Fields

The Field Axioms and their Consequences

Definition 1 (The Field Axioms) A *field* is a set $\mathbb{F}$ with two operations, called *addition* and *multiplication* which satisfy the following axioms (A1–5), (M1–5) and (D).

(A) Axioms for addition

(A1) $x, y \in \mathbb{F} \implies x + y \in \mathbb{F}$
(A2) $x + y = y + x$ for all $x, y \in \mathbb{F}$ (addition is commutative)
(A3) $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{F}$ (addition is associative)
(A4) $\mathbb{F}$ contains an element 0 such that $0 + x = x$ for every $x \in \mathbb{F}$.
(A5) For each $x \in \mathbb{F}$ there is an element $-x \in \mathbb{F}$ such that $x + (-x) = 0$.

(M) Axioms for multiplication

(M1) $x, y \in \mathbb{F} \implies xy \in \mathbb{F}$
(M2) $xy = yx$ for all $x, y \in \mathbb{F}$ (multiplication is commutative)
(M3) $(xy)z = x(yz)$ for all $x, y, z \in \mathbb{F}$ (multiplication is associative)
(M4) $\mathbb{F}$ contains an element $1 \neq 0$ such that $1x = x$ for every $x \in \mathbb{F}$.
(M5) For each $0 \neq x \in \mathbb{F}$ there is an element $\frac{1}{x} \in \mathbb{F}$ such that $x\left(\frac{1}{x}\right) = 1$.

(D) The distributive law

(D) $x(y + z) = xy + xz$ for all $x, y, z \in \mathbb{F}$

Example 2 The rational numbers, $\mathbb{Q}$, real numbers, $\mathbb{R}$, and complex numbers, $\mathbb{C}$ are all fields. The natural numbers $\mathbb{N}$ is not a field — it violates axioms (A4), (A5) and (M5). The integers $\mathbb{Z}$ is not a field — it violates axiom (M5).

Theorem 3 (Consequences of the Field Axioms)

(A) The addition axioms imply

(a) $x + y = x + z \implies y = z$
(b) $x + y = x \implies y = 0$
(c) $x + y = 0 \implies y = -x$
(d) $-(-x) = x$

(M) The multiplication axioms imply

(a) $x \neq 0,\ xy = xz \implies y = z$

(b) $x \neq 0,\ xy = x \implies y = 1$

(c) $x \neq 0,\ xy = 1 \implies y = \frac{1}{x}$

(d) $x \neq 0,\ 1/(1/x) = x$

(F) The field axioms imply

(a) $0x = 0$

(b) $x \neq 0,\ y \neq 0 \implies xy \neq 0$

(c) $(-x)y = -(xy) = x(-y)$

(d) $(-x)(-y) = xy$

Selected Proofs:

(A.a):

$$\begin{aligned} x + y = x + z &\implies -x + (x + y) = -x + (x + z) \\ &\implies (-x + x) + y = (-x + x) + z && \text{(by axiom (A3))} \\ &\implies \big(x + (-x)\big) + y = \big(x + (-x)\big) + z && \text{(by axiom (A2))} \\ &\implies 0 + y = 0 + z && \text{(by axiom (A5))} \\ &\implies y = z && \text{(by axiom (A4))} \end{aligned}$$

(F.a):

$$\begin{aligned} 0 \cdot x &= (0 + 0)x && \text{(by axiom (A4))} \\ &= 0 \cdot x + 0 \cdot x && \text{(by axiom (D))} \\ \implies 0 \cdot x &= 0 && \text{(by part A.b, above)} \end{aligned}$$

(F.b):

$$\begin{aligned} x \neq 0,\ xy = 0 &\implies \tfrac{1}{x}(xy) = \tfrac{1}{x}0 = 0 && \text{(by part F.a and axiom (M2))} \\ &\implies y = 0 && \text{(by axioms (M3), (M2), (M5), (M4))} \end{aligned}$$

(F.c): As a preliminary computation we show that $(-1)x = -x$. By part (A.c), this follows from

$$x + (-1)x = 1 \cdot x + (-1)x = (1 + (-1))x = 0 \cdot x = 0$$

(by axioms (M4), (M2), (D), (A5) and part (F.a)). This implies

$$\begin{aligned} (-x)y &= \big((-1)x\big)y = (-1)(xy) = -(xy) \\ x(-y) &= x\big((-1)y\big) = (-1)(xy) = -(xy) \end{aligned}$$

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Ordered Fields

Definition 4 (Ordered Fields) An *ordered field* is a field $\mathbb{F}$ with a relation, denoted $<$, obeying the

(O) Order axioms

- (O1) For each pair $x, y \in \mathbb{F}$ precisely one of $x < y,\ x = y,\ y < x$ is true.
- (O2) $x < y,\ y < z \implies x < z$
- (O3) $y < z \implies x + y < x + z$
- (O4) $x > 0,\ y > 0 \implies xy > 0$

We also use the notations "$x > y$" for "$y < x$", and "$x \le y$" for "$x < y$ or $x = y$", and "$x \ge y$" for "$y < x$ or $x = y$".
An ordered set is a set with a relation $<$ obeying (O1) and (O2).

Example 5 $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields.

Theorem 6 (Consequences of the Order Axioms) *In every ordered field*
(a) $x > 0 \implies -x < 0$
(b) $x < 0 \implies -x > 0$
(c) $y < z,\ x > 0 \implies xy < xz$
(d) $y < z,\ x < 0 \implies xy > xz$
(e) $x \ne 0 \implies x^2 > 0$
(f) $1 > 0$
(g) $x > 0 \implies \frac{1}{x} > 0$
(h) $0 < x < y \implies 0 < \frac{1}{y} < \frac{1}{x}$

Proof:

(a) $x > 0 \overset{(O3)}{\implies} x + (-x) > 0 + (-x) \overset{(A4),(A5)}{\implies} 0 > -x$

(b) $x < 0 \overset{(O3)}{\implies} x + (-x) < 0 + (-x) \overset{(A4),(A5)}{\implies} 0 < -x$

(c) $y < z \overset{(O3),(A2,5)}{\implies} z - y > 0 \overset{(O4)}{\implies} x(z - y) > 0 \overset{(D)}{\implies} xz - xy > 0 \overset{(O3),(A2,4,5)}{\implies} xz > xy$

(d) $x < 0,\ z - y > 0 \overset{(c)}{\implies} (z - y)x < \overbrace{(z - y)(0)}^{0} \overset{(D)}{\implies} xz - xy < 0 \overset{(O3),(A4),(A5)}{\implies} xy > xz$

(e) $x > 0 \overset{(O4)}{\implies} x^2 > 0$ and $x < 0 \overset{(b)}{\implies} -x > 0 \overset{(O4)}{\implies} x^2 = (x)^2 > 0$

(f) $1 = 1^2 \overset{(e)}{\implies} 1 > 0$

(g) Let $x > 0$. If $\frac{1}{x} < 0$, then $-\frac{1}{x} > 0 \stackrel{(O4)}{\Longrightarrow} \big(-\frac{1}{x}\big)x > 0 \implies -1 > 0 \stackrel{(a)}{\Longrightarrow} 1 < 0$ which contradicts part (e).

(h) Multiply both sides of $x < y$ by $\frac{1}{x}\frac{1}{y} > 0$. This gives, by part (c), $\frac{1}{y} < \frac{1}{x}$. Part (f) gives $\frac{1}{y} > 0$.

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Theorem 7 ($\mathbb{C}$ cannot be ordered.) *There does not exist a relation $<$ making $\mathbb{C}$ into an ordered field.*

Proof: Homework. ■