Theorem (Riesz Representation Theorem) *Let X be a locally compact Hausdorff space and $\Lambda : C_0(X) \to \mathbb{R}$ or $\mathbb{C}$ be a positive linear functional. Then there exists a σ-algebra*$^{\text{step 7}}$ $\mathcal{M}$ *in* X, *which contains all Borel sets*$^{\text{step 7}}$, *and a unique measure* μ *on* $\mathcal{M}$ *such that*

a) $\Lambda f = \int_X f d\mu \qquad \forall f \in C_0(X)$ step10

b) $\mu(K) < \infty \qquad \forall$ compact $K \subset X$ step2

c) *If* $E \in \mathcal{M}$ *then*

$$\mu(E) = \inf\left\{ \mu(V) \;\middle|\; E \subset V,\ V \text{ open} \right\} \qquad \text{definition}$$

d) *If* E *is open*$^{\text{step 3}}$ *or if* $E \in \mathcal{M}$ *with* $\mu(E) < \infty^{\text{step 8}}$ *then*

$$\mu(E) = \sup\left\{ \mu(K) \;\middle|\; K \subset E,\ K \text{ compact} \right\}$$

e) $\langle X, \mathcal{M}, \mu\rangle$ *is a complete measure space. That is, if* $E \in \mathcal{M}$, $A \subset E$, $\mu(E) = 0 \Rightarrow A \in \mathcal{M}$ step 8 and definition of $\mathcal{M}_{\mathrm{F}}$

Formulae for μ^*

V open $\Rightarrow \mu^*(V) = \sup\left\{ \Lambda f \;\middle|\; f \prec V \right\}$ definition

E arbitrary $\Rightarrow \mu^*(E) = \inf\left\{ \mu^*(V) \;\middle|\; V \supset E, V \text{ open} \right\}$ definition

K compact $\Rightarrow \mu^*(K) = \inf\left\{ \Lambda f \;\middle|\; f \succ K \right\}$ after step 2

V open $\Rightarrow \mu^*(V) = \sup\left\{ \mu^*(K) \;\middle|\; K \subset V,\ K \text{ compact} \right\}$ after step 3

$E \in \mathcal{M}_F$ $\Rightarrow \mu^*(E) = \sup\left\{ \mu^*(K) \;\middle|\; K \subset E,\ K \text{ compact} \right\}$ definition

Outline

Definitions:

a) For all open $V \subset X$, define $\mu^*(V) = \sup\left\{ \Lambda f \;\middle|\; f \prec V \right\}$.

b) For all $E \subset X$, define $\mu^*(E) = \inf\left\{ \mu^*(V) \;\middle|\; E \subset V,\ V \text{ open } \right\}$.

c) $\mathcal{M}_F = \Big\{ E \subset X \;\Big|\; \mu^*(E) < \infty$ and

$$\mu^*(E) = \sup\{\mu^*(K) \,|\, K \subset E,\ K \text{compact}\}\Big\}$$

d) $\mathcal{M} = \left\{ E \subset X \;\middle|\; E \cap K \in \mathcal{M}_F \text{ for all compact } K \right\}$

e) μ is the restriction of μ^* to $\mathcal{M}$.

Step 1. μ^* is an outer measure.

Step 2. K compact $\implies K \in \mathcal{M}_F$ and $\mu^*(K) = \inf\left\{ \Lambda f \;\middle|\; K \prec f \right\}$

Step 3. V open $\implies \mu^*(V) = \sup\left\{ \mu^*(K) \;\middle|\; K \subset V,\ K \text{ compact } \right\}$

Step 4. If $E_i \in \mathcal{M}_F$, $1 \le i < \infty$ pairwise disjoint

$$\implies \mu^*\big(\cup_{i=1}^\infty E_i \big) = \textstyle\sum_{i=1}^\infty \mu^*(E_i).$$

If, in addition, $\mu^*\big(\cup_{i=1}^\infty E_i \big) < \infty$, then $\cup_{i=1}^\infty E_i \in \mathcal{M}_F$.

Step 5. If $E \in \mathcal{M}_F$ and $\varepsilon > 0$, then there exists K, compact and V, open, such that $K \subset E \subset V$ and $\mu * (V \setminus K) < \varepsilon$.

Step 6. If $A, B \in \mathcal{M}_F$, then $A \setminus B, A \cup B, A \cap B \in \mathcal{M}_F$

Step 7. $\mathcal{M}$ is a σ–algebra and contains all Borel sets.

Step 8. $\mathcal{M}_F = \left\{ E \in \mathcal{M} \;\middle|\; \mu^*(E) < \infty \right\}$

Step 9. μ is a measure

Step 10. $f \in C_0(X) \implies \Lambda f = \int_X f \; d\mu$.