Tychonoff's Theorem

Theorem (Tychonoff) *If $\{\Omega_\alpha\}_{\alpha\in\mathcal{I}}$ is any family of compact sets, then $\Omega = \bigtimes_{\alpha\in\mathcal{I}} \Omega_\alpha$, with the product topology, is compact.*

Outline of Proof: We use A, B (possibly with subscripts) to denote subsets of Ω and $\mathcal{A}, \mathcal{B}$ (possibly with subscripts) to denote collections of subsets of Ω. Let $\mathcal{A}$ be any collection of closed subsets of Ω which has the finite intersection property (i.e. every finite subcollection has nonempty intersection). We have to show that $\bigcap_{A\in\mathcal{A}} A \neq \emptyset$.

Step 1. There is a collection $\mathcal{B}$ of (not necessarily closed) subsets of Ω that contains $\mathcal{A}$ and is maximal with respect to the finite intersection property (i.e. it is impossible to add another subset of Ω to $\mathcal{B}$ while retaining the finite intersection property).

Step 2. We guess an x that we hope is in $\bigcap_{A\in\mathcal{A}} A$. To do so, let $\pi_\beta : \Omega \to \Omega_\beta$ be the natural projection on the Ω_β axis. If $x \in \Omega$, then x is a function defined on the index set $\mathcal{I}$ and $\pi_\beta(x) = x(\beta)$. For each $\beta \in \mathcal{I}$,

$$\begin{aligned}
&\big\{\ B \subset \Omega \ \big|\ B \in \mathcal{B}\ \big\} \text{ has the finite intersection property} \\
\Longrightarrow &\big\{\ \pi_\beta(B) \subset \Omega_\beta \ \big|\ B \in \mathcal{B}\ \big\} \text{ has the finite intersection property} \\
\Longrightarrow &\big\{\ \overline{\pi_\beta(B)} \ \big|\ B \in \mathcal{B}\ \big\} \text{ has the finite intersection property} \\
\Longrightarrow &\bigcap_{B\in\mathcal{B}} \overline{\pi_\beta(B)} \neq \emptyset \text{ since } \Omega_\beta \text{ is compact}
\end{aligned}$$

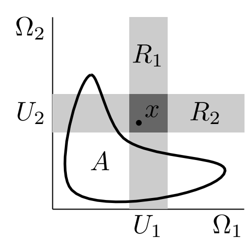
Select, for each $\beta \in \mathcal{I}$, $x_\beta \in \bigcap_{B\in\mathcal{B}} \overline{\pi_\beta(B)}$ and define $x \in \Omega$ by $x(\beta) = x_\beta$.

Step 3. We prove that $x \in \bigcap_{A\in\mathcal{A}} A$. The proof is by contradiction. Suppose that $x \notin A$ for some $A \in \mathcal{A}$. Then, $x \in \Omega \setminus A$, which is an open subset of Ω. Consequently, there is a finite index set $\mathcal{J} \subset \mathcal{I}$ and for each $\alpha \in \mathcal{J}$ there is an open set $U_\alpha \in \Omega_\alpha$ such that

$$x \in \bigtimes_{\alpha\in\mathcal{I}} \left\{ \begin{matrix} U_\alpha & \text{if } \alpha \in \mathcal{J} \\ \Omega_\alpha & \text{if } \alpha \notin \mathcal{J} \end{matrix} \right\} = \bigcap_{\beta\in\mathcal{J}} R_\beta \subset \Omega \setminus A \qquad (*)$$

where

$$R_\beta = \bigtimes_{\alpha\in\mathcal{I}} \left\{ \begin{matrix} U_\alpha & \text{if } \alpha = \beta \\ \Omega_\alpha & \text{if } \alpha \neq \beta \end{matrix} \right\}$$



Step 3a. Show that, for all $B \in \mathcal{B}$ and $\beta \in \mathcal{J}$, $R_\beta \cap B \neq \emptyset$.
Step 3b. Show that $\bigcap_{\beta\in\mathcal{J}} R_\beta \in \mathcal{B}$.
Step 3c. Since $\mathcal{B}$ has the finite intersection property and $A \in \mathcal{A} \subset \mathcal{B}$, $\Big(\bigcap_{\beta\in\mathcal{J}} R_\beta\Big) \cap A \neq \emptyset$. This contradicts $(*)$. ■