## Theorem 2.4.2

If $\vec{F} = \vec{\nabla}\varphi$ is conservative and

$\mathcal{C}$ is a curve that starts at $P_0$ and ends at $P_1$

then $$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \varphi(P_1) - \varphi(P_0).$$

**Proof** Parametrize $\mathcal{C}$: $\vec{r}(t) = (x(t), y(t), z(t))$ $a \le t \le b$

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \frac{d\vec{r}}{dt}(t)\,dt = \int_a^b \vec{\nabla}\varphi(\vec{r}(t))\cdot\frac{d\vec{r}}{dt}(t)\,dt$$

$$= \int_a^b \Big\{ \frac{\partial\varphi}{\partial x}(x(t),y(t),z(t))\frac{dx}{dt}(t) + \frac{\partial\varphi}{\partial y}(x(t),y(t),z(t))\frac{dy}{dt}(t)$$
$$+ \frac{\partial\varphi}{\partial z}(x(t),y(t),z(t))\frac{dz}{dt}(t)\Big\}\,dt$$

(chain rule)

$$= \int_a^b \frac{d}{dt}\big[\varphi(x(t),y(t),z(t))\big]\,dt$$

(FToC)

$$= \varphi(\vec{r}(b)) - \varphi(\vec{r}(a)) = \varphi(P_1) - \varphi(P_0) \quad \blacksquare$$

Consequence: If $\vec{F}$ is conservative, $\int_{\mathcal{C}} \vec{F}\cdot d\vec{r}$ takes the same value for all curves $\mathcal{C}$ that start at any fixed $P_0$ and end at any fixed $P_1$.

This called "path independence".