Mean and Variance of Binomial Random Variables

The probability function for a binomial random variable is

$$b(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x}$$

This is the probability of having x successes in a series of n independent trials when the probability of success in any one of the trials is p. If X is a random variable with this probability distribution,

$$\begin{aligned} E(X) &= \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \\ &= \sum_{x=0}^{n} x \tfrac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \\ &= \sum_{x=1}^{n} \tfrac{n!}{(x-1)!(n-x)!} p^x (1-p)^{n-x} \end{aligned}$$

since the $x = 0$ term vanishes. Let $y = x-1$ and $m = n-1$. Subbing $x = y+1$ and $n = m+1$ into the last sum (and using the fact that the limits $x = 1$ and $x = n$ correspond to $y = 0$ and $y = n-1 = m$, respectively)

$$\begin{aligned} E(X) &= \sum_{y=0}^{m} \tfrac{(m+1)!}{y!(m-y)!} p^{y+1} (1-p)^{m-y} \\ &= (m+1)p \sum_{y=0}^{m} \tfrac{m!}{y!(m-y)!} p^y (1-p)^{m-y} \\ &= np \sum_{y=0}^{m} \tfrac{m!}{y!(m-y)!} p^y (1-p)^{m-y} \end{aligned}$$

The binomial theorem says that

$$(a+b)^m = \sum_{y=0}^{m} \tfrac{m!}{y!(m-y)!} a^y b^{m-y}$$

Setting $a = p$ and $b = 1-p$

$$\sum_{y=0}^{m} \tfrac{m!}{y!(m-y)!} p^y (1-p)^{m-y} = \sum_{y=0}^{m} \tfrac{m!}{y!(m-y)!} a^y b^{m-y} = (a+b)^m = (p+1-p)^m = 1$$

so that

$$\boxed{E(X) = np}$$

Similarly, but this time using $y = x - 2$ and $m = n - 2$

$$\begin{aligned}
E\big(X(X-1)\big) &= \sum_{x=0}^{n} x(x-1)\tbinom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=0}^{n} x(x-1)\tfrac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \\
&= \sum_{x=2}^{n} \tfrac{n!}{(x-2)!(n-x)!} p^x (1-p)^{n-x} \\
&= n(n-1)p^2 \sum_{x=2}^{n} \tfrac{(n-2)!}{(x-2)!(n-x)!} p^{x-2} (1-p)^{n-x} \\
&= n(n-1)p^2 \sum_{y=0}^{m} \tfrac{m!}{y!(m-y)!} p^y (1-p)^{m-y} \\
&= n(n-1)p^2 \big(p + (1-p)\big)^m \\
&= n(n-1)p^2
\end{aligned}$$

So the variance of X is

$$E(X^2) - E(X)^2 = E\big(X(X-1)\big) + E(X) - E(X)^2 = n(n-1)p^2 + np - (np)^2$$
$$= \boxed{np(1-p)}$$