A Lie Group

These notes introduce $SU(2)$ as an example of a compact Lie group.

The Definition

The definition of $SU(2)$ is

$$SU(2) = \left\{ A \;\middle|\; A \text{ a } 2\times 2 \text{ complex matrix}, \det A = 1, AA^* = A^*A = 1\!\!1 \right\}$$

In the name $SU(2)$, the "S" stands for "special" and refers to the condition $\det A = 1$ and the "U" stands for "unitary" and refers to the conditions $AA^* = A^*A = 1\!\!1$. The adjoint matrix A^* is the complex conjugate of the transpose matrix. That is,

$$\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}^* = \begin{bmatrix}\bar\alpha & \bar\gamma\\ \bar\beta & \bar\delta\end{bmatrix}$$

Define the inner product on $\mathbb{C}^2$ by

$$\left\langle \begin{bmatrix}a_1\\ a_2\end{bmatrix}, \begin{bmatrix}b_1\\ b_2\end{bmatrix}\right\rangle = a_1\bar b_1 + a_2\bar b_2$$

The adjoint matrix was defined so that

$$\sum_{i,j=1}^{2} A_{i,j}a_j\bar b_i = \left\langle A\begin{bmatrix}a_1\\ a_2\end{bmatrix}, \begin{bmatrix}b_1\\ b_2\end{bmatrix}\right\rangle = \left\langle \begin{bmatrix}a_1\\ a_2\end{bmatrix}, A^*\begin{bmatrix}b_1\\ b_2\end{bmatrix}\right\rangle = \sum_{i,j=1}^{2} a_j\overline{A^*_{j,i}b_i}$$

Thus the condition $A^*A = 1\!\!1$ is equivalent to

$$\begin{aligned} &\left\langle A^*A\begin{bmatrix}a_1\\ a_2\end{bmatrix}, \begin{bmatrix}b_1\\ b_2\end{bmatrix}\right\rangle = \left\langle \begin{bmatrix}a_1\\ a_2\end{bmatrix}, \begin{bmatrix}b_1\\ b_2\end{bmatrix}\right\rangle \quad \text{for all } \begin{bmatrix}a_1\\ a_2\end{bmatrix}, \begin{bmatrix}b_1\\ b_2\end{bmatrix} \in \mathbb{C}^2\\ \iff &\left\langle A\begin{bmatrix}a_1\\ a_2\end{bmatrix}, A\begin{bmatrix}b_1\\ b_2\end{bmatrix}\right\rangle = \left\langle \begin{bmatrix}a_1\\ a_2\end{bmatrix}, \begin{bmatrix}b_1\\ b_2\end{bmatrix}\right\rangle \quad \text{for all } \begin{bmatrix}a_1\\ a_2\end{bmatrix}, \begin{bmatrix}b_1\\ b_2\end{bmatrix} \in \mathbb{C}^2 \end{aligned}$$

Hence $SU(2)$ is the set of 2×2 complex matrices that have determinant one and preserve the inner product on $\mathbb{C}^2$. (Recall that, for square matrices, $A^*A = 1\!\!1$ is equivalent to $A^{-1} = A^*$, which in turn is equivalent to $AA^* = 1\!\!1$.) By the polarization identity (Problem Set V, #3), preservation of the inner product is equivalent to preservation of the norm

$$\left\| A\begin{bmatrix}a_1\\ a_2\end{bmatrix}\right\| = \left\| \begin{bmatrix}a_1\\ a_2\end{bmatrix}\right\| \quad \text{for all } \begin{bmatrix}a_1\\ a_2\end{bmatrix} \in \mathbb{C}^2$$

Clearly $1\!\!1 \in SU(2)$. If $A, B \in SU(2)$, then $\det(AB) = \det(A)\det(B) = 1$ and $(AB)(AB)^* = ABB^*A^* = A1\!\!1A^* = 1\!\!1$ so that $AB \in SU(2)$. Also, if $A \in SU(2)$, then $A^{-1} = A^* \in SU(2)$. So $SU(2)$ is a group. We may also view $SU(2)$ as a subset of $\mathbb{C}^4$. Then $SU(2)$ inherits a topology from $\mathbb{C}^4$, so that $SU(2)$ is a topological group.

The Pauli Matrices

The matrices

$$\sigma_1 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \qquad \sigma_2 = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} \qquad \sigma_3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

are called the Pauli matrices. They obey $\sigma_\ell = \sigma_\ell^*$ for all $\ell = 1, 2, 3$ and also obey

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1\!\!1 \qquad \sigma_1\sigma_2 = -\sigma_2\sigma_1 = i\sigma_3 \qquad \sigma_2\sigma_3 = -\sigma_3\sigma_2 = i\sigma_1 \qquad \sigma_3\sigma_1 = -\sigma_1\sigma_3 = i\sigma_2 \tag{1}$$

Set, for each $\vec{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$, the matrix

$$\vec{a} \cdot \vec{\sigma} = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$$

Then the product rules (1) can be written

$$\big(\vec{a} \cdot \vec{\sigma}\big)\big(\vec{b} \cdot \vec{\sigma}\big) = \vec{a} \cdot \vec{b}\,\mathbb{1} + i\vec{a} \times \vec{b} \cdot \vec{\sigma} \tag{2}$$

I claim that any 2×2 complex matrix has a unique representation of the form $a_0\mathbb{1}+ia_1\sigma_1+ia_2\sigma_2+ia_3\sigma_3$ for some $a_0, a_1, a_2, a_3 \in \mathbb{C}$. This is easy to see. Since

$$a_0\mathbb{1} + ia_1\sigma_1 + ia_2\sigma_2 + ia_3\sigma_3 = \begin{bmatrix} a_0 + ia_3 & ia_1 + a_2 \\ ia_1 - a_2 & a_0 - ia_3 \end{bmatrix}$$

we have that

$$a_0\mathbb{1} + ia_1\sigma_1 + ia_2\sigma_2 + ia_3\sigma_3 = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \iff a_0 = \tfrac{\alpha+\delta}{2},\ a_1 = \tfrac{\beta+\gamma}{2i},\ a_2 = \tfrac{\beta-\gamma}{2},\ a_3 = \tfrac{\alpha-\delta}{2i}$$

Lemma.

$$SU(2) = \Big\{\ x_0\mathbb{1} + i\vec{x} \cdot \vec{\sigma}\ \Big|\ (x_0, \vec{x}) \in \mathbb{R}^4,\ x_0^2 + \|\vec{x}\|^2 = 1\ \Big\}$$

Proof: Let A be any 2×2 complex matrix and write $A = a_0\mathbb{1} + i\vec{a} \cdot \vec{\sigma}$ with $\vec{a} = (a_1, a_2, a_3)$. Then by (2)

$$\begin{aligned} AA^* &= \big(a_0\mathbb{1} + i\vec{a} \cdot \vec{\sigma}\big)\big(\overline{a_0}\mathbb{1} - i\overline{\vec{a}} \cdot \vec{\sigma}\big) \\ &= |a_0|^2\mathbb{1} + i\overline{a_0}\vec{a} \cdot \vec{\sigma} - ia_0\overline{\vec{a}} \cdot \vec{\sigma} + \vec{a} \cdot \overline{\vec{a}}\mathbb{1} + i\vec{a} \times \overline{\vec{a}} \cdot \vec{\sigma} \\ &= \big(|a_0|^2 + \|\vec{a}\|^2\big)\mathbb{1} + i\big(\overline{a_0}\vec{a} - a_0\overline{\vec{a}} + \vec{a} \times \overline{\vec{a}}\big) \cdot \vec{\sigma} \end{aligned}$$

Hence

$$AA^* = \mathbb{1} \iff |a_0|^2 + \|\vec{a}\|^2 = 1,\ \overline{a_0}\vec{a} - a_0\overline{\vec{a}} + \vec{a} \times \overline{\vec{a}} = 0$$

First, suppose that $\vec{a} \neq \vec{0}$. Since $\vec{a} \times \overline{\vec{a}}$ is orthogonal to both $\vec{a}$ and $\overline{\vec{a}}$, the equation $\overline{a_0}\vec{a} - a_0\overline{\vec{a}} + \vec{a} \times \overline{\vec{a}} = 0$ can be satisfied only if $\vec{a} \times \overline{\vec{a}} = 0$. That is, only if $\vec{a}$ and $\overline{\vec{a}}$ are parallel. Since $\vec{a}$ and $\overline{\vec{a}}$ have the same length, this is the case only if $\overline{\vec{a}} = e^{-2i\theta}\vec{a}$ for some real number θ. This is equivalent to $\overline{e^{-i\theta}\vec{a}} = e^{-i\theta}\vec{a}$ which says that $\vec{x} = e^{-i\theta}\vec{a}$ is real. Subbing $\vec{a} = e^{i\theta}\vec{x}$ back into $\overline{a_0}\vec{a} - a_0\overline{\vec{a}} + \vec{a} \times \overline{\vec{a}} = 0$ gives

$$e^{i\theta}\overline{a_0}\vec{x} - e^{-i\theta}a_0\vec{x} = 0$$

This forces $a_0 = e^{i\theta}x_0$ for some real x_0. If $\vec{a} = \vec{0}$, we may still choose θ so that $a_0 = e^{i\theta}x_0$. We have now shown that

$$AA^* = \mathbb{1} \iff A = e^{i\theta}\big(x_0\mathbb{1} + i\vec{x} \cdot \vec{\sigma}\big) \text{ for some } (x_0, \vec{x}) \in \mathbb{R}^4 \text{ with } |x_0|^2 + \|\vec{x}\|^2 = 1 \text{ and some } \theta \in \mathbb{R}$$

Since

$$\det A = \det e^{i\theta}\big(x_0\mathbb{1} + i\vec{x} \cdot \vec{\sigma}\big) = \det e^{i\theta}\begin{bmatrix} x_0 + ix_3 & ix_1 + x_2 \\ ix_1 - x_2 & x_0 - ix_3 \end{bmatrix} = e^{2i\theta}(x_0^2 + x_1^2 + x_2^2 + x_3^2) = e^{2i\theta}$$

we have that $\det A = 1$ if and only if $e^{i\theta} = \pm1$. If $e^{i\theta} = -1$, we can absorb the -1 into $(x_0, \vec{x})$. ■

As consequences of this Lemma we have that $SU(2)$ is

- ∘ homeomorphic to S^3, the unit sphere in $\mathbb{R}^4$
- ∘ connected
- ∘ simply connected (meaning that every continuous closed curve in $SU(2)$ can be continuously deformed to a point)
- ∘ is a C^∞ manifold (meaning, roughly speaking, that in a neighbourhood of each point, we may choose three of x_0, x_1, x_2, x_3 as coordinates, with the fourth then determined as a C^∞ function of the chosen three)

A topological group that is also a C^∞ manifold (with the maps $(a,b) \mapsto ab$ and $a \mapsto a^{-1}$ C^∞ when expressed in local coordinates) is called a Lie Group.

The Connection between $SU(2)$ and $SO(3)$

Define

$$\begin{aligned} M : \mathbb{R}^3 \to V = \left\{ \ \vec{a}\cdot\vec{\sigma} \ \middle| \ \vec{a} \in \mathbb{R}^3 \ \right\} \subset \{2\times 2 \text{ complex matrices}\} \\ \vec{a} \mapsto M(\vec{a}) = \vec{a}\cdot\vec{\sigma} \end{aligned}$$

This is a linear bijection between $\mathbb{R}^3$ and V. (In fact V is the space of all 2×2 traceless, self–adjoint matrices.)

Each $U \in SU(2)$ determines a linear map $S(U)$ on $\mathbb{R}^3$ by

$$M\big(S(U)\vec{a}\big) = UM(\vec{a})U^{-1} \tag{3}$$

The right hand side is clearly linear in $\vec{a}$. But it is not so clear that $UM(\vec{a})U^{-1}$ is in V, that is, of the form $M(\vec{b})$. To check this, we let $U = x_0\mathbb{1} + i\vec{x}\cdot\vec{\sigma}$ with $(x_0,\vec{x}) \in \mathbb{R}^4$ obeying $\|x_0\|^2 + \|\vec{x}\|^2 = 1$ and compute $UM(\vec{a})U^{-1} = UM(\vec{a})U^*$ explicitly. Applying (2) twice

$$\begin{aligned} UM(\vec{a})U^{-1} &= \big(x_0\mathbb{1} + i\vec{x}\cdot\vec{\sigma}\big)\big(\vec{a}\cdot\vec{\sigma}\big)\big(x_0\mathbb{1} - i\vec{x}\cdot\vec{\sigma}\big) \\ &= \big(x_0\mathbb{1} + i\vec{x}\cdot\vec{\sigma}\big)\big(x_0\vec{a}\cdot\vec{\sigma} - i\vec{a}\cdot\vec{x}\mathbb{1} + \vec{a}\times\vec{x}\cdot\vec{\sigma}\big) \\ &= x_0^2\vec{a}\cdot\vec{\sigma} - ix_0\vec{a}\cdot\vec{x}\mathbb{1} + x_0\vec{a}\times\vec{x}\cdot\vec{\sigma} \\ &\qquad + ix_0\vec{x}\cdot\vec{a}\mathbb{1} - x_0\vec{x}\times\vec{a}\cdot\vec{\sigma} + \vec{a}\cdot\vec{x}\,\vec{x}\cdot\vec{\sigma} + i\vec{x}\cdot(\vec{a}\times\vec{x})\mathbb{1} - \vec{x}\times(\vec{a}\times\vec{x})\cdot\vec{\sigma} \\ &= x_0^2\vec{a}\cdot\vec{\sigma} + 2x_0\vec{a}\times\vec{x}\cdot\vec{\sigma} + \vec{a}\cdot\vec{x}\,\vec{x}\cdot\vec{\sigma} - \vec{x}\times(\vec{a}\times\vec{x})\cdot\vec{\sigma} \end{aligned}$$

since $\vec{x}$ is perpendicular to $(\vec{a}\times\vec{x})$. Using $\vec{c}\times(\vec{a}\times\vec{b}) = (\vec{b}\cdot\vec{c})\vec{a} - (\vec{a}\cdot\vec{c})\vec{b}$,

$$\begin{aligned} UM(\vec{a})U^{-1} &= x_0^2\vec{a}\cdot\vec{\sigma} + 2x_0\vec{a}\times\vec{x}\cdot\vec{\sigma} + \vec{a}\cdot\vec{x}\,\vec{x}\cdot\vec{\sigma} - \|\vec{x}\|^2\vec{a}\cdot\vec{\sigma} + \vec{a}\cdot\vec{x}\,\vec{x}\cdot\vec{\sigma} \\ &= \big(x_0^2 - \|\vec{x}\|^2\big)\vec{a}\cdot\vec{\sigma} + 2x_0\vec{a}\times\vec{x}\cdot\vec{\sigma} + 2\vec{a}\cdot\vec{x}\,\vec{x}\cdot\vec{\sigma} \end{aligned}$$

This shows, not only that $UM(\vec{a})U^{-1} \in V$, but also that, for $U = x_0\mathbb{1} + i\vec{x}\cdot\vec{\sigma}$,

$$S(U)\vec{a} = \big(x_0^2 - \|\vec{x}\|^2\big)\vec{a} - 2x_0\vec{x}\times\vec{a} + 2\vec{a}\cdot\vec{x}\,\vec{x}$$

In fact, we can exactly identify the geometric operation that $S(U)$ implements. If $U = \pm\mathbb{1}$, that is $\vec{x} = \vec{0}$, then it is obvious from (3) that $S(U)\vec{a} = \vec{a}$ for all $\vec{a} \in \mathbb{R}^3$. That is, both $S(\mathbb{1})$ and $S(-\mathbb{1})$ are the identity map on $\mathbb{R}^3$. If $\vec{x} \neq \vec{0}$, there is a unique angle $0 < \theta < 2\pi$ and a unique unit vector $\hat{e}$ such that $x_0 = \cos(\frac{\theta}{2})$ and $\vec{x} = -\sin(\frac{\theta}{2})\,\hat{e}$. If $\vec{a}$ happens to be parallel to $\vec{x}$, that is, $\vec{a} = c\,\vec{x}$,

$$S(U)\vec{a} = \big(x_0^2 - \|\vec{x}\|^2\big)\vec{a} + 2\vec{a}\cdot\vec{x}\,\vec{x} = \big(x_0^2 + \|\vec{x}\|^2\big)\vec{a} = \vec{a}$$

So $S(U)$ leaves the axis $\vec{x}$ invariant. If $\vec{a}$ is not parallel to $\vec{x}$, set

$$\hat{k} = \hat{e} \qquad \hat{\imath} = \tfrac{\vec{a} - \vec{a}\cdot\hat{k}\,\hat{k}}{\|\vec{a} - \vec{a}\cdot\hat{k}\,\hat{k}\|} \qquad \hat{\jmath} = \hat{k}\times\hat{\imath}$$

This is an orthonormal basis for $\mathbb{R}^3$. Since $\vec{a}$ is a linear combination of $\hat{\imath}$ and $\hat{k}$,

$$\vec{a} = \vec{a}\cdot\hat{k}\;\hat{k} + \vec{a}\cdot\hat{\imath}\;\hat{\imath}$$

In terms of this notation

$$\begin{aligned} S(U)\vec{a} &= \cos(\theta)\,\vec{a} + \sin(\theta)\,\hat{e}\times\vec{a} + 2\sin^2(\tfrac{\theta}{2})\,\vec{a}\cdot\hat{e}\,\hat{e} \\ &= \vec{a}\cdot\hat{e}\;\hat{e} + \cos(\theta)\,(\vec{a} - \vec{a}\cdot\hat{e}\,\hat{e}) + \sin(\theta)\,\hat{e}\times\vec{a} \end{aligned}$$

Since

$$\begin{aligned} \vec{a} - \vec{a}\cdot\hat{e}\;\hat{e} &= \vec{a} - \vec{a}\cdot\hat{k}\;\hat{k} = \vec{a}\cdot\hat{\imath}\;\hat{\imath} \\ \hat{e}\times\vec{a} &= \hat{k}\times\vec{a} = \hat{k}\times\big(\vec{a}\cdot\hat{k}\;\hat{k} + \vec{a}\cdot\hat{\imath}\;\hat{\imath}\big) = \vec{a}\cdot\hat{\imath}\;\hat{\jmath} \end{aligned}$$

we have

$$S(U)\vec{a} = \vec{a}\cdot\hat{k}\;\hat{k} + \cos(\theta)\,\vec{a}\cdot\hat{\imath}\;\hat{\imath} + \sin(\theta)\,\vec{a}\cdot\hat{\imath}\;\hat{\jmath}$$

In particular

$$S(U)\hat{k} = \hat{k} \qquad S(U)\hat{\imath} = \cos(\theta)\,\hat{\imath} + \sin(\theta)\,\hat{\jmath}$$

This is exactly the rotation of $\vec{a}$ about the axis $\hat{k} = \hat{e}$ (the $\hat{k}$ component of $\vec{a}$ is unchanged) by an angle θ (the part of $\vec{a}$ perpendicular to $\hat{k}$ has changed by a rotation by θ as in $\mathbb{R}^2$). This shows that

$$S : SU(2) \to SO(3)$$

that S is surjective and that $S(U) = \mathbb{1}_3$, the identity map on $\mathbb{R}^3$, if and only if $U = \pm\mathbb{1}$. Also, by (3),

$$M\big(S(UU')\vec{a}\big) = UU'M(a){U'}^{-1}U^{-1} = U\,M\big(S(U')\vec{a}\big)\,U^{-1} = M\big(S(U)S(U')\vec{a}\big)$$

so that $S(UU') = S(U)S(U')$ and S is a homomorphism. It is not injective, since $S(-\mathbb{1}) = S(\mathbb{1})$. Indeed S is a two to one map since

$$S(U) = S(\tilde{U}) \iff S(U)S(\tilde{U})^{-1} = \mathbb{1}_3 \iff S\big(U\tilde{U}^{-1}\big) = \mathbb{1}_3 \iff U\tilde{U}^{-1} = \pm\mathbb{1} \iff U = \pm\tilde{U}$$

We have now shown that $SO(3)$ is isomorphic to $SU(2)/\{\mathbb{1}, -\mathbb{1}\}$.

The Haar Measure

Recall that

$$SU(2) = \big\{\ x_0\mathbb{1} + i\vec{x}\cdot\vec{\sigma}\ \big|\ (x_0,\vec{x})\in\mathbb{R}^4,\ x_0^2 + \|\vec{x}\|^2 = 1\ \big\}$$

For all $x_1^2 + x_2^2 + x_3^2 < 1$, $x_0 > 0$, we can use $\vec{x}$ as coordinates with $x_0(\vec{x}) = \sqrt{1 - x_1^2 - x_2^2 - x_3^2}$. For all $x_1^2 + x_2^2 + x_3^2 < 1$, $x_0 < 0$, we can use $\vec{x}$ as coordinates with $x_0(\vec{x}) = -\sqrt{1 - x_1^2 - x_2^2 - x_3^2}$. This leaves only $x_1^2 + x_2^2 + x_3^2 = 1$, $x_0 = 0$. We could cover this using other components as coordinates, but as this is a set of measure zero, we won't bother. Denote

$$\begin{aligned} \gamma_+(\vec{x}) &= \ \ \sqrt{1 - x_1^2 - x_2^2 - x_3^2}\;\mathbb{1} + i\vec{x}\cdot\vec{\sigma} \\ \gamma_-(\vec{x}) &= -\sqrt{1 - x_1^2 - x_2^2 - x_3^2}\;\mathbb{1} + i\vec{x}\cdot\vec{\sigma} \end{aligned}$$

We shall now find two functions $\Delta_+(\vec{x})$ and $\Delta_-(\vec{x})$ such that, for all continuous functions f on $SU(2)$

$$\int_{SU(2)} f(\gamma)\,d\mu(\gamma) = \iiint_{\|\vec{x}\|<1} f(\gamma_+(\vec{x}))\,\Delta_+(\vec{x})\,d^3\vec{x} + \iiint_{\|\vec{x}\|<1} f(\gamma_-(\vec{x}))\,\Delta_-(\vec{x})\,d^3\vec{x}$$

where μ is the Haar measure on $SU(2)$.

Define $\vec{z}_+(\vec{y},\vec{x})$ and $\vec{z}_-(\vec{y},\vec{x})$ by

$$\gamma_+\big(\vec{z}_+(\vec{y},\vec{x})\big) = \gamma_+(\vec{y})\gamma_+(\vec{x}) \qquad \gamma_-\big(\vec{z}_-(\vec{y},\vec{x})\big) = \gamma_-(\vec{y})\gamma_+(\vec{x})$$

If you multiply an element of the interior of the upper hemisphere of $SU(2)$ (like $\gamma_+(\vec{y})$ with $\|\vec{y}\| < 1$) by an element of $SU(2)$ that is sufficiently close to the identity (like $\gamma_+(\vec{x})$ with $\|\vec{x}\| \ll 1$) you end up with another element of the interior of the upper hemisphere. Similarly, if you multiply an element of the interior of the lower hemisphere of $SU(2)$ (like $\gamma_-(\vec{y})$ with $\|\vec{y}\| < 1$) by an element of $SU(2)$ that is sufficiently close to the identity (like $\gamma_+(\vec{x})$ with $\|\vec{x}\| \ll 1$) you end up with another element of the interior of the lower hemisphere. Thus both $\vec{z}_+(\vec{y},\vec{x})$ and $\vec{z}_-(\vec{y},\vec{x})$ make sense for all $\vec{y}$ with $\|\vec{y}\| < 1$ provided $\|\vec{x}\|$ is sufficiently small (depending on $\vec{y}$). By the argument of Example 5.ii of the notes "Haar Measure"

$$\Delta_+(\vec{0}) = \Delta_+(\vec{y})\Big|\det\big[\tfrac{\partial z_{+i}}{\partial x_j}(\vec{y},\vec{0})\big]_{1\le i,j\le 3}\Big| \qquad \Delta_+(\vec{0}) = \Delta_-(\vec{y})\Big|\det\big[\tfrac{\partial z_{-i}}{\partial x_j}(\vec{y},\vec{0})\big]_{1\le i,j\le 3}\Big|$$

This will determine both $\Delta_+(\vec{y})$ and $\Delta_-(\vec{y})$ up to the constant $\Delta_+(\vec{0})$. The latter will be determined by the requirement that the measure have total mass one.

We first find $\vec{z}_+$ and $\vec{z}_-$. By (2),

$$\big(y_0\mathbb{1} + i\vec{y}\cdot\vec{\sigma}\big)\big(x_0\mathbb{1} + i\vec{x}\cdot\vec{\sigma}\big) = y_0x_0\mathbb{1} + iy_0\vec{x}\cdot\vec{\sigma} + ix_0\vec{y}\cdot\vec{\sigma} - \vec{x}\cdot\vec{y}\mathbb{1} - i(\vec{y}\times\vec{x})\cdot\vec{\sigma}$$

Thus

$$\begin{aligned}
\vec{z}_+(\vec{y},\vec{x}) &= y_0\vec{x} + x_0\vec{y} - \vec{y}\times\vec{x} \quad &&\text{with}\quad y_0 = \sqrt{1-\|\vec{y}\|^2} \quad &&\text{and}\quad x_0 = \sqrt{1-\|\vec{x}\|^2}\\
\vec{z}_-(\vec{y},\vec{x}) &= y_0\vec{x} + x_0\vec{y} - \vec{y}\times\vec{x} \quad &&\text{with}\quad y_0 = -\sqrt{1-\|\vec{y}\|^2} \quad &&\text{and}\quad x_0 = \sqrt{1-\|\vec{x}\|^2}
\end{aligned}$$

Next we compute the matrices of partial derivatives. Observe that

$$\tfrac{\partial}{\partial x_j} y_0 x_i = y_0\delta_{i,j}$$

$$\tfrac{\partial}{\partial x_j}\sqrt{1-\|\vec{x}\|^2}\ \vec{y}\Big|_{\vec{x}=0} = \tfrac{-x_j}{\sqrt{1-\|\vec{x}\|^2}}\ \vec{y}\Big|_{\vec{x}=0} = \vec{0}$$

$$-\tfrac{\partial}{\partial x_j}\vec{y}\times\vec{x} = \tfrac{\partial}{\partial x_j}\big(x_2y_3 - x_3y_2,\ x_3y_1 - x_1y_3, x_1y_2 - x_2y_1\big) = \begin{cases} (\ \ 0, -y_3,\ \ y_2) & \text{if } j=1\\ (\ \ y_3,\ \ 0, -y_1) & \text{if } j=2\\ (-y_2,\ \ y_1,\ \ 0) & \text{if } j=3\end{cases}$$

Hence, with $y_0 = \pm\sqrt{1-\|\vec{y}\|^2}$ for $\vec{z}_\pm$,

$$\begin{aligned}
\det\big[\tfrac{\partial z_{\pm i}}{\partial x_j}(\vec{y},\vec{0})\big]_{1\le i,j\le 3} &= \det\begin{bmatrix} y_0 & y_3 & -y_2\\ -y_3 & y_0 & y_1\\ y_2 & -y_1 & y_0\end{bmatrix}\\
&= y_0^3 + y_3y_1y_2 - y_2y_3y_1 - \big(-y_0y_1^2 - y_0y_3^2 - y_0y_2^2\big)\\
&= y_0\big(y_0^2 + y_1^2 + y_2^2 + y_3^2\big)\\
&= y_0
\end{aligned}$$

Thus

$$\Delta_+(\vec{y}) = \Delta_-(\vec{y}) = \tfrac{\Delta_+(\vec{0})}{\sqrt{1-y_1^2-y_2^2-y_3^2}}$$

The constant $\Delta_+(\vec{0})$ is determined by the requirement that

$$1 = \iiint_{\|\vec{y}\|<1}\Delta_+(\vec{y})\,d^3\vec{y} + \iiint_{\|\vec{y}\|<1}\Delta_-(\vec{y})\,d^3\vec{y} = 2\Delta_+(\vec{0})\iiint_{\|\vec{y}\|<1}\tfrac{1}{\sqrt{1-y_1^2-y_2^2-y_3^2}}\,d^3\vec{y}$$

Switching to conventional spherical coordinates

$$1 = 2\Delta_+(\vec{0})\int_0^1 d\rho\int_0^{2\pi}d\theta\int_0^{\pi}d\varphi\ \rho^2\sin\varphi\ \tfrac{1}{\sqrt{1-\rho^2}} = 8\pi\Delta_+(\vec{0})\int_0^1\tfrac{\rho^2}{\sqrt{1-\rho^2}}\ d\rho$$

Now making the change of variables $\rho = \sin\alpha$

$$1 = 8\pi\Delta_+(\vec{0})\int_0^{\pi/2}\tfrac{\sin^2\alpha}{\cos\alpha}\ \cos\alpha\ d\alpha = 8\pi\Delta_+(\vec{0})\int_0^{\pi/2}\sin^2\alpha\ d\alpha = 2\pi^2\Delta_+(\vec{0})$$

and $\Delta_+(\vec{x}) = \Delta_-(\vec{x}) = \frac{1}{2\pi^2\sqrt{1-x_1^2-x_2^2-x_3^2}}$.

This is in fact, aside from a constant factor used to normalize the mass of the measure to one, the standard measure on the sphere $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$ that is inherited from the standard Lebesgue measure on $\mathbb{R}^4$. Recall that the standard surface measure on the surface $z = f(x, y)$ is $\sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\, dxdy$. This is derived in second year Calculus courses by cutting up the surface into tiny parallelograms and computing the area of each parallelogram. This same derivation applied to $z = f(x_1, x_2, x_3)$ gives $\sqrt{1 + f_{x_1}(\vec{x})^2 + f_{x_2}(\vec{x})^2 + f_{x_3}(\vec{x})^2}\, d^3\vec{x}$. If $f(\vec{x}) = \pm\sqrt{1 - x_1^2 - x_2^2 - x_3^2}$ then

$$1 + f_{x_1}(\vec{x})^2 + f_{x_2}(\vec{x})^2 + f_{x_3}(\vec{x})^2 = 1 + \tfrac{x_1^2 + x_2^2 + x_3^2}{1 - x_1^2 - x_2^2 - x_3^2} = \tfrac{1}{1 - x_1^2 - x_2^2 - x_3^2}$$

so

$$\sqrt{1 + f_{x_1}(\vec{x})^2 + f_{x_2}(\vec{x})^2 + f_{x_3}(\vec{x})^2}\, d^3\vec{x} = \tfrac{1}{\sqrt{1 - x_1^2 - x_2^2 - x_3^2}}\, d^3\vec{x}$$

as desired.