Haar Measure

Definition 1 A **group** is a set Γ together with an operation $\cdot : \Gamma \times \Gamma \to \Gamma$ obeying

i) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ for all $\alpha, \beta, \gamma \in \Gamma$ (Associativity)

ii) $\exists e \in \Gamma$ such that $\alpha e = e\alpha = e$ for all $\alpha \in \Gamma$ (Existence of identity)

iii) $\forall \alpha \in \Gamma\ \exists \alpha^{-1} \in \Gamma$ such that $\alpha\alpha^{-1} = \alpha^{-1}\alpha = e$ (Existence of inverses)

Definition 2 A **topological group** is a group together with a Hausdorff topology such that the maps

$$\begin{aligned} \Gamma \times \Gamma \to \Gamma &\quad (\alpha, \beta) \mapsto \alpha\beta \\ \Gamma \to \Gamma &\quad \alpha \mapsto \alpha^{-1} \end{aligned}$$

are continuous. A **compact group** is a topological group that is a compact topological space.

Example 3 (Examples of compact topological groups)

i) $U(1) = \left\{ z \in \mathbb{C} \bigm| |z| = 1 \right\} = \left\{ e^{i\theta} \bigm| 0 \le \theta < 2\pi \right\}$ with the usual multiplication in $\mathbb{C}$ and the usual topology in $\mathbb{C}$. That is, $e^{i\theta}e^{i\varphi} = e^{i(\theta+\varphi)}$, the identity is e^{i0}, the inverse is $\left(e^{i\theta}\right)^{-1} = e^{-i\theta}$ and

$$\begin{aligned} d\big(e^{i\theta}, e^{i\varphi}\big) &= d\big(\cos\theta + i\sin\theta, \cos\varphi + i\sin\varphi\big) \\ &= \sqrt{(\cos\theta - \cos\varphi)^2 + (\sin\theta - \sin\varphi)^2} \end{aligned}$$

ii) $SO(3) = \left\{ A \bigm| A \text{ a } 3 \times 3 \text{ real matrix}, \det A = 1, AA^t = \mathbb{1} \right\}$. The product is the usual matrix multiplication, the identity is the usual identity matrix and inverses are the usual matrix inverses. The topology is given by the usual Pythagorean metric on $\mathbb{R}^9$. The requirement that $AA^t = \mathbb{1}$ is equivalent to requiring that the three columns of $A = [\vec{a}_1, \vec{a}_2, \vec{a}_3] \in SO(3)$ be mutually perpendicular unit vectors. That $\det A = \vec{a}_1 \times \vec{a}_2 \cdot \vec{a}_3 = 1$ means that the triple $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is right handed. (This could be taken as the definition of "right handed".) The three columns of A are also the images $A\vec{e}_i$ of the three unit vectors pointing along the positive x, y and z axes, respectively. So $SO(3)$ can be thought of as the set of rotations in $\mathbb{R}^3$ or as the set of possible orientations of a rigid body.

Let

$$Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

be rotations about the $z-$ and $x-$axes, respectively. Any $\gamma \in SO(3)$ has a unique representation of the form

$$\gamma = Z(\varphi_2)X(\theta)Z(\varphi_1)$$

with $0 \le \theta \le \pi$, $0 \le \varphi_1, \varphi_2 < 2\pi$. The argument showing this is in [N]. The angles θ, φ_1 and φ_2 are called the Euler angles. The map $(\varphi_1, \varphi_2, \theta) \mapsto \gamma(\varphi_1, \varphi_2, \theta) = Z(\varphi_2)X(\theta)Z(\varphi_1)$ provides a local coordinate system on a neighbourhood of each point of $SO(3)$. Locally, $SO(3)$ looks like $\mathbb{R}^3$. A group that carries local coordinate systems like this is called a Lie group.

iii) Any group with only finitely many elements, like the group, S_n, of permutations of $\{1, \cdots, n\}$. Just use the discrete topology.

Definition 4 A **Haar measure** on Γ is a measure $\mu : \Sigma \to [0,\infty)$, with Σ a σ–algebra containing all Borel subsets of Γ, such that

i) $\mu(\Gamma) = 1$

ii) $\mu(\gamma S) = \mu(S)$ for all $\gamma \in \Gamma, S \in \Sigma$. Here $\gamma S = \{\ \gamma\alpha \ \big|\ \alpha \in S\ \}$.

We may associate to any measure μ on Γ a bounded linear functional $E : L^1(\Gamma, \Sigma, \mu) \to \mathbb{C}$ by

$$E(f) = \int_\Gamma f(\gamma)\, d\mu(\gamma)$$

In terms of E, the two conditions of Definition 4 translate into $E(1) = 1$ and $E\big(\chi_{\gamma S}\big) = E\big(\chi_S\big)$ for all $\gamma \in \Gamma$ and $S \in \Sigma$, respectively. Define $(L_\alpha f)(\gamma) = f(\alpha^{-1}\gamma)$. In this notation, $E\big(\chi_{\gamma S}\big) = E\big(\chi_S\big)$ becomes $E\big(L_\gamma \chi_S\big) = E\big(\chi_S\big)$. Since simple functions are dense in L^1, the second condition of Definition 4 is equivalent to $E(L_\gamma f) = E(f)$ for all $\gamma \in \Gamma$ and $f \in L^1(\Gamma, \Sigma, \mu)$. We shall, later in these notes, prove the existence of a Haar measure on any compact group Γ, by, first, constructing a positive linear functional $E : C_{\mathbb{R}}(\Gamma) \to \mathbb{R}$ (we can't use $f \in L^1(\Gamma, \Sigma, \mu)$ because μ is not known ahead of time) that obeys $E(1) = 1$ and $E(L_\gamma f) = E(f)$ and then, second, applying the Riesz Representation Theorem.

When a group carries local coordinate systems we can explicitly find the Haar measure in terms of local coordinates. Suppose that Γ is a compact group and that

- $\tilde{\mathcal{O}}$ is an open neighbourhood in $\mathbb{R}^n$
- $\tilde{\mathcal{U}}$ is an open neighbourhood in Γ
- $\tilde\gamma : \tilde{\mathcal{O}} \to \tilde{\mathcal{U}}$ is a homeomorphism (1–1, onto, continuous with continuous inverse)

This is called a coordinate patch. A Lie group is covered by (the $\tilde U$'s of) coordinate patchs. For each coordinate patch, we shall look for a function $\tilde\Delta : \tilde{\mathcal{O}} \to \mathbb{R}$ such that

$$\int_\Gamma f(\gamma)\, d\mu(\gamma) = \int_{\tilde{\mathcal{O}}} f\big(\tilde\gamma(\vec x)\big)\tilde\Delta(\vec x)\, d^n\vec x$$

for all functions $f \in C_{\mathbb{R}}(\Gamma)$ that are supported in $\tilde{\mathcal{U}}$. The measure $\tilde\Delta(\vec x)\, d^n\vec x$ on $\tilde{\mathcal{O}} \subset \mathbb{R}^n$ "represents the Haar measure in local coordinates". The idea is to fix a point in Γ and use $E(L_\gamma f) = E(f)$ to "move" the "measure at the fixed point" all around the group. I'll use the identity element $e \in \Gamma$ as the fixed point in Γ. Suppose that

- $\mathcal{O}$ is an open neighbourhood of the origin in $\mathbb{R}^n$
- $\mathcal{U}$ is an open neighbourhood of e in Γ
- $\gamma : \mathcal{O} \to \mathcal{U}$ is a homeomorphism and obeys $\gamma(\vec 0) = e$

is a coordinate patch for a neighbourhood of e. We shall also look for a function $\Delta : \mathcal{O} \to \mathbb{R}$ such that

$$\int_\Gamma f(\alpha)\, d\mu(\alpha) = \int_{\mathcal{O}} f\big(\gamma(\vec x)\big)\Delta(\vec x)\, d^n\vec x$$

for all functions $f \in C_{\mathbb{R}}(\Gamma)$ that are supported in $\mathcal{U}$.

Fix any $\vec y \in \tilde{\mathcal{O}}$. If $f \in C_{\mathbb{R}}(\Gamma)$ is supported sufficiently near $\tilde\gamma(\vec y) \in \tilde{\mathcal{U}}$, then f is supported in $\tilde{\mathcal{U}}$ and $L_{\tilde\gamma(\vec y)^{-1}} f$ is supported in $\mathcal{U}$ (since $L_{\tilde\gamma(\vec y)^{-1}} f(\alpha) = f\big(\tilde\gamma(\vec y)\alpha\big)$ vanishes unless α is close enough to e). Then, in local coordinates, the requirement that $E(f) = E\big(L_{\gamma(\vec y)^{-1}} f\big)$ becomes

$$\int f\big(\tilde\gamma(\vec x)\big)\tilde\Delta(\vec x)\, d^n\vec x = \int \big(L_{\gamma(\vec y)^{-1}} f\big)\big(\gamma(\vec x)\big)\Delta(\vec x)\, d^n\vec x = \int f\big(\tilde\gamma(\vec y)\gamma(\vec x)\big)\Delta(\vec x)\, d^n\vec x$$

Define $\vec z(\vec y, \vec x)$ by $\tilde\gamma\big(\vec z(\vec y, \vec x)\big) = \tilde\gamma(\vec y)\gamma(\vec x)$. It will be defined for all $\vec y \in \tilde{\mathcal{O}}$ and all $\vec x$ in a sufficiently small neighbourhood of the origin (depending on $\vec y$) that $\tilde\gamma(\vec y)\gamma(\vec x) \in \tilde{\mathcal{U}}$. Then the requirement that $E(f) = E\big(L_{\gamma(\vec y)^{-1}} f\big)$ becomes

$$\int f\big(\tilde\gamma(\vec x)\big)\tilde\Delta(\vec x)\, d^n\vec x = \int f\big(\tilde\gamma(\vec z(\vec y, \vec x))\big)\Delta(\vec x) d^n\vec x$$

Making the change of variables $\vec{x} \to \vec{z}(\vec{y},\vec{x})$ (with $\vec{y}$ held fixed) in the integral on the left hand side, we must have

$$\int f\big(\tilde\gamma(\vec{z}(\vec{y},\vec{x}))\big)\tilde\Delta\big(\vec{z}(\vec{y},\vec{x})\big)\Big|\det\big[\tfrac{\partial z_i}{\partial x_j}(\vec{y},\vec{x})\big]_{1\le i,j\le n}\Big|\, d^n\vec{x} = \int f\big(\tilde\gamma(\vec{z}(\vec{y},\vec{x}))\big)\Delta(\vec{x})\, d^n\vec{x}$$

for all f supported near $\tilde\gamma(\vec{y})$. So $\Delta(\vec{x}) = \tilde\Delta\big(\vec{z}(\vec{y},\vec{x})\big)\Big|\det\big[\frac{\partial z_i}{\partial x_j}(\vec{y},\vec{x})\big]_{1\le i,j\le n}\Big|$ for all $\vec{x}$ sufficiently near $\vec{0}$. In particular, setting $\vec{x}=\vec{0}$, and observing that $\vec{z}(\vec{y},\vec{0})=\vec{y}$,

$$\Delta(\vec{0}) = \tilde\Delta(\vec{y})\Big|\det\big[\tfrac{\partial z_i}{\partial x_j}(\vec{y},\vec{0})\big]_{1\le i,j\le n}\Big|$$

This determines $\tilde\Delta(\vec{y})$ on $\tilde{\mathcal{O}}$, up to the constant $\Delta(\vec{0})$. (By using implicit differentiation, you don't even need to know $\vec{z}(\vec{y},\vec{x})$ explicitly.) Since all of Γ is covered by coordinate patchs, this determines the Haar measure on all of Γ, up to the constant $\Delta(\vec{0})$. The constant is determined by the requirement that $\mu(\Gamma)=1$. The notes entitled "A Lie Group" contain a derivation of the Haar measure of $SU(2)$ using this type of argument.

Example 5 (Examples of Haar measures)

i) For $U(1) = \big\{\ e^{i\theta}\ \big|\ 0\le\theta<2\pi\ \big\}$

$$\int_{U(1)} f(\gamma)\,d\mu(\gamma) = \tfrac{1}{2\pi}\int_0^{2\pi} f\big(e^{i\theta}\big)\,d\theta$$

ii) Applying the "Δ" argument of the last paragraph to $SO(3)$ gives

$$\int_{SO(3)} f(\gamma)\,d\mu(\gamma) = \int_0^{\pi} d\theta\int_0^{2\pi} d\varphi_1\int_0^{2\pi} d\varphi_2\ \tfrac{1}{8\pi^2}\sin\theta\, f\big(\gamma(\varphi_1,\varphi_2,\theta)\big)$$

where $\varphi_1,\varphi_2,\theta$ are the Euler angles. See [N].

iii) If Γ is a finite group with $\#\Gamma$ elements, the Haar measure is $\mu(S)=\frac{\#S}{\#\Gamma}$.

Let Γ be a compact group. Denote by $C_{\mathbb{R}}(\Gamma)$ the real–valued continuous functions on Γ and define, for each $\alpha\in\Gamma$, the maps $L_\alpha, R_\alpha, J : C_{\mathbb{R}}(\Gamma)\to C_{\mathbb{R}}(\Gamma)$ by

$$(L_\alpha f)(\gamma) = f(\alpha^{-1}\gamma)\qquad (R_\alpha f)(\gamma) = f(\gamma\alpha)\qquad (Jf)(\gamma) = f(\gamma^{-1})$$

Theorem 6 (Existence of Haar Measure) *Let Γ be a compact group.*

a) There exists a unique positive linear functional $E : C_{\mathbb{R}}(\Gamma)\to\mathbb{R}$ such that $E(1)=1$ and $E(R_\alpha f)=E(f)$ for all $\alpha\in\Gamma$ and $f\in C_{\mathbb{R}}(\Gamma)$. Furthermore $E(L_\alpha f)=E(Jf)=E(f)$ for all $\alpha\in\Gamma$ and $f\in C_{\mathbb{R}}(\Gamma)$.

b) There is a σ–algebra, Σ, of subsets of Γ that contains all Borel subsets of Γ and is invariant under left and right multiplication and under inversion, i.e.

$$\begin{aligned} S\in\Sigma \quad\Longrightarrow\quad & \gamma S = \big\{\ \gamma\alpha\ \big|\ \alpha\in S\ \big\}\in\Sigma\\ & S\gamma = \big\{\ \alpha\gamma\ \big|\ \alpha\in S\ \big\}\in\Sigma\\ & S^{-1} = \big\{\ \alpha^{-1}\ \big|\ \alpha\in S\ \big\}\in\Sigma \end{aligned}$$

and there is a measure μ on Σ such that

$$\mu(\gamma S) = \mu(S\gamma) = \mu(S^{-1}) = \mu(S)\ \forall S\in\Sigma\qquad \mu(\Gamma)=1\qquad E(f) = \int_\Gamma f(\gamma)\,d\mu(\gamma)$$

Idea of proof: Here is a brief outline of the proof. The details appear later in these notes. For each $f \in C_{\mathbb{R}}(\Gamma)$ set

$$\mathcal{L}(f) = \Big\{ \sum_{i=1}^{n} a_i L_{\alpha_i} f \ \Big|\ n \in \mathbb{N}, \alpha_i \in \Gamma, a_i > 0, \sum_{i=1}^{n} a_i = 1 \Big\} \subset C_{\mathbb{R}}(\Gamma)$$
$$\mathcal{R}(f) = \Big\{ \sum_{i=1}^{n} a_i R_{\alpha_i} f \ \Big|\ n \in \mathbb{N}, \alpha_i \in \Gamma, a_i > 0, \sum_{i=1}^{n} a_i = 1 \Big\} \subset C_{\mathbb{R}}(\Gamma)$$

We denote by $\overline{\mathcal{L}(f)}$ and $\overline{\mathcal{R}(f)}$ the closures of $\mathcal{L}(f)$ and $\mathcal{R}(f)$, respectively, in $C_{\mathbb{R}}(\Gamma)$. Note that any E having the desired properties must always give the same value, $E(f)$, when applied to any element of $\overline{\mathcal{L}(f)}$ or $\overline{\mathcal{R}(f)}$.

Step 1. We use Arzelà–Ascoli to show that $\overline{\mathcal{L}(f)}$ is a compact subset of $C_{\mathbb{R}}(\Gamma)$.

Step 2. We show that $\overline{\mathcal{L}(f)}$ contains a constant function.

Step 3. We show that $\overline{\mathcal{R}(f)}$ contains a constant function.

Step 4. We show that there is a unique constant function $c(f) \in \overline{\mathcal{L}(f)} \cup \overline{\mathcal{R}(f)}$.

Step 5. We define $E(f) = c(f)$ and verify that it has the right properties.

Step 6. We verify that E is unique.

Step 7. We define μ and Σ using the Riesz representation theorem. ■

Representations of Compact Groups

We now look at some applications of Haar measure to the study of representations of compact groups. We use Γ to denote a compact group, μ its Haar measure, V a finite dimensional complex vector space with inner product and $\mathrm{Hom}\,(V)$ the set of linear maps on V. You can think of V as $\mathbb{C}^n$ and of $\mathrm{Hom}\,(V)$ as the set of $n \times n$ complex matrices. A linear map $U \in \mathrm{Hom}\,(V)$ is called unitary if $\langle U\vec{v}, U\vec{w}\rangle = \langle \vec{v}, \vec{w}\rangle$ for all $\vec{v}, \vec{w} \in V$. When $V = \mathbb{C}^n$ with the standard inner product $\langle \vec{v}, \vec{w}\rangle = \sum_{i=1}^{n} v_i \bar{w}_i$, the $n \times n$ matrix U is unitary if and only if $U^*U = \mathbb{1}$, where $U^*_{i,j} = \bar{U}_{j,i}$ is the adjoint matrix of U and $\mathbb{1}$ is the $n \times n$ identity matrix. We use $\mathcal{U}(V)$ to denote the set of unitary linear maps on V.

Definition 7 (Representations)

i) A **representation** of Γ on V is a continuous map $D : \Gamma \to \mathcal{U}(V)$ that obeys $D(\alpha)D(\beta) = D(\alpha\beta)$ for all $\alpha, \beta \in \Gamma$.

ii) Two representations, $D : \Gamma \to \mathcal{U}(V)$ and $D' : \Gamma \to \mathcal{U}(V')$ of Γ on V and V', respectively, are called **equivalent** if there is a unitary map $U : V \to V'$ such that $D'(\gamma) = UD(\gamma)U^{-1}$ for all $\gamma \in \Gamma$.

iii) A representation $D : \Gamma \to \mathcal{U}(V)$ is called **irreducible** if and only if the only invariant subspaces for $\big\{ D(\gamma) \ \big|\ \gamma \in \Gamma \big\}$ are $\{0\}$ and V. That is, if V' is a linear subspace of V and $D(\gamma)\vec{v} \in V'$ for all $\vec{v} \in V'$ and all $\gamma \in \Gamma$, then $V' = \{0\}$ or $V' = V$.

Example 8 (Examples of Representations)

i) For $U(1) = \big\{ z \ \big|\ |z| = 1 \big\}$ and $n \in \mathbb{N}$, define $D : U(1) \to \mathbb{C}$ by $D(z) = z^n$. Let $V = \mathbb{C}$. Then $\mathrm{Hom}\,(V)$ can be thought of as the space of 1×1 complex matrices (i.e. $\mathbb{C}$) and $\mathcal{U}(V)$ as the space of 1×1 unitary matrices (i.e. complex numbers of modulus one). So D is a representation for $U(1)$.

ii) Define $D : SO(3) \to \mathcal{U}(\mathbb{C}^3)$ by $D(A) = A$. This is a representation for $SO(3)$.

iii) Usually we write elements of $\mathbb{C}^9$ in the form $(z_1, \cdots, z_9)$ with the indices labelled 1 through 9. Insead, use labels (m, n) with $1 \le m, n \le 3$. Then a linear map on $\mathbb{C}^9$ is of the form $(M\vec{z})_{(m,n)} = \sum_{m',n'=1}^{3} M_{(m,n),(m',n')} z_{(m',n')}$. Define $D(A)_{(m,n),(m',n')} = A_{m,m'}A_{n,n'}$. This is a representation of $SO(3)$ on $\mathbb{C}^9$.

We first use the Haar measure to show that the continuity and unitarity assumptions built into the definition of "representation" aren't really necessary and that all representations are built out of irreducible representations.

Theorem 9 (Automatic unitarity) *Let* $A : \Gamma \to \mathrm{Hom}\,(V)$ *obey*

- $A(\alpha)A(\beta) = A(\alpha\beta)$ *for all* $\alpha, \beta \in \Gamma$
- $\gamma \mapsto \ell(A(\gamma)v)$ *is measurable and bounded for all* $v \in V,\ \ell \in V^*$
- *for each* $0 \ne v \in V$*,* $A(\gamma)v$ *is not a.e. zero in* γ

Then there exists an inner product on V *such that all* $A(\gamma)$ *are unitary.*

Proof: Let $\langle\ ,\ \rangle_0$ be any inner product on V and let $\{e_i\}_{1\le i\le n}$ be an orthonormal basis for V under this inner product. Let

$$\langle v, w\rangle = \int \langle A(\gamma)v, A(\gamma)w\rangle_0\ d\mu(\gamma)$$

This is well–defined since integrand

$$\langle A(\gamma)v, A(\gamma)w\rangle_0 = \sum_{j=1}^{n} \langle A(\gamma)v, e_j\rangle_0 \langle e_j, A(\gamma)w\rangle_0$$

is bounded and measurable. This is an inner product, with the strict positivity axiom following from the third hypothesis (on nonvanishing). Since

$$\begin{aligned}\langle A(\alpha)v, A(\alpha)w\rangle &= \int \langle A(\alpha)A(\gamma)v, A(\alpha)A(\gamma)w\rangle_0\ d\mu(\gamma) = \int \langle A(\alpha\gamma)v, A(\alpha\gamma)w\rangle_0\ d\mu(\gamma)\\ &= \int \langle A(\gamma')v, A(\gamma')w\rangle_0\ d\mu(\gamma') = \langle v, w\rangle\end{aligned}$$

every $A(\alpha)$ is unitary. ■

Theorem 10 (Automatic continuity) *Let* V *be a finite dimensional Hilbert space and let* $U : \Gamma \to \mathrm{Hom}\,(V)$ *obey*

- $A(\alpha)A(\beta) = A(\alpha\beta)$ *for all* $\alpha, \beta \in \Gamma$
- $\gamma \mapsto \ell(A(\gamma)v)$ *is measurable* $v \in V,\ \ell \in V^*$
- $A(\gamma)$ *is unitary for all* $\gamma \in \Gamma$

Then $\gamma \mapsto A(\gamma)$ *is continuous.*

Proof: For each $f \in C_{\mathbb{R}}(\Gamma)$, let

$$a_f(\gamma) = \int_\Gamma f(\gamma^{-1}\alpha)A(\alpha)\ d\mu(\alpha) \in \mathrm{Hom}\,(V)$$

We claim that f is uniformly continuous in the sense that for each $\varepsilon > 0$ there is a neighbourhood N of e such that $|f(\alpha) - f(\beta)| < \varepsilon$ for all $\alpha, \beta \in \Gamma$ obeying $\beta\alpha^{-1} \in N$. To see this, observe that, since f and multiplication are both continuous, $F(\alpha, \beta) = f(\alpha) - f(\beta\alpha)$ is continuous as a real valued function on $\Gamma \times \Gamma$. For each $\alpha_0 \in \Gamma$, $F(\alpha_0, e) = 0$, so there are neighbourhoods M_{α_0} of α_0 and N_{α_0} of e such that $|F(\alpha, \beta)| < \varepsilon$ for all $(\alpha, \beta) \in M_{\alpha_0} \times N_{\alpha_0}$. By compactness, there is a finite subset $\left\{\ \alpha_i\ \middle|\ 1 \le i \le n\ \right\} \subset \Gamma$ such that $\Gamma \subset \cup_{i=1}^n M_{\alpha_i}$. Set $N_\varepsilon = \cap_{i=1}^n N_{\alpha_i}$. Then if $\beta \in N_\varepsilon$ and $\alpha \in \Gamma$, there is some $1 \le i \le n$ such that $(\alpha, \beta) \in M_{\alpha_i} \times N_\varepsilon \subset M_{\alpha_i} \times N_{\alpha_i}$ so that $|F(\alpha, \beta)| < \varepsilon$, as desired.

The uniform continuity of f implies that if ${\gamma'}^{-1}\gamma \in N_\varepsilon$, then $|f(\gamma^{-1}\alpha) - f({\gamma'}^{-1}\alpha)| < \varepsilon$. For any unitary operator $\|A(\alpha)\| = 1$. As the Haar measure is normalized by $\int_\Gamma d\mu(\alpha) = 1$,

$$\begin{aligned}\|a_f(\gamma) - a_f(\gamma')\| &= \Big\| \int_\Gamma [f(\gamma^{-1}\alpha) - f({\gamma'}^{-1}\alpha)] A(\alpha)\ d\mu(\alpha) \Big\| \le \int_\Gamma |f(\gamma^{-1}\alpha) - f({\gamma'}^{-1}\alpha)|\, \|A(\alpha)\|\ d\mu(\alpha) \\ &\le \int_\Gamma \varepsilon\ d\mu(\alpha) = \varepsilon\end{aligned}$$

Thus $\gamma \mapsto a_f(\gamma)$ is continuous.

By the invariance of $d\mu$

$$a_f(\gamma) = \int_\Gamma f(\gamma^{-1}\alpha) A(\alpha)\ d\mu(\alpha) = \int_\Gamma f(\alpha) A(\gamma\alpha)\ d\mu(\alpha) = A(\gamma) \int_\Gamma f(\alpha) A(\alpha)\ d\mu(\alpha)$$

Thus for any $v \in V$ of the form $v = \int_\Gamma f(\alpha) A(\alpha) v'\ d\mu(\alpha)$ with $f \in C_{\mathbb{R}}(\Gamma)$ and $v' \in V$,

$$\|A(\gamma)v - A(\gamma')v\| = \|a_f(\gamma)v' - a_f(\gamma')v'\| \le \|a_f(\gamma) - a_f(\gamma')\|\, \|v'\|$$

and the map $\gamma \mapsto A(\gamma)v$ is continuous. We claim every $v \in V$ is of this form. Since V is finite dimensional, this will imply that $\gamma \mapsto A(\gamma)$ is continuous. The set of all v's of the form $v = \int_\Gamma f(\alpha)A(\alpha)v'\ d\mu(\alpha)$ is a linear subspace, V', of V. If w is orthogonal to V', then

$$0 = \int_\Gamma f(\alpha) \left\langle A(\alpha)v', w \right\rangle\ d\mu(\alpha) = \int_\Gamma f(\alpha) \left\langle v', A^*(\alpha)w \right\rangle\ d\mu(\alpha)$$

for all $f \in C_{\mathbb{R}}(\Gamma)$ and all $v' \in V$. The uniqueness provision of the Riesz representation theorem, applied to the bounded linear functional $f \mapsto \int_\Gamma f(\alpha) \left\langle v', A^*(\alpha)w \right\rangle\ d\mu(\alpha)$, implies that $\left\langle v', A^*(\alpha)w \right\rangle = 0$ for all $v' \in V$ and a.e. $\alpha \in \Gamma$. Choosing $v' = A^*(\alpha)w$, this implies that $A^*(\alpha)w = 0$ for some $\alpha \in \Gamma$. Since $A(\alpha)$ is unitary, $w = 0$. ■

Lemma 11 *Let $U : \Gamma \to \mathcal{U}(V)$ be a representation of Γ. There are subspaces $W_1, \cdots, W_n$ (with $n \ge 1$) of V such that*

- *$V = W_1 \oplus \cdots \oplus W_n$. That is, $W_i \perp W_j$ for all $i \ne j$ and every vector $v \in V$ has a unique representation $v = w_1 + \cdots + w_n$ with $w_i \in W_i$ for all $1 \le i \le n$.*
- *For each $1 \le i \le n$, the restriction of U to W_i is irreducible.*

Proof: If U is irreducible, then $n = 1$ and $W_1 = V$ works. If U is reducible there is a nontrivial subspace W of V such that $U(\gamma)$ leaves W invariant for every $\gamma \in \Gamma$. But then $U(\gamma)$ also leaves $W^\perp$ invariant, because if $w' \in W^\perp$ and $w \in W$

$$\begin{aligned}\left\langle U(\gamma)w', w \right\rangle &= \left\langle U(\gamma^{-1})U(\gamma)w', U(\gamma^{-1})w \right\rangle \text{ (unitarity)} \\ &= \left\langle w', U(\gamma^{-1})w \right\rangle \text{ (U is a representation)} \\ &= 0 \text{ ($w' \in W^\perp$ and $U(\gamma^{-1})w \in W$)}\end{aligned}$$

As, $V = W \oplus W^\perp$, we are done if the restrictions of U to W and $W^\perp$ are irreducible. If not, we repeat. ■

Lemma 12 *Let Γ be an Abelian group (this means that $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \Gamma$) and let $U : \Gamma \to \mathcal{U}(V)$ be an irreducible representation of Γ. Then $\dim V = 1$.*

Proof: If $U(\gamma)$ is a multiple of the identity matrix for every $\gamma \in \Gamma$, then every subspace of V is invariant. This contradicts the assumption of irreducibility unless $\dim V = 1$. So we may assume without loss of generality that $U(\gamma)$ is not a multiple of the identity matrix for at least one $\gamma \in \Gamma$. Let $U(\alpha)$ not be a multiple of the identity. Let λ be any eigenvalue of $U(\alpha)$ and $W = \{ v \in V \mid U(\alpha)v = \lambda v \}$. This is a proper subspace of V that contains at least one nonzero vector. If $\gamma \in \Gamma$ and $v \in V$, then

$$U(\alpha)U(\gamma)v = U(\alpha\gamma)v = U(\gamma\alpha)v = U(\gamma)U(\alpha)v = \lambda U(\gamma)v \quad \Longrightarrow \quad U(\gamma)v \in W$$

Thus W is an invariant subspace, which contradicts irreducibility. ■

Example 13 Let $\Gamma = U(1)$. This is an Abelian group, so all irreducible representations must be one dimensional. Let $\chi : U(1) \to \mathcal{U}(V)$ be an irreducible representation. Since $\dim V = 1$, every element of $\mathcal{U}(V)$ is of the form $c\mathbb{1}$ with c a complex number of modulus one. It is conventional to simply write c in place of $c\mathbb{1}$. The argument of Theorem 9 shows[1] that $\chi(e^{i\theta})$ must be a C^∞ function of θ. Differentiating $\chi(e^{i(\theta'+\theta)}) = \chi(e^{i\theta'})\chi(e^{i\theta})$ with respect to θ' and then setting $\theta' = 0$ gives $\chi'(e^{i\theta}) = i\chi'(1)\chi(e^{i\theta})$. Solving this differential equation with the initial condition $\chi(1) = 1$ gives $\chi(e^{i\theta}) = e^{i\chi'(1)\theta}$. Since we also need $\lim_{\theta\to 2\pi-} \chi(e^{i\theta}) = \chi(1) = 1$, $\chi'(1)$ must be an integer. We just shown that every irreducible representation of $U(1)$ is of the form $\chi(e^{i\theta}) = e^{in\theta}$ for some integer n.

Lemma 14 (Schur's Lemma) *Let $U : \Gamma \to \mathcal{U}(V)$ and $U' : \Gamma \to \mathcal{U}(V')$ be irreducible representations. Let $T : V \to V'$ obey*

$$TU(\gamma) = U'(\gamma)T$$

for all $\gamma \in \Gamma$. Then either $T = 0$ or U and U' are unitarily equivalent and T is unique up to a constant. In particular, if $U = U'$, then $T = \lambda\mathbb{1}$ for some constant λ.

Proof: We first consider the special case $U = U'$. Any square matrix, whether diagonalizable or not, has at least one eigenvalue and one eigenvector. Let λ be an eigenvalue of T and $W = \{ w \in V \mid Tw = \lambda w \}$. For all $\gamma \in \Gamma$ and all $w \in W$

$$(T - \lambda\mathbb{1})U(\gamma)w = U(\gamma)(T - \lambda\mathbb{1})w = 0 \quad \Longrightarrow \quad U(\gamma)w \in W$$

Thus W is an invariant subspace for U. Since W contains at least one nonzero vector and U is irreducible, $W = V$ and $T = \lambda\mathbb{1}$.

Now for the general case. Taking adjoints of $TU(\gamma) = U'(\gamma)T$ gives $T^*U'(\gamma)^* = U(\gamma)^*T^*$. Since U and U' are unitary, $U(\gamma)^* = U(\gamma)^{-1} = U(\gamma^{-1})$ and $U'(\gamma)^* = U'(\gamma^{-1})$. So $T^*U'(\gamma^{-1}) = U(\gamma^{-1})T^*$. Replacing γ by γ^{-1} gives $T^*U'(\gamma) = U(\gamma)T^*$. Hence

$$TT^*U'(\gamma) = TU(\gamma)T^* = U'(\gamma)TT^* \qquad \text{and} \qquad T^*TU(\gamma) = T^*U'(\gamma)T = U(\gamma)T^*T$$

As both U and U' are irreducible, the special case that we have already dealt with implies that there are constants λ, λ' such that $TT^* = \lambda\mathbb{1}$ and $T^*T = \lambda'\mathbb{1}$. If $\lambda = 0$, then for all $v' \in V'$

$$\|T^*v'\|^2_{V'} = \langle T^*v', T^*v'\rangle_{V'} = \langle TT^*v', v'\rangle_{V'} = \lambda\|v'\|^2_{V'} = 0$$

so that $T^* = 0$ and hence $T = 0$. Similarly, if $\lambda' = 0$, $T = 0$ too. If both λ and λ' are nonzero, then $TT^* = \lambda\mathbb{1}$ forces the kernel of T^* to be $\{0\}$ and $T^*T = \lambda'\mathbb{1}$ forces the kernel of T to be $\{0\}$. Only square matrices can have

[1] Just use C^∞ functions $f(e^{i\theta})$. The reason we couldn't do this originally is that we didn't know how to differentiate functions on Γ.

the kernel of both T and T^* being 0. So V and V' have the same dimension. Then $T^*T = \lambda'\mathbb{1}$ says that $\frac{1}{\lambda'}T^*$ is the inverse of T and $TT^* = \lambda\mathbb{1}$ says that $\frac{1}{\lambda}T^*$ is the inverse of T. Thus $\lambda = \lambda'$. Since $\|T^*v'\|_{V'}^2 = \lambda\|v'\|_{V'}^2$, $\lambda > 0$. Hence $S = \frac{1}{\sqrt{\lambda}}T$ is unitary and

$$TU(\gamma) = U'(\gamma)T \quad \Longrightarrow \quad SU(\gamma) = U'(\gamma)S \quad \Longrightarrow \quad SU(\gamma)S^{-1} = U'(\gamma)$$

gives the equivalence of U and U'.

This leaves only uniqueness up to constants. Let U and U' be equivalent and let $TU(\gamma) = U'(\gamma)T$, $SU(\gamma) = U'(\gamma)S$ for all $\gamma \in \Gamma$. Then, as above, $T^*U'(\gamma) = U(\gamma)T^*$ and

$$ST^*U'(\gamma) = SU(\gamma)T^* = U'(\gamma)ST^*$$

So there is a constant λ such that $ST^* = \lambda\mathbb{1}$ and

$$S = ST^*T/\lambda_T = \tfrac{\lambda}{\lambda_T}T$$

■

Let $\hat{\Gamma}$ be the set of equivalence classes of irreducible representations of the compact group Γ. For each $\alpha \in \hat{\Gamma}$ we select an explicit matrix representative $D^{(\alpha)}_{i,j}(\gamma)$ of the representation. The indices i and j run from 1 to d_α, the dimension of the representation. The **character** of the representation α is defined to be the complex valued function

$$\chi_\alpha(\gamma) = \mathrm{Tr}\, D^{(\alpha)}(\gamma) = \sum_{i=1}^{d_\alpha} D^{(\alpha)}_{i,i}(\gamma)$$

on Γ.

Theorem 15 (Orthogonality) *For any $\alpha, \beta \in \hat{\Gamma}$, $1 \le i, j \le d_\alpha$ and $1 \le k, \ell \le d_\beta$*

$$\int D^{(\alpha)}_{i,j}(\gamma)\overline{D^{(\beta)}_{k,\ell}(\gamma)}\ d\mu(\gamma) = \tfrac{1}{d_\alpha}\delta_{\alpha,\beta}\delta_{i,k}\delta_{j,\ell}$$

Proof: Let B be a $d_\alpha \times d_\beta$ matrix and define

$$C = \int D^{(\alpha)}(\gamma)BD^{(\beta)}(\gamma)^{-1}\ d\mu(\gamma)$$

Then

$$\begin{aligned}
D^{(\alpha)}(\delta)C &= \int D^{(\alpha)}(\delta)D^{(\alpha)}(\gamma)BD^{(\beta)}(\gamma)^{-1}\ d\mu(\gamma)\\
&= \int D^{(\alpha)}(\delta\gamma)BD^{(\beta)}(\gamma^{-1})\ d\mu(\gamma)\\
&= \int D^{(\alpha)}(\gamma)BD^{(\beta)}(\gamma^{-1}\delta)\ d\mu(\gamma)\\
&= \int D^{(\alpha)}(\gamma)BD^{(\beta)}(\gamma^{-1})D^{(\beta)}(\delta)\ d\mu(\gamma)\\
&= CD^{(\beta)}(\delta)
\end{aligned}$$

By Schur's Lemma, $C = 0$ if $\alpha \ne \beta$ and $C = c\mathbb{1}$ if $\alpha = \beta$. In the latter case, $d_\alpha c = \mathrm{Tr}\,(C) = \mathrm{Tr}\,(B)$, since $\mathrm{Tr}\,(DBD^{-1}) = \mathrm{Tr}\,(B)$ by the cyclicity of the trace. Now just take B to be the matrix which has only one nonzero entry, namely $B_{j,\ell} = 1$. Since $D^{(\beta)}$ is unitary,

$$C_{i,k} = \int D^{(\alpha)}_{i,j}(\gamma)\overline{D^{(\beta)}_{k,\ell}(\gamma)}\ d\mu(\gamma) = \tfrac{1}{d_\alpha}\delta_{\alpha,\beta}\delta_{i,k}\mathrm{Tr}\,B$$

as desired. ■

Theorem 16 (Orthogonality of characters) *For any $\alpha, \beta \in \hat{\Gamma}$,*

$$\int \chi_\alpha(\gamma)\overline{\chi_\beta(\gamma)}\ d\mu(\gamma) = \delta_{\alpha,\beta}$$

Proof: Just apply Theorem 15 to

$$\int \chi_\alpha(\gamma)\overline{\chi_\beta(\gamma)}\ d\mu(\gamma) = \sum_{\substack{1\le i\le d_\alpha \\ 1\le j\le d_\beta}} \int D^{(\alpha)}_{i,i}(\gamma)\overline{D^{(\beta)}_{j,j}(\gamma)}\ d\mu(\gamma)$$

■

Theorem 17 (The Peter–Weyl Theorem) *Let Γ be a compact separable group.*

a) The set of finite linear combinations of $\big\{D^{(\alpha)}_{i,j}(\gamma)\big\}_{\alpha\in\hat{\Gamma},1\le i,j\le d_\alpha}$ is dense in the $\|\cdot\|_\infty$ norm in $C(\Gamma)$, the set of all continuous functions on Γ.

b) The set $\big\{\sqrt{d_\alpha}D^{(\alpha)}_{i,j}(\gamma)\big\}_{\alpha\in\hat{\Gamma},1\le i,j\le d_\alpha}$ is an orthonormal basis for $L^2(\Gamma,\mu)$.

Outline of proof: Let $\mathcal{F}$ be the set of finite linear combinations of $D^{(\alpha)}_{i,j}(\gamma)$ with $\alpha \in \hat{\Gamma}$ and $1 \le i,j \le d_\alpha$. By Definition 7.i, $\mathcal{F} \subset C(\Gamma)$. We shall apply the Stone–Weierstrass Theorem to it. The hypotheses of Stone–Weierstrass are

- $\mathcal{F}$ is an algebra. It is obviously closed under addition and multiplication by complex numbers. To show that it is closed under multiplication, it suffices to show that $D^{(\alpha)}_{i,j}(\gamma)D^{(\beta)}_{k,\ell}(\gamma)$ can be expressed as a finite linear combination of $D^{(\alpha')}_{i',j'}(\gamma)$'s. To do so we first build a new representation $D(\gamma)$ of Γ on $\mathbb{C}^{d_\alpha d_\beta}$. We label the components of an element of $\mathbb{C}^{d_\alpha d_\beta}$ by indices running over $\big\{\ (m,n)\in\mathbb{N}^2\ \big|\ m\le d_\alpha, n\le d_\beta\ \big\}$ rather than by indices running over $\big\{\ m\in\mathbb{N}\ \big|\ m\le d_\alpha d_\beta\ \big\}$. We define

$$D_{(m,n),(m',n')}(\gamma) = D^{(\alpha)}_{m,m'}(\gamma)D^{(\beta)}_{n,n'}(\gamma)$$

  Since $D^{(\alpha)}$ and $D^{(\beta)}$ are representations, $D_{(m,n),(m',n')}(\gamma)$ is continuous in γ for each $1\le m,m'\le d_\alpha$, $1\le n,n'\le d_\beta$, $D(\gamma)D(\gamma')=D(\gamma\gamma')$, $D(\gamma)^*=D(\gamma^{-1})$. So D is a legitimate representation, called the tensor product representation, $D^{(\alpha)}\otimes D^{(\beta)}$. It need not be irreducible but, by Lemma 10, there is a unitary matrix U (which maps the standard orthonormal basis of $\mathbb{C}^{d_\alpha d_\beta}$ to an orthonormal basis whose first $\dim W_1$ elements are an orthonormal basis of W_1, whose next $\dim W_2$ elements are an orthonormal basis of W_2 and so on) such that

$$D(\gamma) = U^*\begin{bmatrix} D^{(\alpha_1)}(\gamma) & 0 & 0 & \cdots & 0 \\ 0 & D^{(\alpha_2)}(\gamma) & 0 & \cdots & 0 \\ 0 & 0 & D^{(\alpha_3)}(\gamma) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & D^{(\alpha_n)}(\gamma)\end{bmatrix}U$$

  The $(i,k),(j,\ell)$ matrix element of this equation expresses $D^{(\alpha)}_{i,j}(\gamma)D^{(\beta)}_{k,\ell}(\gamma)$ as a finite linear combination of $D^{(\alpha_m)}_{i_m,j_m}(\gamma)$'s.
- $\mathcal{F}$ vanishes nowhere. This is easy: the map $\gamma\mapsto 1\in\mathbb{C}$ (called the trivial representation) is an irreducible representation and vanishes nowhere.

- $\mathcal{F}$ is self–adjoint. This is also easy. If $D(\gamma)$ is any irreducible representation by $d\times d$ matrices, then so is $\overline{D(\gamma)}$, which is defined by taking the complex conjugate of each matrix element.
- $\mathcal{F}$ separates points. For most concrete compact groups, this is easy. For the matrix groups, like $U(1)$ and $SO(3)$, the representation $\gamma\mapsto\gamma$ separates points. For finite groups, we define the "left regular representation" as follows. Let $V = L^2(\Gamma)$ be the set of all measurable functions on Γ that are L^2 with respect to Haar measure. When Γ is a finite group, V is just the set of all functions on Γ is of dimension $\#\Gamma$. For each $\gamma\in\Gamma$, $f\in V\mapsto L_\gamma f\in V$ is a unitary map on $V=L^2(\Gamma)$, since Haar measure is invariant under left translation. Since $L_\alpha L_\beta = L_{\alpha\beta}$, this is a representation, called the "left regular representation". If $\gamma\neq\gamma'$ and f is any function on Γ with $f(\gamma^{-1})\neq f({\gamma'}^{-1})$, then $L_\gamma f\neq L_{\gamma'}f$, so the left regular representation separates points. It is usually not irreducible, but that doesn't matter – just apply Lemma 11. For the general case, $L^2(\Gamma)$ is infinite dimensional and we don't quite have the machinery to push this argument through. But it can be done.

■

The Existence of Haar measure

Proof of Theorem 6: For each $f\in C_{\mathbb{R}}(\Gamma)$ set

$$\begin{aligned} M(f) &= \max_{\gamma\in\Gamma} f(\gamma) \\ m(f) &= \min_{\gamma\in\Gamma} f(\gamma) \\ v(f) &= M(f)-m(f) \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}(f) &= \Big\{\ \sum_{i=1}^n a_i L_{\alpha_i} f\ \Big|\ n\in\mathbb{N}, \alpha_i\in\Gamma, a_i>0, \sum_{i=1}^n a_i = 1\ \Big\}\subset C_{\mathbb{R}}(\Gamma) \\ \mathcal{R}(f) &= \Big\{\ \sum_{i=1}^n a_i R_{\alpha_i} f\ \Big|\ n\in\mathbb{N}, \alpha_i\in\Gamma, a_i>0, \sum_{i=1}^n a_i = 1\ \Big\}\subset C_{\mathbb{R}}(\Gamma) \end{aligned}$$

We denote by $\overline{\mathcal{L}(f)}$ and $\overline{\mathcal{R}(f)}$ the closures of $\mathcal{L}(f)$ and $\mathcal{R}(f)$, respectively, in $C_{\mathbb{R}}(\Gamma)$. Since $L_\alpha L_\beta = L_{\alpha\beta}$, $R_\alpha R_\beta = R_{\alpha\beta}$ and $JL_\alpha = R_\alpha J$

$$\mathcal{L}(L_\gamma f)=\mathcal{L}(f) \qquad \mathcal{R}(R_\gamma f)=\mathcal{R}(f) \qquad J\mathcal{L}(f)=\mathcal{R}(Jf) \qquad \mathcal{L}(\lambda f)=\lambda\mathcal{L}(f) \qquad \mathcal{R}(\lambda f)=\lambda\mathcal{R}(f)$$

Taking closures

$$\overline{\mathcal{L}(L_\gamma f)}=\overline{\mathcal{L}(f)} \qquad \overline{\mathcal{R}(R_\gamma f)}=\overline{\mathcal{R}(f)} \qquad \overline{\mathcal{L}(\lambda f)}=\lambda\overline{\mathcal{L}(f)} \qquad J\overline{\mathcal{L}(f)}=\overline{\mathcal{R}(Jf)} \qquad \overline{\mathcal{R}(\lambda f)}=\lambda\overline{\mathcal{R}(f)} \tag{1}$$

for all $\gamma\in\Gamma$, $\lambda\in\mathbb{R}$ and $f\in C_{\mathbb{R}}(\Gamma)$. Furthermore all elements of $\overline{\mathcal{L}(f)}$ and $\overline{\mathcal{R}(f)}$ are nonnegative functions whenever f is nonnegative.

Step 1. We show that $\overline{\mathcal{L}(f)}$ *is a compact subset of* $C_{\mathbb{R}}(\Gamma)$.

Note first that

$$\Big\|\sum_{i=1}^n a_i L_{\alpha_i} f\Big\|_\infty \le \sum_{i=1}^n a_i\|L_{\alpha_i}f\|_\infty = \sum_{i=1}^n a_i\|f\|_\infty = \|f\|_\infty \tag{2}$$

In other words, $\overline{\mathcal{L}(f)}$ is a bounded subset of $C_{\mathbb{R}}(\Gamma)$ that is contained in the closed ball of radius $\|f\|_\infty$ centered on the origin. We next claim that $\mathcal{L}(f)$ is an equicontinuous subset of $C_{\mathbb{R}}(\Gamma)$. To see this, pick any $\varepsilon>0$

and choose, as in the proof of Theorem 9, a neighbourhood N_ε of e in Γ such that $|f(\alpha)-f(\beta)|<\varepsilon$ whenever $\alpha^{-1}\beta\in N_\varepsilon$. Then

$$\begin{aligned}\Big|\sum_{i=1}^n a_i(L_{\alpha_i}f)(\alpha)-\sum_{i=1}^n a_i(L_{\alpha_i}f)(\beta)\Big| &\le \sum_{i=1}^n a_i\big|(L_{\alpha_i}f)(\alpha)-(L_{\alpha_i}f)(\beta)\big|\\ &\le \sum_{i=1}^n a_i\big|f(\alpha_i^{-1}\alpha)-f(\alpha_i^{-1}\beta)\big| = \sum_{i=1}^n a_i\varepsilon=\varepsilon\end{aligned}$$

whenever $\alpha^{-1}\beta\in N_\varepsilon$ since $(\alpha_i^{-1}\alpha)^{-1}\alpha_i^{-1}\beta=\alpha^{-1}\beta\in N_\varepsilon$ for all $1\le i\le n$. Hence the closure $\overline{\mathcal{L}(f)}$ is also equicontinuous and is, by Arzelà–Ascoli, a compact subset of $C_{\mathbb{R}}(\Gamma)$.

Step 2. We show that $\overline{\mathcal{L}(f)}$ *contains a constant function.*

The function $v(f)$ is continuous on $C_{\mathbb{R}}(\Gamma)$ and in particular on the compact subset $\overline{\mathcal{L}(f)}$. Therefore $v(f)$ attains its minimum value at a point $f_*\in\overline{\mathcal{L}(f)}$. Either $v(f_*)=0$, and f_* is a constant function, or $v(f_*)\ne 0$ and $M(f_*)>m(f_*)$. We now show that the latter possibility cannot happen. Suppose that $M(f_*)>m(f_*)$. Then $\mathcal{F}=\big\{\ \gamma\in\Gamma\ \big|\ f_*(\gamma)>\frac{M(f_*)+m(f_*)}{2}\ \big\}$ is a nonempty subset of Γ. For each $\alpha\in\Gamma$, $\alpha\mathcal{F}=\big\{\ \alpha\gamma\ \big|\ \gamma\in\mathcal{F}\ \big\}$ is also open and $\{\alpha\mathcal{F}\}_{\alpha\in\Gamma}$ is an open cover of Γ. Since Γ is compact, $\Gamma\subset\alpha_1\mathcal{F}\cup\cdots\cup\alpha_n\mathcal{F}$, for some $n\in\mathbb{N}$ and $\alpha_1,\cdots,\alpha_n\in\Gamma$. Set

$$\tilde f_*(\gamma)=\tfrac{1}{n}\sum_{i=1}^n L_{\alpha_i}f_*(\gamma)$$

Then

- $\tilde f_*\in\overline{\mathcal{L}(f)}$ since $\frac{1}{n}\sum_{i=1}^n L_{\alpha_i}g\in\mathcal{L}(f)$ for all $g\in\mathcal{L}(f)$.
- $M(\tilde f_*)\le M(f_*)$
- $m(\tilde f_*)\ge m(f_*)+\frac{1}{2n}v(f_*)$. To see this, observe that, for any $\gamma\in\Gamma$, there is an $1\le m\le n$ such that $\gamma\in\alpha_m\mathcal{F}$. Then $\alpha_m^{-1}\gamma\in\mathcal{F}$ and, by the definition of $\mathcal{F}$, $f_*(\alpha_m^{-1}\gamma)>\frac{M(f_*)+m(f_*)}{2}$. Hence
$$\begin{aligned}\tilde f_*(\gamma)&=\tfrac{1}{n}f_*(\alpha_m^{-1}\gamma)+\tfrac{1}{n}\sum_{i\ne m}f_*(\alpha_i^{-1}\gamma)>\tfrac{1}{n}\tfrac{M(f_*)+m(f_*)}{2}+\tfrac{1}{n}\sum_{i\ne m}m(f_*)=\tfrac{1}{2n}M(f_*)+\tfrac{2n-1}{2n}m(f_*)\\ &=m(f_*)+\tfrac{1}{2n}v(f_*)\end{aligned}$$

This is true for all $\gamma\in\Gamma$, so $m(\tilde f_*)\ge m(f_*)+\frac{1}{2n}v(f_*)$.

Hence $v(\tilde f_*)=M(\tilde f_*)-m(\tilde f_*)\le M(f_*)-m(f_*)-\frac{1}{2n}v(f_*)<M(f_*)-m(f_*)=v(f_*)$ which is a contradiction. Consequently $v(f_*)=0$ and f_* is a constant function.

Step 3. We show that $\overline{\mathcal{R}(f)}$ *contains a constant function.*

To prove that $\overline{\mathcal{R}(f)}$ is compact, just imitate the argument of step 1. To prove that $\overline{\mathcal{R}(f)}$ contains a constant function, imitate the argument of step 2.

Step 4. We show that *there is a unique constant function* $c(f)\in\overline{\mathcal{L}(f)}\cup\overline{\mathcal{R}(f)}$.

It suffices to show that, if ℓ is any constant function in $\overline{\mathcal{L}(f)}$ and r is any constant function in $\overline{\mathcal{R}(f)}$, then $\ell=r$. Let $\varepsilon>0$. Choose $\alpha_i\in\Gamma, a_i>0$, $1\le i\le m$ and $\beta_j\in\Gamma, b_j>0$, $1\le j\le n$ such that $\sum_{i=1}^m a_i=\sum_{j=1}^n b_j=1$ and

$$\Big\|\ell-\sum_{i=1}^m a_iL_{\alpha_i}f\Big\|_\infty\le\tfrac{\varepsilon}{2}\qquad\Big\|r-\sum_{j=1}^n b_jR_{\beta_j}f\Big\|_\infty\le\tfrac{\varepsilon}{2}$$

In particular, for each $1\le j\le n$, $\sup\limits_{\gamma\in\Gamma}\big|\ell-\sum_{i=1}^m a_if\big(\alpha_i^{-1}\gamma\beta_j\big)\big|\le\frac{\varepsilon}{2}$. Hence

$$\begin{aligned}\Big|\ell-\sum_{j=1}^n\sum_{i=1}^m a_ib_jf\big(\alpha_i^{-1}\gamma\beta_j\big)\Big| &= \Big|\sum_{j=1}^n \ell\, b_j-\sum_{j=1}^n\sum_{i=1}^m a_ib_jf\big(\alpha_i^{-1}\gamma\beta_j\big)\Big|\\ &\le\sum_{j=1}^n b_j\Big|\ell-\sum_{i=1}^m a_if\big(\alpha_i^{-1}\gamma\beta_j\big)\Big|\\ &\le\sum_{j=1}^n b_j\tfrac{\varepsilon}{2}=\tfrac{\varepsilon}{2}\end{aligned}$$

Similarly,

$$\Big| r - \sum_{j=1}^{n}\sum_{i=1}^{m} a_i b_j f\big(\alpha_i^{-1}\gamma\beta_j\big)\Big| \le \tfrac{\varepsilon}{2}$$

By the triangle inequality $|\ell - r| \le \varepsilon$. Since this is true for all $\varepsilon > 0$, $\ell = r$.

Step 5. *Existence and properties of E.* Define $E(f) = c(f)$. By (1) and (2)

$$\big|E(f)\big| \le \|f\|_\infty \quad E(1) = 1 \quad E(\lambda f) = \lambda E(f) \quad E(L_\alpha f) = E(f) \quad E(R_\alpha f) = E(f) \quad E(Jf) = E(f) \tag{3}$$

for all $f \in C_{\rm I\!R}(\Gamma)$, $\lambda \in {\rm I\!R}$ and $\alpha \in \Gamma$. Furthermore $E(f) \ge 0$ whenever $f \ge 0$. The only property of E that remains to be proven is $E(f_1 + f_2) = E(f_1) + E(f_2)$, for all $f_1, f_2 \in C_{\rm I\!R}(\gamma)$. Select $\alpha_i \in \Gamma, a_i > 0$, $1 \le i \le m$ such that $\sum_{i=1}^{m} a_i = 1$ and

$$\big|E(f_1) - \sum_{i=1}^{m} a_i L_{\alpha_i} f_1(\gamma)\big| \le \tfrac{\varepsilon}{2} \tag{4}$$

for all $\gamma \in \Gamma$. Denote $\phi = \sum_{i=1}^{m} a_i L_{\alpha_i} f_2$. Since $\phi \in \mathcal{L}(f_2)$, we have that $\mathcal{L}(\phi) \subset \mathcal{L}(f_2)$ so that $\overline{\mathcal{L}(\phi)} \subset \overline{\mathcal{L}(f_2)}$ and $E(\phi) = E(f_2)$. Select $\beta_j \in \Gamma, b_j > 0$, $1 \le j \le n$ such that $\sum_{j=1}^{n} b_j = 1$ and

$$\big\|E(f_2) - \sum_{j=1}^{n} b_j L_{\beta_j}\phi\big\|_\infty \le \tfrac{\varepsilon}{2}$$

By the definition of ϕ,

$$\begin{aligned}
\Big|E(f_2) - \sum_{j=1}^{n}\sum_{i=1}^{m} a_i b_j f_2\big(\alpha_i^{-1}\beta_j^{-1}\gamma\big)\Big| &= \Big|E(f_2) - \sum_{j=1}^{n}\sum_{i=1}^{m} a_i b_j L_{\beta_j\alpha_i} f_2\big(\gamma\big)\Big| \\
&= \Big|E(f_2) - \sum_{j=1}^{n} b_j L_{\beta_j} \sum_{i=1}^{m} a_i L_{\alpha_i} f_2\big(\gamma\big)\Big| \\
&= \Big|E(f_2) - \sum_{j=1}^{n} b_j L_{\beta_j}\phi(\gamma)\Big| \le \tfrac{\varepsilon}{2}
\end{aligned}$$

Substituting $\beta_j^{-1}\gamma$ for γ in (4) gives

$$\big|E(f_1) - \sum_{i=1}^{m} a_i L_{\alpha_i} f_1(\beta_j^{-1}\gamma)\big| \le \tfrac{\varepsilon}{2} \quad\Longrightarrow\quad \Big|E(f_1) - \sum_{j=1}^{n}\sum_{i=1}^{m} a_i b_j f_1\big(\alpha_i^{-1}\beta_j^{-1}\gamma\big)\Big| \le \tfrac{\varepsilon}{2}$$

By the triangle inequality

$$\Big|E(f_1) + E(f_2) - \sum_{j=1}^{n}\sum_{i=1}^{m} a_i b_j (f_1 + f_2)\big(\alpha_i^{-1}\beta_j^{-1}\gamma\big)\Big| \le \varepsilon$$

Since $\sum_{j=1}^{n}\sum_{i=1}^{m} a_i b_j (f_1 + f_2)\big(\alpha_i^{-1}\beta_j^{-1}\gamma\big) \in \mathcal{L}(f_1 + f_2)$, we have that $E(f_1) + E(f_2) \in \overline{\mathcal{L}(f_1 + f_2)}$. Since there is only one constant function in $\overline{\mathcal{L}(f_1 + f_2)}$, we also have that $E(f_1 + f_2) = E(f_1) + E(f_2)$.

Step 6. *Uniqueness of E.* Let E' be any continuous linear functional on $C_{\rm I\!R}(\Gamma)$ obeying $E'(1) = 1$ and $E'(f) = E'(L_\alpha f)$ for all $f \in C_{\rm I\!R}(\Gamma)$ and $\alpha \in \Gamma$. By linearity, $E'(\phi) = E'(f)$ for all $\phi \in \mathcal{L}(f)$. By continuity, $E'(\phi) = E'(f)$ for all $\phi \in \overline{\mathcal{L}(f)}$. In particular, subbing $\phi = E(f)$, $E(f) = E'(E(f)) = E'(f)$.

Step 7. *Existence and properties of $d\mu$.* Define μ and Σ as in the Riesz representation theorem. Recall that we defined μ^* on open sets by

$$\mu^*(V) = \sup\big\{\ E(f)\ \big|\ f \in C_{\rm I\!R}(\Gamma), 0 \le f \le 1, \operatorname{supp} f \subset V\ \big\}$$

Since $E(1) = 1$, $\mu^*(\Gamma) = 1$. Since $E(L_\gamma f) = E(R_\gamma f) = E(Jf)$ and since γV, $V\gamma^{-1}$ and V^{-1} are all open when V is open, it is clear that

$$\mu^*(\gamma V) = \mu^*(V\gamma^{-1}) = \mu^*(V^{-1})$$

for all open $V \subset \Gamma$. We then defined μ^* on all subset of Γ by

$$\mu^*(S) = \inf \left\{ \ \mu^*(V) \ \middle| \ S \subset V, V \text{ open } \right\}$$

and proved that this was an outer measure. So

$$\mu^*(\gamma S) = \mu^*(S\gamma^{-1}) = \mu^*(S^{-1})$$

Defining Σ to be the σ–algebra of μ^*–measurable sets and μ to be the restriction of μ^* to Σ does it. ∎

Theorem 18 (Arzelà–Ascoli) *Let X be a compact Hausdorff space. If $\mathcal{F}$ is an equicontinuous, pointwise bounded subset of $C(X)$, then $\mathcal{F}$ is totally bounded in the uniform metric and the closure of $\mathcal{F}$ is compact. That is, every sequence in $\overline{\mathcal{F}}$ contains a subsequence that converges uniformly to a point of $\overline{\mathcal{F}}$.*

Proof: Let $\varepsilon > 0$. Since $\mathcal{F}$ is equicontinuous, there is, for each $x \in X$ an open neighbourhood U_x of x such that $|f(x) - f(y)| < \frac{\varepsilon}{4}$ for all $y \in U_x$ and all $f \in \mathcal{F}$. Since X is compact, we can choose $A = \{x_1, \cdots, x_n\} \subset X$ such that $\cup_{i=1}^n U_{x_i} = X$. By pointwise boundedness $B = \left\{ \ f(x_j) \ \middle| \ 1 \le j \le n, \ f \in \mathcal{F} \ \right\}$ is a bounded subset of $\mathbb{C}$. So there exists a finite subset $\{z_1, \cdots, z_m\} \subset \mathbb{C}$ such that every $f(x_j)$ is within a distance $\frac{\varepsilon}{4}$ of some z_i. The set $F(A, B)$ of functions from A to B is finite. For each $\phi \in F(A, B)$, let

$$\mathcal{F}_\phi = \left\{ \ f \in \mathcal{F} \ \middle| \ |f(x_j) - \phi(x_j)| < \tfrac{\varepsilon}{4} \text{ for } 1 \le j \le n \ \right\}$$

If $f, g \in \mathcal{F}_\phi$, then for each $x \in X$, there is a $1 \le j \le n$ such that $x \in U_{x_j}$ and

$$|f(x) - g(x)| \le |f(x) - f(x_j)| + |f(x_j) - \phi(x_j)| + |\phi(x_j) - g(x_j)| + |g(x_j) - g(x)| < \varepsilon$$

Hence $\cup_{\phi \in F(A,B)} \mathcal{F}_\phi$ is a finite cover of $\mathcal{F}$ by sets of diameter at most ε. This is the definition of total boundedness. Since the closure of a totally bounded set is totally bounded and $C(X)$ is complete, $\overline{\mathcal{F}}$ is compact. (See Folland, page 15.) ∎

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