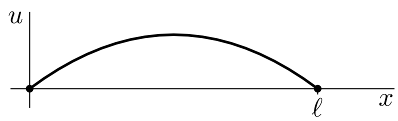
Solution of the Wave Equation by Separation of Variables

The Problem

Let $u(x,t)$ denote the vertical displacement of a string from the x axis at position x and time t. The string has length ℓ. Its left and right hand ends are held fixed at height zero and we are told its initial configuration and speed. For notational convenience, choose a coordinate system so that the left hand end of the string is at $x = 0$ and the right hand end of the string is at $x = \ell$.



We assume that the string is undergoing small amplitude transverse vibrations so that $u(x,t)$ obeys the wave equation

$$\tfrac{\partial^2 u}{\partial t^2}(x,t) = c^2\, \tfrac{\partial^2 u}{\partial x^2}(x,t) \qquad \text{for all } 0 < x < \ell \text{ and } t > 0 \tag{1}$$

The conditions that the left and right hand ends are held at height zero are encoded in the "boundary conditions"

$$u(0,t) = 0 \qquad \text{for all } t > 0 \tag{2}$$
$$u(\ell,t) = 0 \qquad \text{for all } t > 0 \tag{3}$$

As we have been told the position and speed of the string at time 0, there are given functions $f(x)$ and $g(x)$ such that the "initial conditions"

$$u(x,0) = f(x) \qquad \text{for all } 0 < x < \ell \tag{4}$$
$$u_t(x,0) = g(x) \qquad \text{for all } 0 < x < \ell \tag{5}$$

are satisfied. The problem is to determine $u(x,t)$ for all x and t.

Outline of the Method of Separation of Variables

We are going to solve this problem in three steps.

Step 1 In the first step, we find all solutions of (1) that are of the special form $u(x,t) = X(x)T(t)$ for some function $X(x)$ that depends on x but not t and some function $T(t)$ that depends on t but not x. This is where the name "separation of variables" comes from. It is of course too much to expect that all solutions of (1) are of this form. But if we find a bunch of solutions $X_i(x)T_i(t)$ of this form, then since (1) is a linear equation, $\sum_i a_i X_i(x)T_i(t)$ is also a solution for any choice of the constants a_i. (Check this yourself!) If we are lucky (and we shall be lucky), we will be able to choose the constants a_i so that the other conditions (2–5) are also satisfied.

Step 2 We impose the boundary conditions (2) and (3).

Step 3 We impose the initial conditions (4) and (5).

The First Step – Finding Factorized Solutions

The factorized function $u(x,t) = X(x)T(t)$ is a solution to the wave equation (1) if and only if

$$X(x)T''(t) = c^2 X''(x)T(t) \iff \tfrac{X''(x)}{X(x)} = \tfrac{1}{c^2}\tfrac{T''(t)}{T(t)}$$

The left hand side is independent of t. So the right hand side, which is equal to the left hand side, must be independent of t too. The right hand side is independent of x. So the left hand side must be independent of x too. So both sides must be independent of both x and t. So both sides must be constant. Let's call the constant σ. So we have

$$\begin{aligned} &\frac{X''(x)}{X(x)} = \sigma && \frac{1}{c^2}\frac{T''(t)}{T(t)} = \sigma \\ \iff\quad & X''(x) - \sigma X(x) = 0 && T''(t) - c^2\sigma T(t) = 0 \end{aligned} \tag{6}$$

We now have two constant coefficient ordinary differential equations, which we solve in the usual way. We try $X(x) = e^{rx}$ and $T(t) = e^{st}$ for some constants r and s to be determined. These are solutions if and only if

$$\begin{aligned} & \tfrac{d^2}{dx^2}e^{rx} - \sigma e^{rx} = 0 && \tfrac{d^2}{dt^2}e^{st} - c^2\sigma e^{st} = 0 \\ \iff\quad & \big(r^2 - \sigma\big)e^{rx} = 0 && \big(s^2 - c^2\sigma\big)e^{st} = 0 \\ \iff\quad & r^2 - \sigma = 0 && s^2 - c^2\sigma = 0 \\ \iff\quad & r = \pm\sqrt{\sigma} && s = \pm c\sqrt{\sigma} \end{aligned}$$

If $\sigma \neq 0$, we now have two independent solutions, namely $e^{\sqrt{\sigma}x}$ and $e^{-\sqrt{\sigma}x}$, for $X(x)$ and two independent solutions, namely $e^{c\sqrt{\sigma}t}$ and $e^{-c\sqrt{\sigma}t}$, for $T(t)$. If $\sigma \neq 0$, the general solution to (6) is

$$X(x) = d_1 e^{\sqrt{\sigma}x} + d_2 e^{-\sqrt{\sigma}x} \qquad T(t) = d_3 e^{c\sqrt{\sigma}t} + d_4 e^{-c\sqrt{\sigma}t}$$

for arbitrary constants d_1, d_2, d_3 and d_4. If $\sigma = 0$, the equations (6) simplify to

$$X''(x) = 0 \qquad T''(t) = 0$$

and the general solution is

$$X(x) = d_1 + d_2 x \qquad T(t) = d_3 + d_4 t$$

for arbitrary constants d_1, d_2, d_3 and d_4. We have now found a huge number of solutions to the wave equation (1). Namely

$$\begin{aligned} u(x,t) &= \big(d_1 e^{\sqrt{\sigma}x} + d_2 e^{-\sqrt{\sigma}x}\big)\big(d_3 e^{c\sqrt{\sigma}t} + d_4 e^{-c\sqrt{\sigma}t}\big) && \text{for arbitrary } \sigma \neq 0 \text{ and arbitrary } d_1, d_2, d_3, d_4 \\ u(x,t) &= \big(d_1 + d_2 x\big)\big(d_3 + d_4 t\big) && \text{for arbitrary } d_1, d_2, d_3, d_4 \end{aligned}$$

The Second Step – Imposition of the Boundary Conditions

If $X_i(x)T_i(t)$, $i = 1, 2, 3, \cdots$ all solve the wave equation (1), then $\sum_i a_i X_i(x)T_i(t)$ is also a solution for any choice of the constants a_i. This solution satisfies the boundary condition (2) if and only if

$$\sum_i a_i X_i(0)T_i(t) = 0 \qquad \text{for all } t > 0$$

This will certainly be the case if $X_i(0) = 0$ for all i. In fact, if the a_i's are nonzero and the $T_i(t)$'s are independent, then (2) is satisfied if and only if all of the $X_i(0)$'s are zero. For us, it will be good enough to simply restrict our attention to X_i's for which $X_i(0) = 0$, so I am not even going to define what "independent" means[(1)]. Similarly, $u(x,t) = \sum_i a_i X_i(x)T_i(t)$ satisfies the boundary condition (3) if and only if

$$\sum_i a_i X_i(\ell)T_i(t) = 0 \qquad \text{for all } t > 0$$

[(1)] It forbids, for example, $T_1 = 7T_2$ or $T_1 = 3T_2 + 5T_3$.

and this will certainly be the case if $X_i(\ell) = 0$ for all i. We are now going to go through the solutions that we found in Step 1 and discard all of those that fail to satisfy $X(0) = X(\ell) = 0$.

First, consider $\sigma = 0$ so that $X(x) = d_1 + d_2 x$. The condition $X(0) = 0$ is satisfied if and only if $d_1 = 0$. The condition $X(\ell) = 0$ is satisfied if and only if $d_1 + \ell d_2 = 0$. So the conditions $X(0) = X(\ell) = 0$ are both satisfied only if $d_1 = d_2 = 0$, in which case $X(x)$ is identically zero. There is nothing to be gained by keeping an identically zero $X(x)$, so we discard $\sigma = 0$ completely.

Next, consider $\sigma \neq 0$ so that $d_1 e^{\sqrt{\sigma}x} + d_2 e^{-\sqrt{\sigma}x}$. The condition $X(0) = 0$ is satisfied if and only if $d_1 + d_2 = 0$. So we require that $d_2 = -d_1$. The condition $X(\ell) = 0$ is satisfied if and only if

$$0 = d_1 e^{\sqrt{\sigma}\ell} + d_2 e^{-\sqrt{\sigma}\ell} = d_1\big(e^{\sqrt{\sigma}\ell} - e^{-\sqrt{\sigma}\ell}\big)$$

If d_1 were zero, then $X(x)$ would again be identically zero and hence useless. So instead, we discard any σ that does not obey

$$e^{\sqrt{\sigma}\ell} - e^{-\sqrt{\sigma}\ell} = 0 \iff e^{\sqrt{\sigma}\ell} = e^{-\sqrt{\sigma}\ell} \iff e^{2\sqrt{\sigma}\ell} = 1$$

In the last step, we multiplied both sides of $e^{\sqrt{\sigma}\ell} = e^{-\sqrt{\sigma}\ell}$ by $e^{\sqrt{\sigma}\ell}$. One σ that obeys $e^{2\sqrt{\sigma}\ell} = 1$ is $\sigma = 0$. But we are now considering only $\sigma \neq 0$. Fortunately, there are infinitely many complex numbers(2) that work. In fact $e^{2\sqrt{\sigma}\ell} = 1$ if and only if there is an integer k such that

$$2\sqrt{\sigma}\ell = 2k\pi\imath \iff \sqrt{\sigma} = k\tfrac{\pi}{\ell}\imath \iff \sigma = -k^2\tfrac{\pi^2}{\ell^2}$$

With $\sqrt{\sigma} = k\frac{\pi}{\ell}\imath$ and $d_2 = -d_1$,

$$\begin{aligned} X(x)T(t) &= d_1\big(e^{\imath\frac{k\pi}{\ell}x} - e^{-\imath\frac{k\pi}{\ell}x}\big)\big(d_3 e^{\imath\frac{ck\pi}{\ell}t} + d_4 e^{-\imath\frac{ck\pi}{\ell}t}\big) \\ &= 2\imath d_1 \sin\big(\tfrac{k\pi}{\ell}x\big)\big[(d_3+d_4)\cos\big(\tfrac{ck\pi}{\ell}t\big) + \imath(d_3-d_4)\sin\big(\tfrac{ck\pi}{\ell}t\big)\big] \\ &= \sin\big(\tfrac{k\pi}{\ell}x\big)\big[\alpha_k\cos\big(\tfrac{ck\pi}{\ell}t\big) + \beta_k\sin\big(\tfrac{ck\pi}{\ell}t\big)\big] \end{aligned}$$

where $\alpha_k = 2\imath d_1(d_3+d_4)$ and $\beta_k = -2d_1(d_3-d_4)$. Note that, to this point, d_1, d_3 and d_4 are allowed to be any complex numbers so that α_k and β_k are allowed to be any complex numbers.

The Third Step – Imposition of the Initial Conditions

We now know that

$$u(x,t) = \sum_{k=1}^{\infty} \sin\big(\tfrac{k\pi}{\ell}x\big)\big[\alpha_k\cos\big(\tfrac{ck\pi}{\ell}t\big) + \beta_k\sin\big(\tfrac{ck\pi}{\ell}t\big)\big]$$

obeys the wave equation (1) and the boundary conditions (2) and (3), for any choice of the constants $\alpha_k,\ \beta_k$. It remains only to see if we can choose the α_k's and β_k's to satisfy

$$f(x) = u(x,0) \ = \sum_{k=1}^{\infty} \alpha_k \sin\big(\tfrac{k\pi}{\ell}x\big) \tag{4'}$$

$$g(x) = u_t(x,0) = \sum_{k=1}^{\infty} \beta_k \tfrac{ck\pi}{\ell}\sin\big(\tfrac{k\pi}{\ell}x\big) \tag{5'}$$

But any (reasonably smooth) function, $h(x)$, defined on the interval $0 < x < \ell$, has a unique representation(3)

$$h(x) = \sum_{k=1}^{\infty} b_k \sin\tfrac{k\pi x}{\ell} \tag{7}$$

(2) See, for example, the notes "Complex Numbers and Exponentials".

(3) See, for example, the notes "Fourier Series".

as a linear combination of $\sin\frac{k\pi x}{\ell}$'s and we also know the formula

$$b_k = \tfrac{2}{\ell}\int_0^\ell h(x)\sin\tfrac{k\pi x}{\ell}\ dx$$

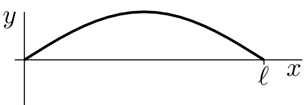
for the coefficients. We can make (7) match (4′) by choosing $h(x) = f(x)$ and $b_k = \alpha_k$. This tells us that $\alpha_k = \frac{2}{\ell}\int_0^\ell f(x)\sin\frac{k\pi x}{\ell}\ dx$. Similarly, we can make (7) match (5′) by choosing $h(x) = g(x)$ and $b_k = \beta_k\frac{ck\pi}{\ell}$. This tells us that $\frac{ck\pi}{\ell}\beta_k = \frac{2}{\ell}\int_0^\ell g(x)\sin\frac{k\pi x}{\ell}\ dx$. So we have a solution:

$$\boxed{u(x,t) = \sum_{k=1}^\infty \sin\left(\tfrac{k\pi}{\ell}x\right)\left[\alpha_k\cos\left(\tfrac{ck\pi}{\ell}t\right) + \beta_k\sin\left(\tfrac{ck\pi}{\ell}t\right)\right]} \tag{8}$$

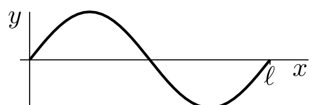
with

$$\boxed{\alpha_k = \tfrac{2}{\ell}\int_0^\ell f(x)\sin\tfrac{k\pi x}{\ell}\ dx \qquad \beta_k = \tfrac{2}{ck\pi}\int_0^\ell g(x)\sin\tfrac{k\pi x}{\ell}\ dx}$$

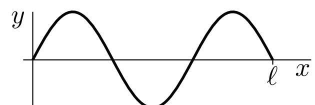
While the sum (8) can be very complicated, each term, called a "mode", is quite simple. For each fixed t, the mode $\sin\left(\frac{k\pi}{\ell}x\right)\left[\alpha_k\cos\left(\frac{kc\pi}{\ell}t\right) + \beta_k\sin\left(\frac{kc\pi}{\ell}t\right)\right]$ is just a constant times $\sin\left(\frac{k\pi}{\ell}x\right)$. As x runs from 0 to ℓ, the argument of $\sin\left(\frac{k\pi}{\ell}x\right)$ runs from 0 to $k\pi$, which is k half–periods of sin. Here are graphs, at fixed t, of the first three modes, called the fundamental tone, the first harmonic and the second harmonic.



fundamental



$1^{\rm st}$ harmonic



$2^{\rm nd}$ harmonic

For each fixed x, the mode $\sin\left(\frac{k\pi}{\ell}x\right)\left[\alpha_k\cos\left(\frac{kc\pi}{\ell}t\right) + \beta_k\sin\left(\frac{kc\pi}{\ell}t\right)\right]$ is just a constant times $\cos\left(\frac{kc\pi}{\ell}t\right)$ plus a constant times $\sin\left(\frac{kc\pi}{\ell}t\right)$. As t increases by one second, the argument, $\frac{kc\pi}{\ell}t$, of both $\cos\left(\frac{kc\pi}{\ell}t\right)$ and $\sin\left(\frac{kc\pi}{\ell}t\right)$ increases by $\frac{kc\pi}{\ell}$, which is $\frac{kc}{2\ell}$ cycles (i.e. periods). So the fundamental oscillates at $\frac{c}{2\ell}$cps, the first harmonic oscillates at $2\frac{c}{2\ell}$cps, the second harmonic oscillates at $3\frac{c}{2\ell}$cps and so on. If the string has density ρ and tension T, then we have seen[4] that $c = \sqrt{\frac{T}{\rho}}$. So to increase the frequency of oscillation of a string you increase the tension and/or decrease the density and/or shorten the string.

Example 1 As a concrete example, suppose that

$$\begin{aligned}
&\tfrac{\partial^2 u}{\partial t^2}(x,t) = c^2\,\tfrac{\partial^2 u}{\partial x^2}(x,t) && \text{for all } 0 < x < 1 \text{ and } t > 0\\
&u(0,t) = u(1,t) = 0 && \text{for all } t > 0\\
&u(x,0) = x(1-x) && \text{for all } 0 < x < 1\\
&u_t(x,0) = 0 && \text{for all } 0 < x < 1
\end{aligned}$$

This is a special case of equations (1–5) with $\ell = 1$, $f(x) = x(1-x)$ and $g(x) = 0$. So, by (8),

$$u(x,y) = \sum_{k=1}^\infty \sin(k\pi x)\left[\alpha_k\cos(ck\pi t) + \beta_k\sin(ck\pi t)\right]$$

with

$$\alpha_k = 2\int_0^1 x(1-x)\sin(k\pi x)\ dx \qquad \beta_k = 2\int_0^1 0\,\sin(k\pi x)\ dx = 0$$

[4] See, for example, the notes "Derivation of the Wave Equation".

Using[5]

$$\begin{aligned}
\int_0^1 x\sin(k\pi x)\ dx &= \int_0^1 -\tfrac{1}{\pi}\tfrac{d}{dk}\cos(k\pi x)\ dx \quad = -\tfrac{1}{\pi}\tfrac{d}{dk}\int_0^1 \cos(k\pi x)\ dx \quad = -\tfrac{1}{\pi}\tfrac{d}{dk}\tfrac{1}{k\pi}\sin(k\pi x)\Big|_0^1 \\
&= -\cos(k\pi)\tfrac{1}{k\pi} \\
\int_0^1 x^2\sin(k\pi x)\ dx &= \int_0^1 -\tfrac{1}{\pi^2}\tfrac{d^2}{dk^2}\sin(k\pi x)\ dx = -\tfrac{1}{\pi^2}\tfrac{d^2}{dk^2}\int_0^1 \sin(k\pi x)\ dx = \tfrac{1}{\pi^2}\tfrac{d^2}{dk^2}\tfrac{1}{k\pi}\cos(k\pi x)\Big|_0^1 \\
&= \cos(k\pi)\tfrac{2-k^2\pi^2}{k^3\pi^3} - \tfrac{2}{k^3\pi^3}
\end{aligned}$$

we have

$$\begin{aligned}
\alpha_k &= 2\int_0^1 x(1-x)\sin(k\pi x)\ dx = 2\big[-\cos(k\pi)\tfrac{1}{k\pi} - \cos(k\pi)\tfrac{2-k^2\pi^2}{k^3\pi^3} + \tfrac{2}{k^3\pi^3}\big] = \tfrac{4}{k^3\pi^3}[1-\cos(k\pi)] \\
&= \begin{cases} \tfrac{8}{k^3\pi^3} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even} \end{cases}
\end{aligned}$$

and

$$u(x,y) = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \tfrac{8}{k^3\pi^3}\sin(k\pi x)\cos(ck\pi t)$$

Example 2 As a second concrete example, suppose that

$$\begin{aligned}
&\tfrac{\partial^2 u}{\partial t^2}(x,t) = c^2\,\tfrac{\partial^2 u}{\partial x^2}(x,t) && \text{for all } 0<x<1 \text{ and } t>0 \\
&u(0,t) = u(1,t) = 0 && \text{for all } t>0 \\
&u(x,0) = \sin(5\pi x) + 2\sin(7\pi x) && \text{for all } 0<x<1 \\
&u_t(x,0) = 0 && \text{for all } 0<x<1
\end{aligned}$$

This is again a special case of equations (1–5) with $\ell = 1$. So, by (8),

$$u(x,y) = \sum_{k=1}^{\infty}\sin(k\pi x)\big[\alpha_k\cos(ck\pi t) + \beta_k\sin(ck\pi t)\big]$$

This time it is very inefficient to use the integral formulae to evaluate α_k and β_k. It is easier to observe directly, just by matching coefficients, that

$$\sin(5\pi x) + 2\sin(7\pi x) = u(x,0) \ = \sum_{k=1}^{\infty}\alpha_k\sin(k\pi x) \qquad \Rightarrow \alpha_k = \begin{cases} 1 & \text{if } k=5 \\ 2 & \text{if } k=7 \\ 0 & \text{if } k\neq 5,7 \end{cases}$$

$$0 = u_t(x,0) = \sum_{k=1}^{\infty} ck\pi\beta_k\sin(k\pi x) \Rightarrow \beta_k = 0$$

So

$$u(x,y) = \sin(5\pi x)\cos(5c\pi t) + 2\sin(7\pi x)\cos(7c\pi t)$$

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[5] Note that we cannot impose the condition that k is an integer until after evaluating the $\frac{d}{dk}$ derivatives.

Using Fourier Series to Solve the Wave Equation

We can also use Fourier series to derive the solution (8) to the wave equation (1) with boundary conditions (2,3) and initial conditions (4,5). The basic observation is that, for each fixed $t \ge 0$, the unknown $u(x,t)$ is a function of the one variable x and this function vanishes at $x = 0$ and $x = \ell$. Thus, by the Fourier series theorem and the odd periodic extension trick, $u(x,t)$ has, for each fixed t, a unique expansion

$$u(x,t) = \sum_{k=1}^{\infty} b_k(t) \sin\left(\tfrac{k\pi x}{\ell}\right) \tag{9}$$

By using other periodic extensions, like the even periodic extension, we can get other expansions of $u(x,t)$ too. But the odd periodic expansion (9) is particularly useful because, with it, the boundary conditions (2,3) are automatically satisfied. If we substitute $x = 0$ into the right hand side of (9) we necessarily get zero, regardless of the value of $b_k(t)$, because every term contains a factor of $\sin(0) = 0$. Similarly, if we substitute $x = \ell$ into the right hand side of (9) we again necessarily get zero, for any $b_k(t)$, because $\sin(k\pi) = 0$ for every integer k.

The solution $u(x,t)$ is completely determined by the, as yet unknown, coefficients $b_k(t)$. Furthermore these coefficients can be found by substituting $u(x,t) = \sum_{k=1}^{\infty} b_k(t) \sin\left(\frac{k\pi x}{\ell}\right)$ into the three remaining requirements (1), (4), (5) on $u(x,t)$. First the wave equation (1):

$$\begin{aligned} 0 = \tfrac{\partial^2 u}{\partial t^2}(x,t) - c^2 \tfrac{\partial^2 u}{\partial x^2}(x,t) &= \sum_{k=1}^{\infty} b_k''(t) \sin\left(\tfrac{k\pi x}{\ell}\right) + \sum_{k=1}^{\infty} \tfrac{k^2\pi^2 c^2}{\ell^2} b_k(t) \sin\left(\tfrac{k\pi x}{\ell}\right) \\ &= \sum_{k=1}^{\infty} \left[b_k''(t) + \tfrac{k^2\pi^2 c^2}{\ell^2} b_k(t) \right] \sin\left(\tfrac{k\pi x}{\ell}\right) \end{aligned}$$

This says that, for each fixed $t \ge 0$, the function 0, viewed as a function of x, has Fourier series expansion $\sum_{k=1}^{\infty} \left[b_k''(t) + \frac{k^2\pi^2 c^2}{\ell^2} b_k(t) \right] \sin\left(\frac{k\pi x}{\ell}\right)$. Applying (9) with $h(x)$ being the zero function and with b_k replaced by $\left[b_k''(t) + \frac{k^2\pi^2 c^2}{\ell^2} b_k(t) \right]$ then forces

$$b_k''(t) + \tfrac{k^2\pi^2 c^2}{\ell^2} b_k(t) = \tfrac{2}{\ell} \int_0^{\ell} 0 \sin\left(\tfrac{k\pi x}{\ell}\right) dx = 0 \qquad \text{for all } k, t \tag{1$'$}$$

Substituting into (4) and (5) gives

$$\begin{aligned} u(0,t) &= \sum_{k=1}^{\infty} b_k(0) \sin\left(\tfrac{k\pi x}{\ell}\right) = f(x) \\ \tfrac{\partial u}{\partial t}(0,t) &= \sum_{k=1}^{\infty} b_k'(0) \sin\left(\tfrac{k\pi x}{\ell}\right) = g(x) \end{aligned}$$

By uniqueness of Fourier coefficients, once again,

$$b_k(0) = \tfrac{2}{\ell} \int_0^{\ell} f(x) \sin\left(\tfrac{k\pi x}{\ell}\right) dx \tag{4$'$}$$

$$b_k'(0) = \tfrac{2}{\ell} \int_0^{\ell} g(x) \sin\left(\tfrac{k\pi x}{\ell}\right) dx \tag{5$'$}$$

For each fixed k, equations (1′), (4′) and (5′) constitute one second order constant coefficient ordinary differential equation and two initial conditions for the unknown function $b_k(t)$.

You already know how to solve constant coefficient ordinary differential equations. The function $b_k(t) = e^{rt}$ satisfies the ordinary differential equation ($1'$) if and only if

$$r^2 + \tfrac{k^2\pi^2c^2}{\ell^2} = 0$$

which in turn is true if and only if

$$r = \pm i\tfrac{k\pi c}{\ell}$$

so that the general solution to ($1'$) is

$$b_k(t) = C_k e^{i\frac{k\pi c}{\ell}t} + D_k e^{-i\frac{k\pi c}{\ell}t}$$

with C_k and D_k arbitrary constants. Using $e^{\pm i\frac{k\pi c}{\ell}t} = \cos\left(\tfrac{k\pi c}{\ell}t\right) \pm i\sin\left(\tfrac{k\pi c}{\ell}t\right)$ we may rewrite the solution as

$$b_k(t) = \alpha_k \cos\left(\tfrac{ck\pi}{\ell}t\right) + \beta_k \sin\left(\tfrac{ck\pi}{\ell}t\right)$$

with $\alpha_k = C_k + D_k$ and $\beta_k = iC_k - iD_k$ again arbitrary constants. They are determined by the initial conditions ($4'$) and ($5'$).

$$\begin{aligned}
(4') &\implies b_k(0) = \alpha_k \quad\;\; = \tfrac{2}{\ell}\int_0^\ell f(x)\sin\left(\tfrac{k\pi x}{\ell}\right)dx \\
(5') &\implies b_k'(0) = \tfrac{ck\pi}{\ell}\beta_k = \tfrac{2}{\ell}\int_0^\ell g(x)\sin\left(\tfrac{k\pi x}{\ell}\right)dx \implies \beta_k = \tfrac{2}{ck\pi}\int_0^\ell g(x)\sin\left(\tfrac{k\pi x}{\ell}\right)dx
\end{aligned}$$

This gives us the solution (8) once again.

