$\mathbb{Q}$ Does Not Obey the Least Upper Bound Axiom

Recall that a field $\mathbb{F}$ obeys the least upper bound axiom if every subset $E \subset \mathbb{F}$, that is nonempty and bounded above, has a least upper bound in $\mathbb{F}$. In these notes we prove that the set of rational numbers, $\mathbb{Q}$, does not obey the least upper bound axiom.

We do so by studying the subset

$$E = \left\{ \ q \in \mathbb{Q} \ \middle| \ q^2 < 2 \ \right\} \subset \mathbb{Q}$$

Now

- E is nonempty, since $0 \in E$,
- and every number in E is smaller than 2 (for example) since if $q \geq 2$, then $q^2 \geq 4 > 2$ so that $q \notin E$. So E has an upper bound, namely 2.

So if $\mathbb{Q}$ were to obey the least upper bound axiom, E would have to have a least upper bound in $\mathbb{Q}$. We prove that this is in fact not the case.

It is very reasonable to guess that E does not have a least upper bound in $\mathbb{Q}$, because it is reasonable to guess that the supremum of E would have to be $\sqrt{2}$, which is not a rational number. (We'll prove this shortly.) To prove in detail that E has no least upper bound in $\mathbb{Q}$, we'll argue by contradiction. We will

- assume that E does have a least upper bound in $\mathbb{Q}$, call it α, and then
- prove that $\alpha^2 = 2$, and then
- prove that there is no rational number α that obeys $\alpha^2 = 2$.

Now we start the proof. Assume that E has a least upper bound in $\mathbb{Q}$ and call it α. Note that $1 \leq \alpha \leq 2$, because $1 \in E$ and 2 is an upper bound for E. To prove that $\alpha^2 = 2$, we prove that it is impossible to have $\alpha^2 < 2$ (that will be Step 1) and that it is also impossible to have $\alpha^2 > 2$ (that will be Step 2).

Step 1 We first prove that if $\alpha^2 < 2$, then α cannot be an upper bound for E at all and, in particular cannot be the least upper bound. To do so, it suffices to construct a rational number, q, which is in E, but which is strictly bigger than α. To ensure that $q > \alpha$ we choose q to be of the form $\alpha + \varepsilon$ with ε a strictly positive rational number. For q to be in E, we need to choose ε small enough that

$$q^2 = (\alpha + \varepsilon)^2 = \alpha^2 + 2\alpha\varepsilon + \varepsilon^2$$

is smaller than 2. We already know that $\alpha \leq 2$ and we may always choose ε to be smaller than, for example, 1. Then

$$q^2 \leq \alpha^2 + 4\varepsilon + \varepsilon = \alpha^2 + 5\varepsilon$$

This is strictly smaller than 2 if

$$\alpha^2 + 5\varepsilon < 2 \iff \varepsilon < \frac{2-\alpha^2}{5}$$

So the number $q = \alpha + \frac{2-\alpha^2}{6}$ is a rational number in E that is strictly larger[(1)] than α. So α is not an upper bound for E.

Step 2 We next prove that if $\alpha^2 > 2$, then α cannot be the least upper bound for E. To do so it suffices to construct another rational number, $\tilde{q}$, which is an upper bound for E and that is strictly smaller than α. To ensure that $\tilde{q} < \alpha$ we choose $\tilde{q}$ to be of the form $\alpha - \varepsilon$ with ε a strictly positive rational number. We choose ε small enough that

$$\tilde{q}^2 = (\alpha - \varepsilon)^2 = \alpha^2 - 2\alpha\varepsilon + \varepsilon^2$$

is larger than 2. As $\alpha \le 2$, we have $-2\alpha\varepsilon \ge -4\varepsilon$ so that

$$\tilde{q}^2 \ge \alpha^2 - 4\varepsilon + \varepsilon^2 > \alpha^2 - 4\varepsilon$$

This is strictly larger than 2 if

$$\alpha^2 - 4\varepsilon > 2 \iff \varepsilon < \frac{\alpha^2 - 2}{4}$$

So the number $\tilde{q} = \alpha - \frac{\alpha^2-2}{6}$ is a rational number that is strictly smaller than α, is strictly larger than 0 (since $\frac{\alpha^2-2}{6} \le \frac{2^2-2}{6} = \frac{1}{3} < 1 \le \alpha$) and obeys $\tilde{q}^2 > 2$. To see that $\tilde{q}$ is an upper bound for E, observe that

$$q \ge \tilde{q} \ge 0 \implies q^2 \ge \tilde{q}^2 > 2$$

This shows that any number larger than $\tilde{q}$ cannot be in E.

This completes step 2. By this stage we know that if α is a least upper bound for E, then α must obey $\alpha^2 = 2$.

Step 3 We finally show that there is no rational number α that obeys $\alpha^2 = 2$. By definition, any rational number α has to be of the form $\frac{m}{n}$ with m an integer, n a natural number and with m and n having no common factors. But, if $\alpha^2 = 2$, this implies that

$$\begin{aligned} \left(\tfrac{m}{n}\right)^2 = 2 \quad &\implies m^2 = 2n^2 \quad \implies 2 \text{ is a factor of } m, \text{ i.e. } \tfrac{m}{2} \in \mathbb{Z} \\ \implies 4\left(\tfrac{m}{2}\right)^2 = 2n^2 &\implies 2\left(\tfrac{m}{2}\right)^2 = n^2 \implies 2 \text{ is a factor of } n \end{aligned}$$

So 2 is a common factor for m and n, which is a contradiction.

(1) We have chosen $\varepsilon = \frac{2-\alpha^2}{6}$. Note that $\varepsilon = \frac{2-\alpha^2}{6} < \frac{2}{6} < 1$, as desired.