Lecture Notes on Measure Theory and Integration
5 — Product Measures and the Fubini-Tonelli Theorem

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5 Product Measures and Fubini-Tonelli

Our goal in this section is to prove the Fubini-Tonelli theorem[1], which says that, under appropriate hypotheses,

$$\begin{aligned}\int_{X\times Y} f(x,y)\ d\mu\times\nu(x,y) &= \int_X \Bigg[\overbrace{\int_Y f(x,y)\,d\nu(y)}^{g(x)}\Bigg] d\mu(x)\\ &= \int_Y \Bigg[\underbrace{\int_X f(x,y)\,d\mu(x)}_{h(y)}\Bigg] d\nu(y)\end{aligned}$$

5.1 Product Measures

Let $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be measure spaces. Our first task is to define the product.

Definition 5.1 (Product Measure)**.** Let $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be measure spaces.

(a) Define the set of finite disjoint unions of measurable rectangles in $X\times Y$ to be

$$\mathcal{R}=\Big\{\ \textstyle\bigcup_{j=1}^n A_j\times B_j\ \Big|\ n\in\mathbb{N},\ A_j\in\mathcal{M},\ B_j\in\mathcal{N},\ (A_j\times B_j)\cap(A_k\times B_k)=\emptyset,\ \text{for all } 1\le j,k\le n \text{ with } j\ne k\ \Big\}$$

$\mathcal{R}$ is nonempty. We will shortly show that it is closed under complements and finite unions and so is an algebra.

(b) Define $\mathcal{M}\otimes\mathcal{N}=\mathcal{M}(\mathcal{R})$ to be the σ-algebra generated by $\mathcal{R}$.

(c) Define $\pi:\mathcal{R}\to[0,\infty]$ by

$$\pi\Big(\textstyle\bigcup_{j=1}^n A_j\times B_j\Big)=\displaystyle\sum_{j=1}^n \mu(A_j)\nu(B_j)$$

for all $n\in\mathbb{N}$ and all $A_j\in\mathcal{M},\ B_j\in\mathcal{N},\ 1\le j\le n$ with $(A_j\times B_j)\cap(A_k\times B_k)=\emptyset$ for all $j\ne k$. In this definition, we use the convention that $0\times\infty=0$. We will shortly show that π is a well-defined premeasure.

[1] The special case of this theorem, for continuous functions on rectangles, was known to Euler in the 18th century. Lebesgue extended this to bounded measurable functions in 1904. Fubini's version was 1907, and Tonelli's version was 1909.

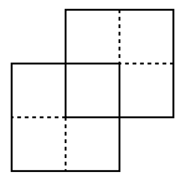
(d) Let π^* be the outer measure generated by π. By Theorem 2.36,

$$\mu \times \nu = \pi^* \restriction \mathcal{M} \otimes \mathcal{N}$$

is a measure which extends π. We will shortly show that if μ and ν are σ-finite, then $\mu \times \nu$ is σ-finite. Then it is the unique measure on $\mathcal{M} \otimes \mathcal{N}$ such that

$$\mu \times \nu(A \times B) = \mu(A)\, \nu(B) \ \forall\ A \in \mathcal{M},\ B \in \mathcal{N}$$

Remark 5.2. (a) Any finite union of measurable rectangles can also be expressed as a finite disjoint union of measurable rectangles.



So

$$\mathcal{R} = \left\{ \textstyle\bigcup_{j=1}^{n} A_j \times B_j \ \middle|\ n \in \mathbb{N},\ A_j \in \mathcal{M},\ B_j \in \mathcal{N},\ \text{ for all } 1 \le j, k \le n \right\}$$

too. As $(A \times B)^c = \big(X \times B^c\big) \cup \big(A^c \times B\big)$, $\mathcal{R}$ written in this form is obviously an algebra.

(b) That π is a well–defined premeasure (in part (c) of the definition) is a consequence of the observation that, if

$$\bigcup_{j=1}^{\infty} A_j \times B_j = \bigcup_{k=1}^{n} \tilde{A}_k \times \tilde{B}_k$$

are disjoint unions of measurable rectangles, then

$$\begin{aligned} \sum_{k=1}^{n} \chi_{\tilde{A}_k}(x)\chi_{\tilde{B}_k}(y) &= \sum_{k=1}^{n} \chi_{\tilde{A}_k \times \tilde{B}_k}(x,y) = \chi_{\cup_k \tilde{A}_k \times \tilde{B}_k}(x,y) = \chi_{\cup_j A_j \times B_j}(x,y) \\ &= \sum_{j=1}^{\infty} \chi_{A_j \times B_j}(x,y) = \sum_{j=1}^{\infty} \chi_{A_j}(x)\, \chi_{B_j}(y) \end{aligned}$$

So integrating $d\mu(x)$ gives

$$\sum_{k=1}^{n} \mu(\tilde{A}_k)\, \chi_{\tilde{B}_k}(y) = \sum_{j=1}^{\infty} \mu(A_j)\, \chi_{B_j}(y)$$

by the monotone convergence theorem. Then integrating $d\nu(y)$ gives

$$\underbrace{\sum_{k=1}^{n} \mu(\tilde{A}_k)\,\nu(\tilde{B}_k)}_{\pi\left(\bigcup_{k=1}^{n} \tilde{A}_k \times \tilde{B}_k\right)} = \sum_{j=1}^{\infty} \underbrace{\mu(A_j)\nu(B_j)}_{\pi(A_j \times B_j)}$$

again by the monotone convergence theorem.

(c) For the σ-finite statement in part (d) of the definition, observe that if

$$\big\{A_j\big\}_{j\in\mathbb{N}} \subset \mathcal{M},\ \big\{B_k\big\}_{k\in\mathbb{N}} \subset \mathcal{N},\ X = \bigcup_{j\in\mathbb{N}} A_j,\ Y = \bigcup_{k\in\mathbb{N}} B_k,\ \mu(A_j) < \infty,\ \nu(B_k) < \infty$$

then

$$\big\{A_j \times B_k\big\}_{j,k\in\mathbb{N}} \subset \mathcal{M} \otimes \mathcal{N},\ X \times Y = \bigcup_{j,k\in\mathbb{N}} A_j \times B_k$$

and

$$\mu \times \nu(A_j \times B_k) = \mu(A_j)\,\nu(B_k) < \infty$$

Proposition 5.3 (Tensor product of Borel σ-algebras)**.** *Let X and Y be separable*[2] *metric spaces with metrics d_X and d_Y respectively. Then $X \times Y$ is a metric space with metric $D\big((x,y),(x',y')\big) = \sqrt{d_X(x,x')^2 + d_Y(y,y')^2}$ and $\mathcal{B}_{X\times Y} = \mathcal{B}_X \otimes \mathcal{B}_Y$. (By definition, a metric space is separable if and only if it contains a countable dense subset. For example, $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$, so that $\mathbb{R}$ is separable. Applying this proposition to $X = Y = \mathbb{R}$ gives that $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^2}$.)*

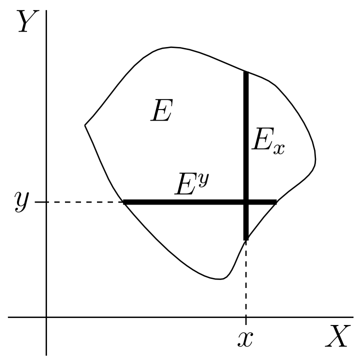
Proof. Problem Set 9, #5. □

Proposition 5.4 (Slices — sets)**.** *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces, $x \in X$ and $y \in Y$.*

(a) If $E \in \mathcal{M} \otimes \mathcal{N}$, then

$$E_x = \big\{\ y' \in Y \ \big|\ (x,y') \in E\ \big\} \in \mathcal{N}$$
$$E^y = \big\{\ x' \in X \ \big|\ (x',y) \in E\ \big\} \in \mathcal{M}$$

[2] For a counterexample in the nonseparable case, see Exercise 29 on page 231 of Folland.



(b) If $f : X \times Y \to \mathbb{R}$ is $\mathcal{M} \otimes \mathcal{N}$-measurable, then the function $f_x : Y \to \mathbb{R}$ defined by $f_x(y) = f(x, y)$ is $\mathcal{N}$-measurable and the function $f^y : X \to \mathbb{R}$ defined by $f^y(x) = f(x, y)$ is $\mathcal{M}$-measurable.

Proof. (a) Let

$$\mathcal{P} = \left\{ \ E \subset X \times Y \ \middle| \ E_x \in \mathcal{N} \text{ for all } x \in X, \ E^y \in \mathcal{M} \text{ for all } y \in Y \ \right\}$$

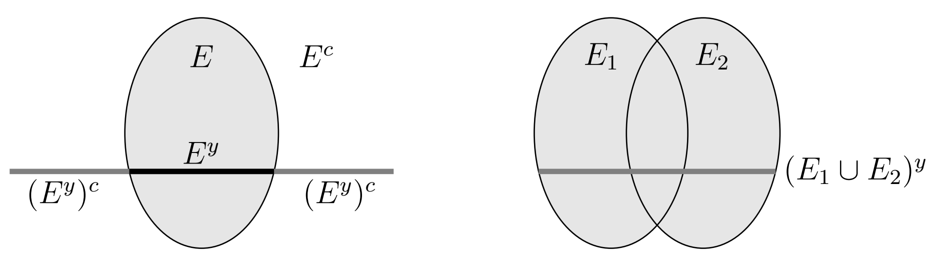
Then

- $A \times B \in \mathcal{P}$ for all $A \in \mathcal{M}$ and $B \in \mathcal{N}$, since

$$(A \times B)_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases} \qquad\qquad (A \times B)^y = \begin{cases} A & \text{if } y \in B \\ \emptyset & \text{if } y \notin B \end{cases}$$

- $\mathcal{P}$ is closed under complements since, if $E \in \mathcal{P}$, then

$$\big(E^c\big)_x = \big(\overbrace{E_x}^{\in \mathcal{N}}\big)^c \in \mathcal{N}$$
$$\big(E^c\big)^y = \big(\overbrace{E^y}^{\in \mathcal{M}}\big)^c \in \mathcal{M}$$



- $\mathcal{P}$ is closed under countable unions since, if $E_n \in \mathcal{P}$ for all $n \in \mathbb{N}$, then

$$\left(\bigcup_n E_n\right)_x = \bigcup_n \overbrace{(E_n)_x}^{\in\mathcal{N}} \in \mathcal{N}$$

$$\left(\bigcup_n E_n\right)^y = \bigcup_n \overbrace{(E_n)^y}^{\in\mathcal{M}} \in \mathcal{M}$$

So $\mathcal{P}$ is a σ-algebra which contains $\mathcal{R}$, and hence contains $\mathcal{M}(\mathcal{R}) = \mathcal{M} \otimes \mathcal{N}$.

(b) Let $B \in \mathcal{B}_{\mathbb{R}}$. As f is $\mathcal{M} \otimes \mathcal{N}$-measurable, $f^{-1}(B) \in \mathcal{M} \otimes \mathcal{N}$. So

$$f_x^{-1}(B) = \left\{\ y \in Y\ \middle|\ f_x(y) = f(x,y) \in B\ \right\} = \underbrace{\overbrace{f^{-1}(B)}^{\in\mathcal{M}\otimes\mathcal{N}}{}_x}_{\in\mathcal{N}}$$

$$\left(f^y\right)^{-1}(B) = \left\{\ x \in X\ \middle|\ f^y(x) = f(x,y) \in B\ \right\} = \underbrace{\overbrace{f^{-1}(B)}^{\in\mathcal{M}\otimes\mathcal{N}}{}^y}_{\in\mathcal{M}}$$

$\square$

5.2 Technical Aside — Monotone Classes

Definition 5.5. Let X be a nonempty set. A collection $\mathcal{C} \subset \mathcal{P}(X)$ of subsets of X is called a **monotone class** if

- it is closed under countable increasing unions (that is, if $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{C}$ and $E_1 \subset E_2 \subset E_3 \subset \cdots$, then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{C}$) and
- it is closed under countable decreasing intersections (that is, if $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{C}$ and $E_1 \supset E_2 \supset E_3 \supset \cdots$, then $\bigcap_{n=1}^{\infty} E_n \in \mathcal{C}$).

Remark 5.6.

(a) Monotone classes are closely related to σ–algebras. In fact, for us, their only use will be to help verify that a certain collection of subsets is a σ–algebra.

(b) Every σ–algebra is a monotone class, because σ–algebras are closed under arbitrary countable unions and intersections.

(c) If, for every index i in some index set $\mathcal{I}$, $\mathcal{C}_i$ is a monotone class, then $\bigcap_{i\in\mathcal{I}} \mathcal{C}_i$ is also a monotone class. In particular, for any $\mathcal{E} \subset \mathcal{P}(X)$, the collection

$$\mathcal{C}(\mathcal{E}) = \bigcap_{\substack{\mathcal{C}\ \text{monotone class} \\ \mathcal{E}\subset\mathcal{C}}} \mathcal{C}$$

is a monotone class, called the monotone class generated by $\mathcal{E}$. It is the smallest monotone class that contains $\mathcal{E}$. So if $\mathcal{C}$ is any monotone class that contains $\mathcal{E}$, then $\mathcal{C}(\mathcal{E}) \subset \mathcal{C}$.

Lemma 5.7. *Let X be a nonempty set. If $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra, then*

$$\mathcal{C}(\mathcal{A}) = \mathcal{M}(\mathcal{A})$$

That is, the monotone class generated by $\mathcal{A}$ is the same as the σ–algebra generated by $\mathcal{A}$.

Proof.

- $\mathcal{C}(\mathcal{A}) \subset \mathcal{M}(\mathcal{A})$:

  $\mathcal{M}(\mathcal{A})$ is a σ-algebra, and hence a monotone class, that contains $\mathcal{A}$. So, this follows by part (c) of Remark 5.6.

- $\mathcal{M}(\mathcal{A}) \subset \mathcal{C}(\mathcal{A})$:

  It suffices to prove that $\mathcal{C}(\mathcal{A})$ is a σ–algebra, because then we will know that $\mathcal{C}(\mathcal{A})$ is a σ–algebra containing $\mathcal{A}$ and hence $\mathcal{M}(\mathcal{A})$, which is the smallest σ–algebra containing $\mathcal{A}$.

  By Problem Set 1, # 6, any algebra that is closed under countable increasing unions is a σ–algebra. So it suffices to prove that $\mathcal{C}(\mathcal{A})$ is an algebra (i.e. that $\mathcal{C}(\mathcal{A})$ is nonempty and closed under complements and finite intersections). So it suffices to prove

  $$E, F \in \mathcal{C}(\mathcal{A}) \implies E \setminus F,\ F \setminus E,\ E \cap F \in \mathcal{C}(\mathcal{A}) \tag{$*$}$$

  (since X is automatically in $\mathcal{A}$, which is an algebra, and hence is automatically in $\mathcal{C}(\mathcal{A})$ and is an allowed choice for E). Define, for each $E \in \mathcal{C}(\mathcal{A})$,

  $$\mathcal{D}(E) = \left\{ F \in \mathcal{C}(\mathcal{A}) \;\middle|\; E \setminus F, F \setminus E, E \cap F \in \mathcal{C}(\mathcal{A}) \right\}$$

  We wish to show that

  $$E \in \mathcal{C}(A) \implies \mathcal{C}(\mathcal{A}) \subset \mathcal{D}(E)$$

  To do so, it suffices to show that $\mathcal{D}(E)$ is a monotone class that contains $\mathcal{A}$. We first prove some properties of $\mathcal{D}(E)$.

  (a) $E \in \mathcal{C}(A) \implies \emptyset, E \in \mathcal{D}(E)$.

  (b) For $E, F \in \mathcal{C}(\mathcal{A})$,

  $$\begin{aligned} F \in \mathcal{D}(E) &\iff E \setminus F, F \setminus E, E \cap F \in \mathcal{C}(\mathcal{A}) \\ &\iff E \in \mathcal{D}(F) = \left\{ F' \in \mathcal{C}(\mathcal{A}) \;\middle|\; F \setminus F', F' \setminus F, F \cap F' \in \mathcal{C}(\mathcal{A}) \right\} \end{aligned}$$

(c) $\mathcal{D}(E)$ is closed under countable increasing unions. To see this, let the sequence $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(E)$ obey $F_1 \subset F_2 \subset F_3 \subset \cdots$ and set $F = \bigcup_{n=1}^\infty F_n$. Then

- $\big\{E \setminus F_n = E \cap F_n^c\big\}_{n\in\mathbb{N}} \subset \mathcal{C}(\mathcal{A})$ is decreasing,
- $\big\{F_n \setminus E = F_n \cap E^c\big\}_{n\in\mathbb{N}} \subset \mathcal{C}(\mathcal{A})$ is increasing and
- $\big\{E \cap F_n\big\}_{n\in\mathbb{N}} \subset \mathcal{C}(\mathcal{A})$ is increasing,

so that

$$\begin{aligned}
E \setminus F &= E \cap \Big(\textstyle\bigcup_{n=1}^\infty F_n\Big)^c = E \cap \Big(\textstyle\bigcap_{n=1}^\infty F_n^c\Big) &&= \bigcap_{n=1}^\infty (E \cap F_n^c) \in \mathcal{C}(\mathcal{A})\\
F \setminus E &= \Big(\textstyle\bigcup_{n=1}^\infty F_n\Big) \cap E^c = \bigcup_{n=1}^\infty (F_n \cap E^c) &&= \bigcup_{n=1}^\infty (F_n \setminus E) \in \mathcal{C}(\mathcal{A})\\
E \cap F &= E \cap \Big(\textstyle\bigcup_{n=1}^\infty F_n\Big) &&= \bigcup_{n=1}^\infty (E \cap F_n) \in \mathcal{C}(\mathcal{A})
\end{aligned}$$

since $\mathcal{C}(\mathcal{A})$ is closed under countable decreasing intersections and countable increasing unions. So $F \in \mathcal{D}(E)$.

(d) $\mathcal{D}(E)$ is closed under countable decreasing intersections. To see this, let $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(E)$ obey $F_1 \supset F_2 \supset F_3 \supset \cdots$ and set $F = \bigcap_{n=1}^\infty F_n$. As in part (c)

$$\begin{aligned}
E \setminus F &= E \cap \Big(\textstyle\bigcap_{n=1}^\infty F_n\Big)^c = E \cap \Big(\textstyle\bigcup_{n=1}^\infty F_n^c\Big) &&= \bigcup_{n=1}^\infty (E \cap F_n^c) \in \mathcal{C}(\mathcal{A})\\
F \setminus E &= \Big(\textstyle\bigcap_{n=1}^\infty F_n\Big) \cap E^c = \bigcap_{n=1}^\infty (F_n \cap E^c) &&= \bigcap_{n=1}^\infty (F_n \setminus E) \in \mathcal{C}(\mathcal{A})\\
E \cap F &= E \cap \Big(\textstyle\bigcap_{n=1}^\infty F_n\Big) &&= \bigcap_{n=1}^\infty (E \cap F_n) \in \mathcal{C}(\mathcal{A})
\end{aligned}$$

So $F \in \mathcal{D}(E)$.

We are now ready to prove $(*)$, or equivalently, that $\mathcal{C}(\mathcal{A}) \subset \mathcal{D}(E)$ for all $E \in \mathcal{C}(\mathcal{A})$. Let $E \in \mathcal{C}(\mathcal{A})$. By properties (c) and (d), $\mathcal{D}(E)$ is a monotone class, so it suffices to prove that $\mathcal{A} \subset \mathcal{D}(E)$. But

$$\begin{aligned}
F \in \mathcal{A} &\implies \mathcal{A} \subset \mathcal{D}(F) && \text{by the definition of } \mathcal{D}(F)\text{, since } \mathcal{A} \text{ is an algebra} \\
&\implies \mathcal{C}(\mathcal{A}) \subset \mathcal{D}(F) && \text{since } \mathcal{D}(\mathcal{F}) \text{ is a monotone class} \\
&\implies E \in \mathcal{D}(F) && \text{since } E \in \mathcal{C}(\mathcal{A}) \\
&\implies F \in \mathcal{D}(E) && \text{by property (b)}
\end{aligned}$$

□

5.3 Fubini-Tonelli

Proposition 5.8 (Slices — measure)**.** *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces and let $E \in \mathcal{M} \otimes \mathcal{N}$. Then the function $f : X \to [0, \infty]$ defined by $f(x) = \nu(E_x)$ is $\mathcal{M}$-measurable and the function $g : Y \to [0, \infty]$ defined by $g(y) = \mu(E^y)$ is $\mathcal{N}$-measurable and*

$$\mu \times \nu(E) = \int \nu(E_x)\ d\mu(x) = \int \mu(E^y)\ d\nu(y)$$

Proof.
Case 1: μ, ν finite:
Set

$$\mathcal{C} = \left\{\ E \in \mathcal{M} \otimes \mathcal{N} \ \middle|\ \text{conclusions of the theorem are true}\ \right\}$$

We will show that

$\mathcal{C}$ is a monotone class that contains $\mathcal{R}$ (the set of finite unions of measurable rectangles)

which will then imply that

$$\mathcal{M} \otimes \mathcal{N} = \mathcal{M}(\mathcal{R}) = \mathcal{C}(\mathcal{R}) \subset \mathcal{C}$$

We have

- $A \in \mathcal{M},\ B \in \mathcal{N} \implies A \times B \in \mathcal{C}$ since

$$\nu\big((A \times B)_x\big) = \nu\left(\begin{Bmatrix} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{Bmatrix}\right) = \chi_A(x)\,\nu(B)$$

$$\mu\big((A \times B)^y\big) = \chi_B(y)\,\mu(A)$$

implies

$$\int \nu\big((A\times B)_x\big)\,d\mu(x) = \mu(A)\,\nu(B) = \mu\times\nu(A\times B)$$
$$\int \mu\big((A\times B)^y\big)\,d\nu(x) = \mu(A)\,\nu(B) = \mu\times\nu(A\times B)$$

- $\mathcal{C}$ is closed under finite disjoint unions since

$$\nu\big((\underbrace{E\cup F}_{disjoint})_x\big) = \nu(\underbrace{E_x\cup F_x}_{disjoint}) = \nu(E_x)+\nu(F_x)$$

  Similarly for $\mu\big((E\cup F)^y\big)$. So $\mathcal{R}\subset\mathcal{C}$.
- $\mathcal{C}$ is closed under countable increasing unions:

  If $E_1\subset E_2\subset E_3\subset\cdots$ are all in $\mathcal{C}$ and $E=\bigcup_{n\in\mathbb{N}}E_n$ then

$$\big\{f_n(x)=\nu\big((E_n)_x\big)\big\}_{n\in\mathbb{N}} \text{ increases pointwise to } f(x)=\nu\big(E_x\big)$$

  by continuity from below, so f is measurable and by the monotone convergence theorem

$$\int \overbrace{\nu\big(E_x\big)}^{f(x)}\,d\mu(x) = \lim_{n\to\infty}\int \overbrace{\nu\big((E_n)_x\big)}^{f_n(x)}\,d\mu(x) \overset{E_n\in\mathcal{C}}{=} \lim_{n\to\infty}\mu\times\nu(E_n) = \mu\times\nu(E)$$

  by continuity from below. Similarly for $\int\mu\big(E^y\big)\,d\nu(y)$.
- $\mathcal{C}$ is closed under countable decreasing intersections:

  If $E_1\supset E_2\supset E_3\supset\cdots$ are all in $\mathcal{C}$ and $E=\bigcap_{n\in\mathbb{N}}E_n$ then

$$\big\{f_n(x)=\nu\big((E_n)_x\big)\big\}_{n\in\mathbb{N}} \text{ decreases pointwise to } f(x)=\nu\big(E_x\big)$$

  by continuity from above. (We used $\nu\big((E_1)_x\big)\le\nu(Y)<\infty$ here.) So f is measurable and $0\le f_n(x)\le f_1(x)\in L^1(X,\mathcal{M},\mu)$ and, by the dominated convergence theorem,

$$\begin{aligned}\int \nu\big(E_x\big)\,d\mu(x) &= \int f(x)\,d\mu(x) = \lim_{n\to\infty}\int f_n(x)\,d\mu(x)\\ &= \lim_{n\to\infty}\int \nu\big((E_n)_x\big)\,d\mu(x) \overset{E_n\in\mathcal{C}}{=} \lim_{n\to\infty}\mu\times\nu(E_n)\\ &= \mu\times\nu(E)\end{aligned}$$

  by continuity from above. Similarly for $\int\mu\big(E^y\big)\,d\nu(y)$.

So $\mathcal{C}$ is a monotone class that contains the algebra $\mathcal{R}$ and hence contains $\mathcal{C}(\mathcal{R})=\mathcal{M}(\mathcal{R})=\mathcal{M}\otimes\mathcal{N}$.

Case 2: μ, ν σ-finite:

We can write $X \times Y$ as a countable increasing union of rectangles $\{X_n \times Y_n\}_{n\in\mathbb{N}}$ of finite measure. Then, if $E \in \mathcal{M} \otimes \mathcal{N}$, we can apply the previous case to $E \cap (X_n \times Y_n)$.

$$\begin{aligned}\mu \times \nu\big(E \cap (X_n \times Y_n)\big) &= \int \mu\Big(\big(E \cap (X_n \times Y_n)\big)^y \Big)\, d\nu(y) \\ &= \int \mu\big(E^y \cap X_n\big)\, \chi_{Y_n}(y)\, d\nu(y)\end{aligned}$$

In the limit as $n \to \infty$,

- the left hand side converges to $\mu \times \nu(E)$ by continuity from below and
- the right had side converges to $\int \mu(E^y)\, d\nu(y)$ by the monotone covergence theorem.

□

THEOREM 5.9 (Fubini[3]-Tonelli[4]). *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces.*

(a) (Tonelli) If the function $f : X \times Y \to [0, \infty]$ is $\mathcal{M} \otimes \mathcal{N}$-measurable, then

- *the function $g : X \to [0, \infty]$ defined by $g(x) = \int f(x, y)\ d\nu(y)$ is $\mathcal{M}$-measurable, and*
- *the function $h : Y \to [0, \infty]$ defined by $h(y) = \int f(x, y)\ d\mu(x)$ is $\mathcal{N}$-measurable*

and

$$\begin{aligned}\int f(x,y)\ d\mu \times \nu(x,y) &= \int \left[\int f(x,y)\ d\nu(y) \right] d\mu(x) \\ &= \int \left[\int f(x,y)\ d\mu(x) \right] d\nu(y)\end{aligned}$$

(b) (Fubini) If $f \in L^1(\mu \times \nu)$ then

- *the function $f_x : Y \to \mathbb{R}$ defined by $f_x(y) = f(x, y)$ is in $L^1(\nu)$ for almost all $x \in X$,*
- *$g(x) = \int f(x, y)\ d\nu(y) \in L^1(\mu)$*

[3]Guido Fubini (1879–1943) was an Italian mathematician. By way of comparison, Henri Lebesgue (1875–1941) was a French mathematician and Bernhard Riemann (1826–1866) was a German mathematician.

[4]Leonida Tonelli (1885–1946) was an Italian mathematician.

- the function $f^y : X \to \mathbb{R}$ defined by $f^y(x) = f(x, y)$ is in $L^1(\mu)$ for almost all $y \in Y$,
- $h(y) = \int f(x, y)\ d\mu(x) \in L^1(\nu)$

and

$$\begin{aligned}\int f(x,y)\ d\mu\times\nu(x,y) &= \int\left[\int f(x,y)\ d\nu(y)\right]\ d\mu(x)\\ &= \int\left[\int f(x,y)\ d\mu(x)\right]\ d\nu(y)\end{aligned}$$

Proof. (a) (Tonelli)

Case 1: $f = \chi_E$ with $E \in \mathcal{M} \otimes \mathcal{N}$:

This is Proposition 5.8.

Case 2: $f \geq 0$ simple:

This follows from Case 1 by linearity.

Case 3: $f \in L^+(X \times Y, \mathcal{M} \otimes \mathcal{N})$:

Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative simple functions that increase pointwise to f. For example, we could take

$$f_n = \sum_{m=0}^{4^n-1} \frac{m}{2^n}\chi_{f^{-1}(I_{m,n})} + 2^n\chi_{f^{-1}([2^n,\infty])}(x) \qquad \text{where } I_{m,n} = f^{-1}\left(\left[\tfrac{m}{2^n}, \tfrac{m+1}{2^n}\right)\right)$$

Then, by the monotone convergence theorem, the limit as $n \to \infty$ of

$$\overbrace{\int \overbrace{f_n(x,y)}^{\text{increases to } f}\ d\mu\times\nu(x,y)}^{\text{increases to } \int f(x,y)\ d\mu\times\nu(x,y)} = \overbrace{\int\left[\overbrace{\int \overbrace{f_n(x,y)}^{\text{increases to } f}\ d\nu(y)}^{\text{increases to } \int f(x,y)\ d\nu(y)}\right]\ d\mu(x)}^{\text{increases to } \int\left[\int f(x,y)\ d\nu(y)\right]\ d\mu(x)}$$

gives

$$\int f(x,y)\,d\mu\times\nu(x,y) = \int\left[\int f(x,y)d\nu(y)\right]\,d\mu(x)$$

Similarly for the other order.

(b) (Fubini)

Write

$$\begin{aligned} h(y) &= \int f^y(x)\ d\mu(x) = \int f(x,y)\ d\mu(x) \\ g(x) &= \int f_x(y)\ d\nu(y) = \int f(x,y)\ d\nu(y) \end{aligned}$$

By Tonelli,

$$\begin{aligned} f(x,y) \in L^1 &\implies \int |f(x,y)|\ d\mu \times \nu < \infty \\ &\implies \int \Bigg[\overbrace{\int |f(x,y)|\ d\mu(x)}^{<\infty \text{ a.e. } y} \Bigg]\ d\nu(y) < \infty \\ &\implies f^y \in L^1(\mu) \text{ a.e. } y \text{ and } |h(y)| \le \int |f(x,y)|\ d\mu(x) \in L^1 \\ &\implies f^y \in L^1(\mu) \text{ a.e. } y \text{ and } h(y) \in L^1(\nu) \end{aligned}$$

Similarly

$$f_x \in L^1(\nu) \text{ a.e. } x \text{ and } g(x) \in L^1(\mu)$$

Now just apply the Tonelli theorem to the positive and negative parts of f, that is, to $\max\{f(x,y)\,,\,0\}$ and $\max\{-f(x,y)\,,\,0\}$.

$\square$