Dini's Theorem

Theorem (Dini's Theorem) *Let K be a compact metric space. Let $f : K \to \mathbb{R}$ be a continuous function and $f_n : K \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of continuous functions. If $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to f and if*

$$f_n(x) \ge f_{n+1}(x) \qquad \text{for all } x \in K \text{ and all } n \in \mathbb{N}$$

then $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f.

Proof: Set, for each $n \in \mathbb{N}$, $g_n(x) = f_n(x) - f(x)$. Then $\{g_n\}_{n\in\mathbb{N}}$ is a sequence of continuous functions on the compact metric space K that converges pointwise to 0. Furthermore $g_n(x) \ge g_{n+1}(x) \ge 0$ for all $x \in K$ and $n \in \mathbb{N}$. Set $M_n = \sup\{g_n(x) \,\big|\, x \in K\}$. We must prove that $\lim_{n\to\infty} M_n = 0$.

We use the finite open subcover property of K. Let $\varepsilon > 0$ and define $\mathcal{O}_n = g_n^{-1}\big((-\infty, \varepsilon)\big)$. Since g_n is continuous, this is an open set. Since $g_n(x) \ge g_{n+1}(x)$, $\mathcal{O}_n \subset \mathcal{O}_{n+1}$. For each $x \in K$, $\lim_{n\to\infty} g_n(x) = 0$ so that there is an $n \in \mathbb{N}$ with $g_n(x) < \varepsilon$ so that $x \in \mathcal{O}_n$. Thus $\bigcup_{n=1}^\infty \mathcal{O}_n = K$. Since K is compact, there is a finite collection of $\mathcal{O}_n$'s that also covers K. Since $\mathcal{O}_n \subset \mathcal{O}_{n+1}$, the $\mathcal{O}_n$ in that finite collection with the largest index covers K. Thus there is an $N \in \mathbb{N}$ with $\mathcal{O}_N = K$. That is $g_N(x) < \varepsilon$ for all $x \in K$. Thus $M_N \le \varepsilon$. Since M_n decreases with n and every $M_n \ge 0$, this forces $\lim_{n\to\infty} M_n = 0$. ■

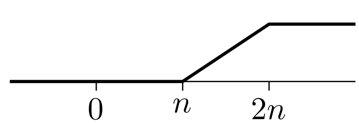
Here are examples that show that the hypotheses
(a) K is compact
(b) f is continuous
(c) $f_n(x)$ decreases as n increases
are each necessary.

Example (a) The set $K = (0, 1)$ is not compact. As $n \to \infty$, the sequence $f_n(x) = x^n$ of continuous functions decreases pointwise to zero. But the convergence is not uniform because

$$\sup\big\{\, x^n \,\big|\, 0 < x < 1 \,\big\} = 1$$

for all $n \in \mathbb{N}$.

Example (a') The set $K = \mathbb{R}$ is not compact. As $n \to \infty$, the sequence

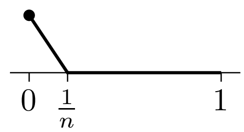
$f_n(x) =$ 

(the right hand plateau is at height 1) of continuous functions decreases pointwise to zero. But the convergence is not uniform because

$$\sup\left\{\ f_n(x)\ \middle|\ x\in\mathbb{R}\ \right\}=1$$

for all $n\in\mathbb{N}$.

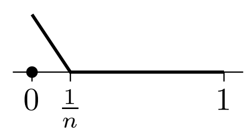
Example (b) The set $K=[0,1]$ is compact. As $n\to\infty$, the sequence

$f_n(x) =$ 

of continuous functions (the dot is at height 1) decreases pointwise to the discontinuous function

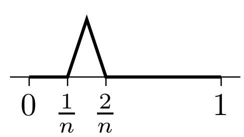
$f(x) =$ 

But the convergence is not uniform because

$\sup\left\{\ f_n(x)-f(x)\ \middle|\ x\in[0,1]\ \right\}=\sup$  $= 1$

for all $n\in\mathbb{N}$.

Example (c) The set $K=[0,1]$ is compact. As $n\to\infty$, the sequence

$f_n(x) =$ 

of continuous functions (the triangle has height 1) converges pointwise to zero. But the convergence is not uniform because

$$\sup\left\{\ f_n(x)\ \middle|\ x\in\mathbb{R}\ \right\}=1$$

for all $n\in\mathbb{N}$.

