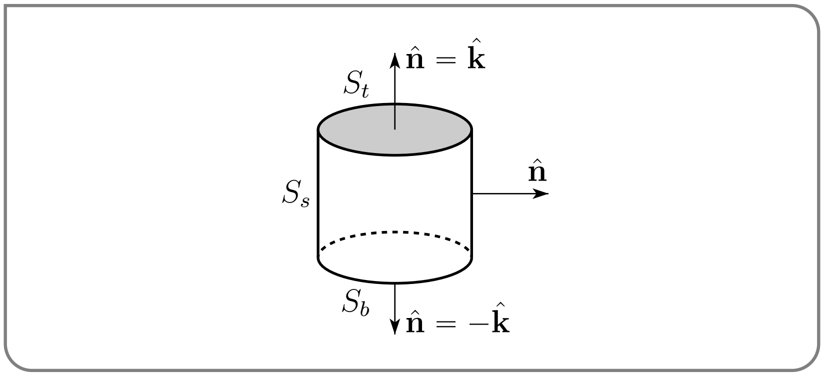
Problem: Evaluate $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS$ where

$$\mathbf{F}(x,y,z) = \hat{\boldsymbol{\jmath}} + (x+z)\,\hat{\mathbf{k}}$$

and S is the boundary of $V = \{\,(x,y,z)\,\big|\,0 \leqslant x^2+y^2 \leqslant 9,\ 0 \leqslant z \leqslant 5\,\}$, and $\hat{\mathbf{n}}$ is the outward normal to S.

Solution. The volume V looks like a tin can of radius 3 and height 5. It is natural to



decompose its surface S into three parts

$$\begin{aligned} S_t &= \{\,(x,y,z)\,\big|\,0 \leqslant x^2+y^2 \leqslant 9,\ z=5\,\} = \text{the top}\\ S_b &= \{\,(x,y,z)\,\big|\,0 \leqslant x^2+y^2 \leqslant 9,\ z=0\,\} = \text{the bottom}\\ S_s &= \{\,(x,y,z)\,\big|\,x^2+y^2 = 9,\ 0 \leqslant z \leqslant 5\,\} = \text{the side} \end{aligned}$$

We'll compute the flux through each of the three parts separately and then add them together.

The Bottom: On S_b, $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$ and $dS = dxdy$. So

$$\iint_{S_b} \mathbf{F}\cdot \overbrace{\hat{\mathbf{n}}}^{=-\hat{\mathbf{k}}}\,dS = -\iint_{\substack{x^2+y^2\leqslant 9\\ z=0}} (x + \overbrace{z}^{=0})\,dxdy = -\iint_{x^2+y^2\leqslant 9} \overbrace{x}^{\text{odd}}\,dxdy = 0$$

The Top: On S_t, $\hat{\mathbf{n}} = \hat{\mathbf{k}}$ and $dS = dxdy$. So

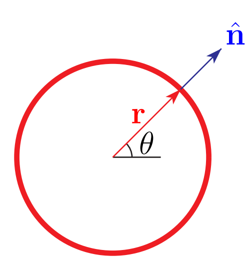
$$\iint_{S_t} \mathbf{F}\cdot \overbrace{\hat{\mathbf{n}}}^{=\hat{\mathbf{k}}}\,dS = \iint_{\substack{x^2+y^2\leqslant 9\\ z=5}} (\overbrace{x}^{\text{odd}} + \overbrace{z}^{5})\,dxdy = \iint_{x^2+y^2\leqslant 9} 5\,dxdy = 5\pi(3)^2 = 45\pi$$

The Side: We can parametrize the side by using cylindrical coordinates.

$$\mathbf{r}(\theta,z) = 3\cos\theta\,\hat{\boldsymbol{\imath}} + 3\sin\theta\,\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}} \qquad 0 \leqslant \theta < 2\pi,\ 0 \leqslant z \leqslant 5$$

Then,

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial \theta} &= -3\sin\theta\,\hat{\boldsymbol{\imath}} + 3\cos\theta\,\hat{\boldsymbol{\jmath}}\\ \frac{\partial \mathbf{r}}{\partial z} &= \hat{\mathbf{k}}\\ \hat{\mathbf{n}}\,dS &= \frac{\partial \mathbf{r}}{\partial \theta}\times\frac{\partial \mathbf{r}}{\partial z}\,d\theta\,dz\\ &= (3\cos\theta\,\hat{\boldsymbol{\imath}} + 3\sin\theta\,\hat{\boldsymbol{\jmath}})\,d\theta\,dz \end{aligned}$$



top view

So

$$\begin{aligned}
\mathbf{F}\big(x(\theta,z)\,,\,y(\theta,z)\,,\,z(\theta,z)\big) &= \hat{\pmb{\jmath}} + (3\cos\theta + z)\,\hat{\mathbf{k}} \\
\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= 3\sin\theta\,\mathrm{d}\theta\,\mathrm{d}z \\
\iint_{S_s} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= 3\int_0^{2\pi}\mathrm{d}\theta\int_0^5 \mathrm{d}z\ \sin\theta = 15\int_0^{2\pi}\mathrm{d}\theta\ \sin\theta \\
&= 0
\end{aligned}$$

and the total flux is

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_{S_t} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \iint_{S_b} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \iint_{S_s} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 0 + 45\pi + 0 = 45\pi$$