Uniqueness of Limits

Let X be a topological space. We shall say the X has the ULP (this stands for "unique limit property") if, for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$,

$$\lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} x_n = y \qquad \Longrightarrow \qquad x = y$$

I have just made up the notation ULP to save typing. It is not standard. We have proven in class that if X is Hausdorff, then it automatically has the ULP. Here is a counterexample that proves that X can have the ULP without being Hausdorff.

Example. Let X be an uncountable set. Define a subset $Y \subset X$ to be open if either $Y = \emptyset$ or $X \setminus Y$ is countable. Since any finite union of countable sets is still countable, this is a legitimate topology on X. I claim that

$$\lim_{n\to\infty} x_n = x \iff \exists N \text{ such that } n > N \Longrightarrow x_n = x \tag{1}$$

To prove the $\Longrightarrow$ part of this equivalence, set $U = X \setminus \{\ x_n \mid x_n \neq x\ \}$. This is an open set which contains x. So if $\lim\limits_{n\to\infty} x_n = x$, there exists an N such that $x_n \in U$, and hence $x_n = x$, for all $n > N$. The equivalence (1) clearly implies that X has the ULP. On the other hand, let x and y be any two distinct elements of X and let U and V be open sets containing x and y, respectively. Since $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$ is countable, it cannot contain all of X. Hence $U \cap V$ cannot be empty. Thus X is not Hausdorff.

On the other hand, there is a Theorem which says "A topological space X is Hausdorff if and only if every net in X converges to at most one point". A "net" is a generalization of "sequence" in which the subscript may take more than countably many values. See Folland §4.3, Exercise 32. Also

Theorem. *If X has the ULP and is first countable, meaning that it has a countable base at each point (see Problem Set I, #2c), then X is Hausdorff.*

Proof: Suppose that X is not Hausdorff. Let x and y be two distinct points of X such that

$$\text{If } U \text{ and } V \text{ are open subsets of } X \text{ with } x \in U \text{ and } y \in V, \text{ then } U \cap V \neq \emptyset. \tag{2}$$

Let $\{U_n\}_{n\in\mathbb{N}}$ be a countable base at x and $\{V_n\}_{n\in\mathbb{N}}$ be a countable base at y. We may assume without loss of generality that $U_1 \supset U_2 \supset U_3 \cdots$ (otherwise, replace U_n by $U_1 \cap U_2 \cap \cdots \cap U_n$) and $V_1 \supset V_2 \supset V_3 \cdots$. By (2), there exists, for each $n \in \mathbb{N}$, a point $x_n \in U_n \cap V_n$. Then $x_n \in U_N$ for all $n \geq N$ and $x_n \in V_N$ for all $n \geq N$. If U is any open set containing x, there is a U_N with $U_N \subset U$. Thus $x_n \in U$ for all $n \geq N$ and $\lim\limits_{n\to\infty} x_n = x$. Similarly, $\lim\limits_{n\to\infty} x_n = y$, violating the ULP. ■