a+i\omega}$ (a constant, $\mathrm{Re}\,a<0$)
	$e^{-a\lvert t\rvert}$	$\dfrac{2a}{a^2+\omega^2}$ (a constant, $\mathrm{Re}\,a>0$)
Boxcar in time	$\mathrm{rect}(t)=\begin{cases}1 & \text{if } \lvert t\rvert<\frac{1}{2}\\ 0 & \text{if } \lvert t\rvert>\frac{1}{2}\end{cases}$	$\mathrm{sinc}\big(\frac{\omega}{2}\big)=\frac{2}{\omega}\sin\big(\frac{\omega}{2}\big)$
General boxcar	$r_{HWC}(t)=\begin{cases}H & \text{if } \lvert t-C\rvert<\frac{W}{2}\\ 0 & \text{if } \lvert t-C\rvert>\frac{W}{2}\end{cases}$	$HWe^{-i\omega C}\,\mathrm{sinc}\big(\frac{W\omega}{2}\big)=e^{-i\omega C}\frac{2H}{\omega}\sin\frac{W\omega}{2}$
Boxcar in frequency	$\frac{1}{2\pi}\mathrm{sinc}\big(\frac{t}{2}\big)=\frac{1}{\pi t}\sin\big(\frac{t}{2}\big)$	$\mathrm{rect}(\omega)=\begin{cases}1 & \text{if } \lvert\omega\rvert<1/2\\ 0 & \text{if } \lvert\omega\rvert>1/2\end{cases}$
Impulse in time	$\delta(t-t_0)$	$e^{-i\omega t_0}$
	$\delta(t-t_0)\,x(t)$	$e^{-i\omega t_0}\,x(t_0)$
Single frequency	$e^{i\omega_0 t}$	$2\pi\,\delta(\omega-\omega_0)$