The Unilateral z–Transform and Generating Functions

Recall from "Discrete–Time Linear, Time Invariant Systems and z-Transforms" that the behaviour of a discrete–time LTI system is determined by its impulse response function $h[n]$ and that the z–transform of $h[n]$ is

$$H(z) = \sum_{k=-\infty}^{\infty} z^{-k} h[k]$$

If the LTI system is causal, then $h[n] = 0$ for all $n < 0$ and

$$H(z) = \sum_{k=0}^{\infty} z^{-k} h[k]$$

Definition 1 (Unilateral z–Transform) The unilateral z–transform of the discrete–time signal $x[n]$ (whether or not $x[n] = 0$ for negative n's) is defined to be

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} z^{-n} x[n]$$

When there is any danger of confusing the regular z–transform with the unilateral z–transform,

$$X(z) = \sum_{n=-1}^{\infty} z^{-n} x[n]$$

is called the **bilateral z–transform**.

Example 2 The signal $x[n] = a^n u[n]$ is zero for all $n < 0$. So the unilateral z–transform of $x[n]$ is the same as the ordinary z–transform. So, as we saw in Example 7 of "Discrete–Time Linear, Time Invariant Systems and z-Transforms",

$$\mathcal{X}(z) = X(z) = \sum_{n=0}^{\infty} z^{-n} a^n = \tfrac{1}{1-z^{-1}a}$$

provided that $|z^{-1}a| < 1$, or equivalently $|z| > |a|$. Since the unilateral z–transform of any $x[n]$ is always equal to the bilateral z–transform of the right–sided signal $x[n]u[n]$, the region of convergence of a unilateral z–transform is always the *exterior* of a circle.

Example 3 If $x[n] = a^{n+1}u[n+1]$, then $x[-1] = 1$ is not zero and the unilateral and bilateral z–transforms of $x[n]$ are not the same. The unilateral z–transform is

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} z^{-n} a^{n+1} u[n+1] = \sum_{n=0}^{\infty} z^{-n} a^{n+1} = \tfrac{a}{1-z^{-1}a}$$

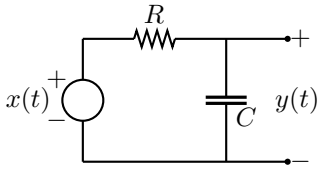
while the bilateral z–transform is

$$X(z) = \sum_{n=-\infty}^{\infty} z^{-n} a^{n+1} u[n+1] = \sum_{n=-1}^{\infty} z^{-n} a^{n+1} = \tfrac{z}{1-z^{-1}a}$$

In this particular example, the difference between the two transforms is $X(z) - \mathcal{X}(z) = \frac{z-a}{1-z^{-1}a} = \frac{z(1-z^{-1}a)}{1-z^{-1}a} = z$, which is the $n = -1$ term that is present in $X(z)$, but not in $\mathcal{X}(z)$.

Unilateral z–transforms are often used to analyze causal systems that are specified by linear constant coefficient difference equations with nonzero initial conditions (i.e. systems that are not initially at rest). Here is a simple example.

Example 4 We have already seen, in Example 1, of "Discrete–Time Linear, Time Invariant Systems and z-Transforms", that continous time input and output signals for the RC circuit



are related by the differential equation

$$\tfrac{dy}{dt}(t) + \tfrac{1}{RC}y(t) = \tfrac{1}{RC}x(t)$$

We have also seen, in Example 3, of "Discrete–Time Linear, Time Invariant Systems and z-Transforms", that if we fix some step size $\Delta > 0$ and approximate $\frac{dy}{dt}(t) \approx \frac{y(t)-y(t-\Delta)}{\Delta}$ then the differential equation, after a little simplication, becomes

$$y[n] = \big(1 + \tfrac{\Delta}{RC}\big)^{-1}\big(y[n-1] + \tfrac{\Delta}{RC}x[n]\big)$$

where $x[n] = x(n\Delta)$, $y[n] = y(n\Delta)$. For concreteness, suppose that $\frac{\Delta}{RC} = 1$ so that

$$y[n] - \tfrac{1}{2}y[n-1] = \tfrac{1}{2}x[n] \tag{1}$$

Suppose that we now run an experiment, starting at time $n = 0$. We input the signal $x[n] = \alpha u[n]$. But there is already a charge on the capacitor before we turn on the input signal, so that $y[-1] = \beta$. We want to find the output signal $y[n]$. One way to do so is to take the unilateral z–transform of both sides of (1). That is, we multiply both sides of (1) by z^{-n} and sum n from 0 to ∞.

$$\sum_{n=0}^{\infty} z^{-n}y[n] - \tfrac{1}{2}\sum_{n=0}^{\infty} z^{-n}y[n-1] = \tfrac{1}{2}\sum_{n=0}^{\infty} z^{-n}x[n] \tag{2}$$

The first term on the left hand side is just $\mathcal{Y}(z)$ and the term on the right hand side is just

$$\tfrac{1}{2}\mathcal{X}(z) = \tfrac{\alpha}{2}\sum_{n=0}^{\infty} z^{-n} = \tfrac{\alpha}{2}\tfrac{1}{1-z^{-1}}$$

To identify the second term on the left hand side, we make the change of summation variables $m = n - 1$.

$$\tfrac{1}{2}\sum_{n=0}^{\infty} z^{-n}y[n-1] = \tfrac{1}{2}\sum_{m=-1}^{\infty} z^{-(m+1)}y[m] = \tfrac{1}{2}y[-1] + \tfrac{1}{2z}\sum_{m=0}^{\infty} z^{-m}y[m] = \tfrac{\beta}{2} + \tfrac{1}{2z}\mathcal{Y}(z)$$

So (2) is

$$\mathcal{Y}(z) - \tfrac{\beta}{2} - \tfrac{1}{2z}\mathcal{Y}(z) = \tfrac{\alpha}{2}\tfrac{1}{1-z^{-1}} \implies \mathcal{Y}(z) = \frac{\frac{\beta}{2}}{1-\frac{1}{2z}} + \frac{\frac{\alpha}{2}}{\big(1-\frac{1}{2z}\big)\big(1-\frac{1}{z}\big)}$$

Using Example 2, we can immediately identify the inverse transform of the first term as $\frac{\beta}{2}\frac{1}{2^n}u[n]$. For the second term, we need to do a preliminary partial fractions step

$$\frac{\frac{\alpha}{2}}{\big(1-\frac{1}{2z}\big)\big(1-\frac{1}{z}\big)} = \frac{\alpha}{1-\frac{1}{z}} - \frac{\frac{\alpha}{2}}{1-\frac{1}{2z}}$$

Again using Example 2,

$$y[n] = \tfrac{\beta}{2}\tfrac{1}{2^n}u[n] + \alpha u[n] - \tfrac{\alpha}{2}\tfrac{1}{2^n}u[n] = \alpha\big(1 - \tfrac{1}{2^{n+1}}\big)u[n] + \beta\tfrac{1}{2^{n+1}}u[n]$$

The first part of the solution, $\alpha\big(1-\frac{1}{2^{n+1}}\big)u[n]$ is called the zero–state response. It is a response to the applied input signal $x[n]$ and is present even if $\beta = 0$ so that the initial condition or intial state, $y[-1]$. is zero. It is like the particular solution to an ordinary differential equation. The second part of the solution, $\beta\frac{1}{2^{n+1}}u[n]$, is called the zero-input response. It is a response to the initial state of the system and is present even if $\alpha = 0$ so that the input signal is zero. It is like the complementary solution to a differential equation.

The unilateral z–transform is also used outside of the context of LTI systems. Then the standard notation and terminology is a little different. In particular z^{-1} is replaced by some variable, like x, without the inverse.

Definition 5 (Generating function) The generating function of the sequence of (real or complex) numbers $\{a_0,\ a_1,\ a_2,\ \cdots\}$ is defined to be

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

Here are some examples.

Example 6 Let $a_n = 1 + 2 + \cdots + n$ be the sum of the first n integers. You probably learned that $a_n = \frac{n(n+1)}{2}$ in high school. Here is how you can use generating functions to derive this formula, without having to know or guess it ahead of time. The same method will work for sums of squares, cubes or higher powers. We can express a_{n+1} in terms of a_n by

$$a_{n+1} = (1 + 2 + \cdots + n) + (n + 1) = a_n + (n + 1)$$

If we define $a_0 = 0$, then substituting $n = 0$ into the above recursion relation gives $a_1 = a_0 + 1 = 1$, which is what we want. So all of the a_n's are determined by the recursion relation

$$a_0 = 0, \qquad a_{n+1} = a_n + (n + 1)$$

Define the generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$. Then multiplying the recursion relation by x^n and summing n from 0 to ∞ gives

$$\sum_{n=0}^{\infty} a_{n+1}x^n = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty}(n + 1)x^n \tag{3}$$

To simplify the left hand side, we substitute $m = n + 1$

$$\sum_{n=0}^{\infty} a_{n+1}x^n = \sum_{m=1}^{\infty} a_m x^{m-1} = \tfrac{1}{x}\sum_{m=0}^{\infty} a_m x^m \qquad (\text{since } a_0 = 0)$$

This is just $\frac{1}{x}A(x)$. The last sum can be easily computed using the trick $\frac{d}{dx}x^{n+1} = (n + 1)x^n$:

$$\sum_{n=0}^{\infty}(n + 1)x^n = \tfrac{d}{dx}\sum_{n=0}^{\infty} x^{n+1} = \tfrac{d}{dx}\tfrac{x}{1-x} = \tfrac{1}{1-x} + \tfrac{x}{(1-x)^2} = \tfrac{1}{(1-x)^2}$$

So (3) is

$$\tfrac{1}{x}A(x) = A(x) + \tfrac{1}{(1-x)^2} \iff A(x) = \tfrac{1}{1/x-1}\tfrac{1}{(1-x)^2} \iff A(x) = \tfrac{x}{(1-x)^3}$$

To identify the coefficient of x^n in $\frac{x}{(1-x)^3}$ (which is a_n after all) we start with

$$\tfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n \tag{4}$$

We can convert $\frac{1}{1-x}$ into $\frac{x}{(1-x)^3}$ by first differentiating twice and then multiplying by x. Applying $\frac{d}{dx}$ twice to (4) and them multiplying by $\frac{x}{2}$ gives

$$\begin{aligned}
\tfrac{1}{(1-x)^2} &= \sum_{n=0}^{\infty} nx^{n-1} && \text{and then}\\
\tfrac{2}{(1-x)^3} &= \sum_{n=0}^{\infty} n(n-1)x^{n-2} && \text{and then}\\
\tfrac{x}{(1-x)^3} &= \sum_{n=0}^{\infty} \tfrac{n(n-1)}{2}x^{n-1} \overset{m=n-1}{=} \sum_{m=-1}^{\infty} \tfrac{(m+1)m}{2}x^m
\end{aligned}$$

We are free to drop the $m=-1$ term from the last sum, because it is zero. So

$$\sum_{m=0}^{\infty} a_m x^m = A(x) = \tfrac{x}{(1-x)^3} = \sum_{m=0}^{\infty} \tfrac{(m+1)m}{2}x^m$$

and a_m, which is the coefficient of x^m in $\frac{x}{(1-x)^3}$ is $\frac{(m+1)m}{2}$, as expected.

Example 7 We now use the method of Example 6 to compute $a_n = 1^3 + 2^3 + \cdots + n^3$. The a_n's are determined by the recursion relation

$$a_0 = 0, \qquad a_n = a_{n-1} + n^3$$

Define the generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$. Then multiplying the recursion relation by x^n and summing n from 1 to ∞ gives

$$\sum_{n=1}^{\infty} a_n x^n = x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + \sum_{n=1}^{\infty} n^3 x^n \text{ or } A(x) = xA(x) + \sum_{n=1}^{\infty} n^3 x^n$$

To compute the last sum we use $x\frac{d}{dx}x^n = nx^n$.

$$\begin{aligned}
\sum_{n=1}^{\infty} n^3 x^n &= \big(x\tfrac{d}{dx}\big)^3 \sum_{n=0}^{\infty} x^n = \big(x\tfrac{d}{dx}\big)^3 \tfrac{1}{1-x} = \big(x\tfrac{d}{dx}\big)^2\big(x\tfrac{d}{dx}\tfrac{1}{1-x}\big) = \big(x\tfrac{d}{dx}\big)^2 \tfrac{x}{(1-x)^2} = \big(x\tfrac{d}{dx}\big)\big(x\big[\tfrac{1}{(1-x)^2} + \tfrac{2x}{(1-x)^3}\big]\big)\\
&= \big(x\tfrac{d}{dx}\big)\tfrac{x+x^2}{(1-x)^3} = x\big[\tfrac{1+2x}{(1-x)^3} + \tfrac{3x+3x^2}{(1-x)^4}\big] = \tfrac{x+4x^2+x^3}{(1-x)^4}
\end{aligned}$$

So

$$A(x) = xA(x) + \tfrac{x+4x^2+x^3}{(1-x)^4} \iff A(x) = \tfrac{x+4x^2+x^3}{(1-x)^5}$$

Again, we can convert $\frac{1}{1-x}$, whose series expansion we know, into $\frac{x+4x^2+x^3}{(1-x)^5}$ by differentiating a bunch of times and then multiplying by a polynomial in x. Repeatedly applying $\frac{d}{dx}$ to

$$\begin{aligned}
\tfrac{1}{1-x} &= \sum_{n=0}^{\infty} x^n && \text{gives}\\
\tfrac{1}{(1-x)^2} &= \sum_{n=0}^{\infty} n x^{n-1} && \text{and then}\\
\tfrac{2}{(1-x)^3} &= \sum_{n=0}^{\infty} n(n-1) x^{n-2} && \text{and then}\\
\tfrac{6}{(1-x)^4} &= \sum_{n=0}^{\infty} n(n-1)(n-2) x^{n-3} && \text{and then}\\
\tfrac{24}{(1-x)^5} &= \sum_{n=0}^{\infty} n(n-1)(n-2)(n-3) x^{n-4}
\end{aligned}$$

so that

$$\begin{aligned}
\tfrac{x}{(1-x)^5} &= \sum_{n=0}^{\infty} \tfrac{n(n-1)(n-2)(n-3)}{24} x^{n-3} \overset{m=n-3}{=} \sum_{m=-3}^{\infty} \tfrac{(m+3)(m+2)(m+1)m}{24} x^m\\
\tfrac{4x^2}{(1-x)^5} &= \sum_{n=0}^{\infty} \tfrac{n(n-1)(n-2)(n-3)}{6} x^{n-2} \overset{m=n-2}{=} \sum_{m=-2}^{\infty} \tfrac{(m+2)(m+1)m(m-1)}{6} x^m\\
\tfrac{x^3}{(1-x)^5} &= \sum_{n=0}^{\infty} \tfrac{n(n-1)(n-2)(n-3)}{24} x^{n-1} \overset{m=n-1}{=} \sum_{m=-1}^{\infty} \tfrac{(m+1)m(m-1)(m-2)}{24} x^m
\end{aligned}$$

We are free to drop the $m < 0$ terms from these sums because thay are all zero. So adding gives that the coefficient of x^m in $\frac{x+4x^2+x^3}{(1-x)^5}$ is

$$\begin{aligned}
a_m &= \frac{m(m+1)[(m-1)(m-2)+4(m+2)(m-1)+(m+3)(m+2)]}{24} = \frac{m(m+1)[6m^2+6m]}{24}\\
&= \frac{m^2(m+1)^2}{4}
\end{aligned}$$

Example 8 (Fibonacci Numbers) The sequence of Fibonacci numbers was first described by the Indian mathematicians Gopala and Hemachandra in 1150, who were investigating bin packing problems. In the West, it was first studied by Leonardo da Pisa (1175 – 1250) who is better known by his nickname Fibonacci (which is a contraction of "filius Bonacci", which in turn is latin for "son of Bonaccio"). In his book Liber Abaci[(1)] he introduced the n^{th} Fibonacci number as giving the number of pairs in a rabbit population after n months if it is assumed that

* in the first month there is just one new–born pair,
* new–born pairs are fertile commencing in their second month
* each month every fertile pair produces a new pair, and
* the rabbits never die

Suppose that in month $n-1$ we have F_{n-1} pairs of rabbits (both fertile and newly born) and in month n we have F_n pairs. In month $n+1$ we will have the F_n pairs of month n and an additional F_{n-1} pairs of newborn offspring from the F_{n-1} fertile pairs. Thus

$$F_{n+1} = F_{n-1} + F_n, \qquad F_0 = 0, \qquad F_1 = 1 \tag{5}$$

(1) "Liber Abaci" means "Book of Calculations" and was published in 1202. In this book Fibonacci also introduced the decimal–number system to the Latin–speaking world.

For the appearance of Fibonacci numbers in other natural settings (like the number of spirals in a sunflower) see

`http://ccins.camosun.bc.ca/~jbritton/fibslide/jbfibslide.htm`

`http://en.wikipedia.org/wiki/Fibonacci_number`

We'll now use a generating function to find an explicit formula for F_n. Define

$$F(x) = \sum_{n=0}^{\infty} F_n x^n$$

Multiplying both sides of (5) by x^n and summing over $n \geq 1$ gives

$$\tfrac{1}{x}\sum_{n=1}^{\infty} F_{n+1}x^{n+1} = x\sum_{n=1}^{\infty} F_{n-1}x^{n-1} + \sum_{n=1}^{\infty} F_n x^n \tag{6}$$

Since $F_0 = 0$ and $F_1 = 1$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} F_{n+1}x^{n+1} &\overset{m=n+1}{=} \sum_{m=2}^{\infty} F_m x^m = \sum_{m=0}^{\infty} F_m x^m - F_0 - F_1 x = F(x) - x \\
\sum_{n=1}^{\infty} F_{n-1}x^{n-1} &\overset{m=n-1}{=} \sum_{m=0}^{\infty} F_m x^m = F(x) \\
\sum_{n=1}^{\infty} F_n x^n &= \sum_{n=0}^{\infty} F_n x^n - F_0 = F(x)
\end{aligned}$$

Substituting these into (6) gives

$$\tfrac{1}{x}\big(F(x) - x\big) = xF(x) + F(x) \iff F(x) - x = x^2F(x) + xF(x)$$

We can now easily solve for

$$F(x) = -\tfrac{x}{x^2+x-1}$$

To identify the coefficient of x^n in $-\frac{x}{x^2+x-1}$ we first simplify it using partial fractions. Let $r_\pm = \frac{1}{2}\big(-1 \pm \sqrt{5}\big)$ be the two roots of $x^2 + x - 1$. To solve for the coefficient a in the partial fraction expansion

$$-\tfrac{x}{x^2+x-1} = -\tfrac{x}{(x-r_+)(x-r_-)} = \tfrac{a}{x-r_+} + \tfrac{b}{x-r_-} \tag{7}$$

multiply the equation by $(x - r_+)$, simplify and then set $x = r_+$ (actually, take the limit as $x \to r_+$). To solve for b, multiply (7) by $(x - r_-)$, simplify and then set $x = r_-$. This gives

$$\begin{aligned}
a\tfrac{x-r_+}{x-r_+} + b\tfrac{x-r_+}{x-r_-} = -\tfrac{x(x-r_+)}{(x-r_+)(x-r_-)} &\implies a + b\tfrac{x-r_+}{x-r_-} = -\tfrac{x}{x-r_-} \overset{x=r_+}{\implies} a = -\tfrac{r_+}{r_+-r_-} \\
a\tfrac{x-r_-}{x-r_+} + b\tfrac{x-r_-}{x-r_-} = -\tfrac{x(x-r_-)}{(x-r_+)(x-r_-)} &\implies a\tfrac{x-r_-}{x-r_+} + b = -\tfrac{x}{x-r_+} \overset{x=r_-}{\implies} b = \tfrac{r_-}{r_+-r_-}
\end{aligned}$$

So

$$\begin{aligned}
F(x) &= \frac{1}{r_+ - r_-}\Big[-\frac{r_+}{x-r_+} + \frac{r_-}{x-r_-}\Big] = \frac{1}{r_+ - r_-}\Big[\frac{1}{1-(x/r_+)} - \frac{1}{1-(x/r_-)}\Big] \\
&= \frac{1}{r_+ - r_-}\sum_{n=0}^{\infty}\Big[\Big(\frac{x}{r_+}\Big)^n - \Big(\frac{x}{r_-}\Big)^n\Big]
\end{aligned}$$

which tells us that

$$F_n = \frac{1}{r_+ - r_-}\Big[\frac{1}{r_+^n} - \frac{1}{r_-^n}\Big]$$

To simplify $\frac{1}{r_+}$ and $\frac{1}{r_-}$ we can use the obervation that r_+r_- is the constant term in $(x-r_+)(x-r_-) = x^2+x-1$, so that $r_+r_- = -1$. So $\frac{1}{r_+} = -r_-$ and $\frac{1}{r_-} = -r_+$ and

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Note that $\frac{1}{2}(1+\sqrt{5}) > 1$ so that the first term grows with n while $\frac{1}{2}|1-\sqrt{5}| \approx 0.62 < 1$ so that the second term decreases with n. Thus, for large n

$$F_n \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

By the way, $\frac{1+\sqrt{5}}{2}$ is the golden ratio.

Generating functions are often used to compute averages, variances and standard deviations. Suppose that you take some measurement, N, whose answer has to be a positive integer. For example, you count the number of calls to a call centre, or the number of hits on a web site in a prescribed period of time. Suppose further that the probability that the measurement yields the number n is p_n. The numbers $\{p_n\}_{n\geq 0}$ are called a probability distribution. They obey

$$0 \leq p_n \leq 1 \qquad \sum_{n=0}^{\infty} p_n = 1$$

By definition, the average value of N is

$$\mu = \sum_{n=0}^{\infty} np_n$$

and the variance of N is the average value of $(N-\mu)^2$, which is

$$\sigma^2 = \sum_{n=0}^{\infty}(n-\mu)^2 p_n = \sum_{n=0}^{\infty} n^2 p_n - 2\mu\Big[\sum_{n=0}^{\infty} np_n\Big] + \mu^2\Big[\sum_{n=0}^{\infty} p_n\Big] = \sum_{n=0}^{\infty} n^2 p_n - 2\mu[\mu] + \mu^2[1] = \sum_{n=0}^{\infty} n^2 p_n - \mu^2$$

The standard deviation of N is σ. The standard deviation is a measure of the width of the probability distribution. For example, for the normal or Gaussian (bell curve) probability distribution, 68% of all outcomes will be within one standard deviation of the mean and 95% will be within two standard deviations of the mean. If we can determine the generating function

$$F(x) = \sum_{n=0}^{\infty} p_n x^n$$

for the probability distribution, then we can easily find the mean, variance and standard deviation. Since $x\frac{d}{dx}x^n = nx^n$, we have

$$x\tfrac{d}{dx}F(x) = \sum_{n=0}^{\infty} np_n x^n \implies \mu = \sum_{n=0}^{\infty} np_n x^n\Big|_{x=1} = F'(1)$$

and

$$\big(x\tfrac{d}{dx}\big)^2 F(x) = \sum_{n=0}^{\infty} n^2 p_n x^n \implies \sum_{n=0}^{\infty} n^2 p_n = \sum_{n=0}^{\infty} n^2 p_n x^n\Big|_{x=1} = \big(x\tfrac{d}{dx}\big)^2 F(x)\Big|_{x=1} = x\tfrac{d}{dx}\big[xF'(x)\big]\Big|_{x=1}$$
$$= \big[xF'(x) + x^2F''(x)\big]_{x=1} = F'(1) + F''(1)$$

We conclude that

$$\mu = F'(1) \qquad \text{and} \qquad \sigma^2 = F''(1) + F'(1) - F'(1)^2$$

Example 9 (The Poisson Distribution) Suppose that we perform an experiment in which we count the number of occurrences of some event in a specified time interval or in a specified region of space. For example, we could be counting the number of telephone calls arriving at a call centre in a specified one hour period. It is possible to show[(2)] that if

(1) the events occur at a point in time or space
(2) the number of events occurring in one region (for example the first quarter hour) is independent of the number occurring in any disjoint region (for example the second quarter hour)
(3) the probability of more than one event occurring at exactly the same point is negligible
(4) the probability of n events in region #1 is the same as the probability of n events in region #2, when the regions have the same size
(5) for N very large, the average number of events occurring in a subregion whose size is one N^{th} of the full region is approximately $\frac{\lambda}{N}$

then the probability that n events occur is $p_n = \frac{e^{-\lambda}\lambda^n}{n!}$. This is called the Poisson distribution. It has been observed experimentally in

- radioactive disintegration (α–decay)
- flying bomb hits on London during the second world war
- chromosome interchanges in cells due to irradiation by x-rays
- connections to wrong phone numbers
- positions of bacterial colonies in a Petri dish

The generating function for this distribution is

$$F(x) = \sum_{n=0}^{\infty} p_n x^n = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda x)^n}{n!} = e^{-\lambda} e^{\lambda x} = e^{\lambda(x-1)}$$

Note, as a check, that $F(1) = 1$, as we know must be the case for the generating function of any probability distribution, because $F(1) = \sum_{n=0}^{\infty} p_n x^n\big|_{n=1} = \sum_{n=0}^{\infty} p_n$. We have

$$\begin{aligned} F'(x) &= \lambda e^{\lambda(x-1)} & F''(x) &= \lambda^2 e^{\lambda(x-1)} \\ F'(1) &= \lambda & F''(1) &= \lambda^2 \end{aligned}$$

so that

$$\mu = F'(1) = \lambda \qquad \text{and} \qquad \sigma^2 = F''(1) + F'(1) - F'(1)^2 = \lambda \qquad \text{and} \qquad \sigma = \sqrt{\lambda}$$

(2) This is done, for example, in STAT 251 and in MATH 302.