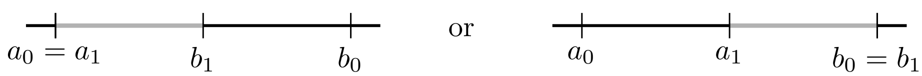
The Bolzano–Weierstrass Theorem

Theorem (The Bolzano–Weierstrass Theorem) *Every bounded sequence of real numbers has a convergent subsequence i.e. a subsequential limit.*

Proof: Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of real numbers with $|s_n| \le L$ for all $N \in \mathbb{N}$.

Step 1 (The Search Procedure):

- Set $a_0 = -L$ and $b_0 = L$. Note that $|b_0 - a_0| = 2L$.
- Divide the interval $[a_0, b_0]$ into two halves. At least one half must contain infinitely many s_n's. Pick one such half and call it $[a_1, b_1]$. Note that $|b_1 - a_1| = \frac{1}{2}|b_0 - a_0| = L$, and that $a_1 \in \{a_0, a_0 + \frac{b_0 - a_0}{2}\}$. So $a_1 \ge a_0$. There are infinitely many s_n's in $[a_1, b_1]$. Select one, say s_{i_1}.



- Divide the interval $[a_1, b_1]$ into two halves. At least one half must contain infinitely many s_n's. Pick one such half and call it $[a_2, b_2]$. Note that $|b_2 - a_2| = \frac{1}{2}|b_1 - a_1| = \frac{1}{2}L$, and that $a_2 \in \{a_1, a_1 + \frac{b_1 - a_1}{2}\}$. So $a_2 \ge a_1$. There are infinitely many s_n's in $[a_2, b_2]$. Select one with $n > i_1$, say s_{i_2}.
- Continue.

In this way we generate

$$\underbrace{[a_0, b_0]}_{\text{length } 2L} \supset \underbrace{\overbrace{[a_1, b_1]}^{s_{i_1}\in}}_{\text{length } L} \supset \underbrace{\overbrace{[a_2, b_2]}^{s_{i_2}\in}}_{\text{length } \frac{1}{2}L} \supset \cdots$$

Note that $|b_n - a_n| = \frac{1}{2}|b_{n-1} - a_{n-1}|$ for all $n \in \mathbb{N}$ so that $|b_n - a_n| = \frac{1}{2^n}|b_0 - a_0| = \frac{L}{2^{n-1}}$ for all $n \in \mathbb{N}$. Also $a_0 \le a_1 \le a_2 \le \cdots$.

Step 2 (Guess the Limit): The sequence $a_1, a_2, a_3, \cdots$ is monotone increasing and bounded above by $b_0 = L$. So it converges. Call the limit s.

Step 3 (Prove that $\lim\limits_{n\to\infty} s_n = s$): Let $\varepsilon > 0$. Since $(a_n)_{n\in\mathbb{N}}$ converges to s,

$$\exists\, N_1 \in \mathbb{N} \ \text{ such that } \quad \big|a_n - s\big| < \tfrac{\varepsilon}{2} \text{ whenever } n \ge N_1$$

Since s_{i_n} lies in the interval $[a_n, b_n]$, and the length of the interval $[a_n, b_n]$ is $\frac{L}{2^{n-1}}$, the distance from s_{i_n} to a_n is at most $\frac{L}{2^{n-1}}$, which converges to zero as $n \to \infty$,

$$\exists\, N_2 \in \mathbb{N} \ \text{ such that } \quad \big|s_{i_n} - a_n\big| < \tfrac{\varepsilon}{2} \text{ whenever } n \ge N_2$$

Choose $N = \max\{N_1, N_2\}$. Then

$$n \ge N \implies \big|s_{i_n} - s\big| \le \big|s_{i_n} - a_n\big| + \big|a_n - s\big| < \varepsilon$$

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