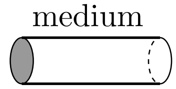
The Heat Equation (One Space Dimension)

In these notes we derive the heat equation for one space dimension. This partial differential equation describes the flow of heat energy, and consequently the behaviour of the temperature, in an idealized long thin rod, under the assumptions that heat energy neither enters nor leaves the rod through its sides and that heat energy is neither created nor destroyed (for example by chemical reactions) in the interior of the rod.

Let $T(x,t)$ be the temperature at time t at the point x in the rod. We derive the heat equation for T from two physical "laws", that we assume are valid:

- The amount of heat energy required to raise the temperature of a body by dT degrees is $sm\,dT$ where, m is the mass of the body and s is a positive physical constant determined by the material contained in the body. It is called the specific heat of the body.
- The rate at which heat energy crosses a surface is proportional to the surface area and to the temperature gradient at the surface. The constant of proportionality is called the thermal conductivity and is denoted κ.

So consider an infinitesmal hunk of bar of cross–sectional area A and mass density ρ. To



make it easy to get the signs right, we pretend that the temperature is increasing from left to right. The temperature gradient at the right hand end is $\frac{\partial T}{\partial x}(x+dx,t)$, so the rate at which heat energy crosses the right hand end is $\kappa A\frac{\partial T}{\partial x}(x+dx,t)$. Similarly, the rate at which heat energy crosses the left hand end is $\kappa A\frac{\partial T}{\partial x}(x,t)$. When the temperature is increasing from left to right, both $\kappa A\frac{\partial T}{\partial x}(x+dx,t)$ and $\kappa A\frac{\partial T}{\partial x}(x,t)$ are positive. Since heat flows from hot regions to cold regions, heat energy should be entering the hunk through the right end and exiting the hunk through the left end. So in an infinitesmal time interval dt, the net amount of heat that enters the hunk of rod is the amount that enters through the right end minus the amount that departs through the left end, which is

$$\kappa A\tfrac{\partial T}{\partial x}(x+dx,t)\ dt-\kappa A\tfrac{\partial T}{\partial x}(x,t)\ dt$$

In this same time interval the temperature in the hunk of rod changes by $\frac{\partial T}{\partial t}(x,t)\,dt$. The mass of the hunk of rod is $\rho A\,dx$, so by our first physical law, the amount of heat energy required to increase the temperature by $\frac{\partial T}{\partial t}(x,t)\,dt$ is $s\rho A\,dx\,\frac{\partial T}{\partial t}(x,t)\,dt$. Assuming that the rod is not generating or destroying heat itself, this must be same as the amount of heat that entered the hunk in the time interval dt. That is,

$$s\ \rho A\,dx\ \tfrac{\partial T}{\partial t}(x,t)\,dt=\kappa A\left[\tfrac{\partial T}{\partial x}(x+dx,t)-\tfrac{\partial T}{\partial x}(x,t)\right]dt$$

Dividing both sides by $dx\ dt$ and taking the limits $dx, dt \to 0$ gives

$$s\ \rho A\ \tfrac{\partial T}{\partial t}(x,t) = \kappa A\,\tfrac{\partial^2 T}{\partial x^2}(x,t) \qquad \text{or} \qquad \tfrac{\partial T}{\partial t}(x,t) = \alpha^2\,\tfrac{\partial^2 T}{\partial x^2}(x,t) \tag{1}$$

where the thermal diffusivity $\alpha^2 = \frac{\kappa}{s\rho}$. This is the heat equation.