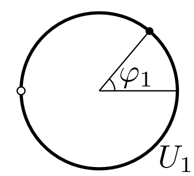
Examples of Manifolds

Example 1 (Open Subset of $\mathbb{R}^n$) Any open subset, $\mathcal{O}$, of $\mathbb{R}^n$ is a manifold of dimension n. One possible atlas is $\mathcal{A} = \{(\mathcal{O}, \varphi_{\rm id})\}$, where $\varphi_{\rm id}$ is the identity map. That is, $\varphi_{\rm id}(\mathbf{x}) = \mathbf{x}$. Of course one possible choice of $\mathcal{O}$ is $\mathbb{R}^n$ itself.

Example 2 (The Circle) The circle $S^1 = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$ is a manifold of dimension one. One possible atlas is $\mathfrak{A} = \{(U_1, \varphi_1),\ (U_1, \varphi_2)\}$ where

$$U_1 = S^1 \setminus \{(-1,0)\} \qquad \varphi_1(x,y) = \arctan \tfrac{y}{x} \text{ with } -\pi < \varphi_1(x,y) < \pi$$
$$U_2 = S^1 \setminus \{(1,0)\} \qquad \varphi_2(x,y) = \arctan \tfrac{y}{x} \text{ with } 0 < \varphi_2(x,y) < 2\pi$$



Example 3 (S^n) The n–sphere $S^n = \{ \mathbf{x} = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1 \}$ is a manifold of dimension n. One possible atlas is $\mathfrak{A}_1 = \{ (U_i, \varphi_i),\ (V_i, \psi_i) \mid 1 \le i \le n+1 \}$ where, for each $1 \le i \le n+1$,

$$U_i = \{ (x_1, \cdots, x_{n+1}) \in S^n \mid x_i > 0 \} \qquad \varphi_i(x_1, \cdots, x_{n+1}) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1})$$
$$V_i = \{ (x_1, \cdots, x_{n+1}) \in S^n \mid x_i < 0 \} \qquad \psi_i(x_1, \cdots, x_{n+1}) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1})$$

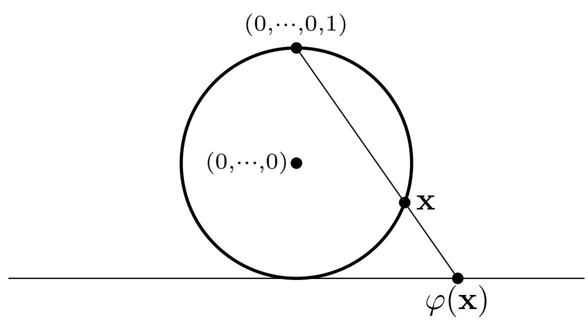
So both φ_i and ψ_i project onto $\mathbb{R}^n$, viewed as the hyperplane $x_i = 0$. Another possible atlas is

$$\mathfrak{A}_2 = \left\{ \left(S^n \setminus \{(0, \cdots, 0, 1)\}, \varphi\right),\ \left(S^n \setminus \{(0, \cdots, 0, -1)\}, \psi\right) \right\}$$

where

$$\varphi(x_1, \cdots, x_{n+1}) = \left(\tfrac{2x_1}{1-x_{n+1}}, \cdots, \tfrac{2x_n}{1-x_{n+1}}\right)$$
$$\psi(x_1, \cdots, x_{n+1}) = \left(\tfrac{2x_1}{1+x_{n+1}}, \cdots, \tfrac{2x_n}{1+x_{n+1}}\right)$$

are the stereographic projections from the north and south poles, respectively.



Both φ and ψ have range $\mathbb{R}^n$. So we can think of S^n as $\mathbb{R}^n$ plus an additional single "point at infinity".

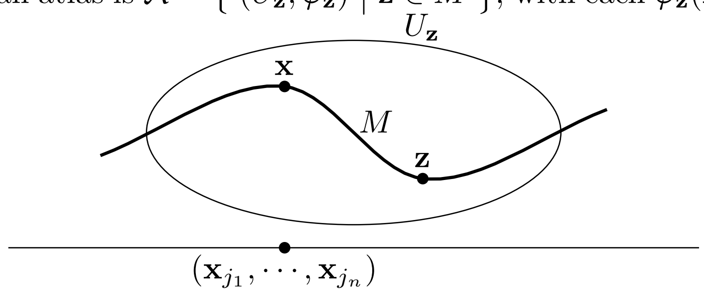
Example 4 (Surfaces) Any smooth n–dimensional surface in $\mathbb{R}^{n+m}$ is an n–dimensional manifold. When we say that M is an n–dimensional surface in $\mathbb{R}^{n+m}$, we mean that M is a subset of $\mathbb{R}^{n+m}$ with the property that for each $\mathbf{z} \in M$, there are

- ◦ a neighbourhood $U_{\mathbf{z}}$ of $\mathbf{z}$ in $\mathbb{R}^{n+m}$
- ◦ n integers $1 \le j_1 < j_2 < \cdots < j_n \le n+m$
- ◦ and m C^∞ functions $f_k(\mathbf{x}_{j_1}, \cdots, \mathbf{x}_{j_n})$, $k \in \{1, \cdots, n+m\} \setminus \{j_1, \cdots, j_n\}$

such that the point $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_{n+m}) \in U_{\mathbf{z}}$ is in M if and only if $\mathbf{x}_k = f_k(\mathbf{x}_{j_1}, \cdots, \mathbf{x}_{j_n})$ for all $k \in \{1, \cdots, n+m\} \setminus \{j_1, \cdots, j_n\}$. That is, we may express the part of M that is near $\mathbf{z}$ as

$$\begin{aligned}
\mathbf{x}_{i_1} &= f_1\big(\mathbf{x}_{j_1},\ \mathbf{x}_{j_2},\ \cdots,\ \mathbf{x}_{j_n}\big) \\
\mathbf{x}_{i_2} &= f_2\big(\mathbf{x}_{j_1},\ \mathbf{x}_{j_2},\ \cdots,\ \mathbf{x}_{j_n}\big) \\
&\ \ \vdots \\
\mathbf{x}_{i_m} &= f_m\big(\mathbf{x}_{j_1},\ \mathbf{x}_{j_2},\ \cdots,\ \mathbf{x}_{j_n}\big) \\
&\text{where } \{i_1, \cdots, i_m\} = \{1, \cdots, n+m\} \setminus \{j_1, \cdots, j_n\}
\end{aligned}$$

for some C^∞ functions $f_1,\ \cdots,\ f_m$. We may use $\mathbf{x}_{j_1},\ \mathbf{x}_{j_2},\ \cdots,\ \mathbf{x}_{j_n}$ as coordinates for M in $M \cap U_{\mathbf{z}}$. Of course, an atlas is $\mathcal{A} = \big\{\ (U_{\mathbf{z}}, \varphi_{\mathbf{z}})\ \big|\ \mathbf{z} \in M\ \big\}$, with each $\varphi_{\mathbf{z}}(\mathbf{x}) = (\mathbf{x}_{j_1}, \cdots, \mathbf{x}_{j_n})$.



Equivalently, M is an n–dimensional surface in $\mathbb{R}^{n+m}$, if, for each $\mathbf{z} \in M$, there are

- ◦ a neighbourhood $U_{\mathbf{z}}$ of $\mathbf{z}$ in $\mathbb{R}^{n+m}$
- ◦ and m C^∞ functions $g_k : U_{\mathbf{z}} \to \mathbb{R}$, such that the vectors $\big\{\ \nabla g_k(\mathbf{z})\ \big|\ 1 \le k \le m\ \big\}$ are linearly independent

such that the point $\mathbf{x} \in U_{\mathbf{z}}$ is in M if and only if $g_k(\mathbf{x}) = 0$ for all $1 \le k \le m$. To get from the implicit equations for M given by the g_k's to the explicit equations for M given by the f_k's one need only invoke (possible after renumbering the components of $\mathbf{x}$) the

Implicit Function Theorem

Let $m, n \in \mathbb{N}$ and let $U \subset \mathbb{R}^{n+m}$ be an open set. Let $\mathbf{g} : U \to \mathbb{R}^m$ be C^∞ with $\mathbf{g}(\mathbf{z}) = 0$ for some $\mathbf{z} \in U$. Assume that $\det \big[\frac{\partial\, \mathbf{g}_i}{\partial \mathbf{x}_{n+j}}(\mathbf{z})\big]_{1 \le i,j \le m} \ne 0$. Write $\mathbf{a} = (\mathbf{z}_1, \cdots, \mathbf{z}_n)$ and $\mathbf{b} = (\mathbf{z}_{n+1}, \cdots, \mathbf{z}_{n+m})$. Then there exist open sets $V \subset \mathbb{R}^{n+m}$ and $W \subset \mathbb{R}^n$ with $\mathbf{a} \in W$ and $\mathbf{z} = (\mathbf{a}, \mathbf{b}) \in V$ such that

$$\text{for each } \mathbf{x} \in W, \text{ there is a unique } (\mathbf{x}, \mathbf{y}) \in V \text{ such that } \mathbf{g}(\mathbf{x}, \mathbf{y}) = 0.$$

If the $\mathbf{y}$ above is denoted $\mathbf{f}(\mathbf{x})$, then $\mathbf{f} : W \to \mathbb{R}^m$ is C^∞, $\mathbf{f}(\mathbf{a}) = \mathbf{b}$ and $\mathbf{g}\big(\mathbf{x}, \mathbf{f}(\mathbf{x})\big) = 0$ for all $\mathbf{x} \in W$.

The n–sphere S^n is the n–dimensional surface in $\mathbb{R}^{n+1}$ given implicitly by the equation $g(\mathbf{x}_1, \cdots, \mathbf{x}_{n+1}) = \mathbf{x}_1^2 + \cdots + \mathbf{x}_{n+1}^2 - 1 = 0$. In a neighbourhood of the north pole (for example, the northern hemisphere), S^n is given explicitly by the equation $\mathbf{x}_{n+1} = \sqrt{\mathbf{x}_1^2 + \cdots + \mathbf{x}_n^2}$.

If you think of the set, M_3, of all 3×3 real matrices as $\mathbb{R}^9$ (because a 3×3 matrix has 9 matrix elements) then

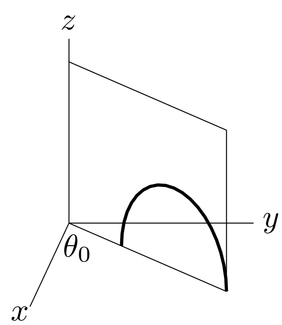
$$SO(3) = \left\{ \ A \in M_3 \ \middle|\ A^t A = \mathbb{1},\ \det A = 1 \ \right\}$$

is a 3–dimensional surface[(1)] in $\mathbb{R}^9$. $SO(3)$ is the group of all rotations about the origin in $\mathbb{R}^3$ and is also the set of all orientations of a rigid body,

Example 5 (The Torus) The torus T^2 is the two dimensional surface

$$T^2 = \left\{ \ (x, y, z) \in \mathbb{R}^3 \ \middle|\ \left(\sqrt{x^2 + y^2} - 1\right)^2 + z^2 = \tfrac{1}{4} \ \right\}$$

in $\mathbb{R}^3$. In cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, the equation of the torus is $(r-1)^2 + z^2 = \frac{1}{4}$. Fix any θ, say θ_0. Recall that the set of all points in $\mathbb{R}^3$ that have $\theta = \theta_0$ is like one page in an open book. It is a half–plane that starts at the z axis. The intersection of the torus with that half plane is a circle of radius $\frac{1}{2}$ centred on $r = 1$, $z = 0$. As φ runs from 0 to 2π, the point $r = 1 + \frac{1}{2}\cos\varphi$, $z = \frac{1}{2}\sin\varphi$,



$\theta = \theta_0$ runs over that circle. If we now run θ from 0 to 2π, the circle on the page sweeps out the whole torus. So, as φ runs from 0 to 2π and θ runs from 0 to 2π, the point $(x, y, z) = \big((1 + \frac{1}{2}\cos\varphi)\cos\theta, (1 + \frac{1}{2}\cos\varphi)\sin\theta, \frac{1}{2}\sin\varphi\big)$ runs over the whole torus. So we may build coordinate patches for T^2 using θ and φ (with ranges $(0, 2\pi)$ or $(-\pi, \pi)$) as coordinates.

(1) Note that $A^t A = \mathbb{1}$ forces $\det A \in \{-1, 1\}$. If you fix any $B \in SO(3)$, then, just by continuity, all matrices A that obey $A^t A = \mathbb{1}$ and are sufficiently close to B automatically obey $\det A = 1$. So the equation $\det A = 1$ is redundant. Since $A^t A$ is automatically symmetric, the requirement $A^t A = \mathbb{1}$ gives at most 6 independent equations. In fact they are independent.

Example 6 (The Cartesian Product) If M is a manifold of dimension m with atlas $\mathfrak{A}$ and N is a manifold of dimension n with atlas $\mathfrak{B}$ then

$$M \times N = \big\{\ (x,y)\ \big|\ x \in M,\ y \in N\ \big\}$$

is an $(m+n)$–dimensional manifold with atlas

$$\big\{\ \big(U \times V, \varphi \oplus \psi\big)\ \big|\ (U,\varphi) \in \mathfrak{A},\ (V,\psi) \in \mathfrak{B}\ \big\} \qquad \text{where } \varphi \oplus \psi\big((x,y)\big) = \big(\varphi(x), \psi(y)\big)$$

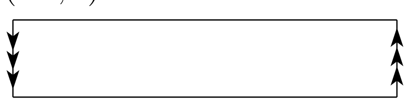
For example, $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$, $S^1 \times \mathbb{R}$ is a cylinder, $S^1 \times S^1$ is a torus and the configuration space of a rigid body is $\mathbb{R}^3 \times SO(3)$ (with the $\mathbb{R}^3$ components giving the location of the centre of mass of the body and the $SO(3)$ components giving the orientation).

Example 7 (The Möbius Strip) Take a length of ribbon. Put a half twist in it and glue the ends together. The result is a Möbius Strip. Mathematically, you can think of it as the set $[0,1] \times (-1,1)$ but with the points $(0,t)$ and $(1,-t)$ identified (i.e. pretend that they are the same point) for all $-1 < t < 1$. We can view the Möbius Strip as a manifold with the set points $M = [0,1) \times (-1,-1)$ and the two patch atlas $\mathfrak{A} = \{(U_1,\varphi_1),\ (U_1,\varphi_2)\}$ where

$$U_1 = \big(\tfrac{1}{8}, \tfrac{7}{8}\big) \qquad \varphi_1(x,y) = (x,y)$$

$$U_2 = \big[0, \tfrac{1}{4}\big) \cup \big(\tfrac{3}{4}, 1\big) \quad \varphi_2(x,y) = \begin{cases} (x,y) & \text{if } 0 \le x < \frac{1}{4} \\ (x-1,-y) & \text{if } \frac{3}{4} < x < 1 \end{cases}$$

The range of φ_2 is $\big(-\tfrac{1}{4}, \tfrac{1}{4}\big) \times (-1,1)$.



Example 8 (Projective n–space, $\mathbb{P}^n$) The projective n–space, $\mathbb{P}^n$, is the set of all lines through the origin in $\mathbb{R}^{n+1}$. If $\vec{x} \in \mathbb{R}^{n+1}$ is nonzero, then there is a unique line $L_{\vec{x}}$ through the origin in $\mathbb{R}^{n+1}$ that contains $\vec{x}$. Namely $L_{\vec{x}} = \big\{\ \lambda\vec{x}\ \big|\ \lambda \in \mathbb{R}\ \big\}$. If $\vec{x}, \vec{y} \in \mathbb{R}^{n+1}$ are both nonzero, then $L_{\vec{x}} = L_{\vec{y}}$ if and only if there is a $\lambda \in \mathbb{R} \setminus \{0\}$ such that $\vec{y} = \lambda\vec{x}$. One choice of atlas for $\mathbb{P}^n$ is $\mathfrak{A} = \big\{\ (U_i, \varphi_i)\ \big|\ 1 \le i \le n+1\ \big\}$ with

$$U_i = \big\{\ L_{\vec{x}}\ \big|\ \vec{x} \in \mathbb{R}^{n+1},\ x_i \ne 0\ \big\} \qquad \varphi(L_{\vec{x}}) = \big(\tfrac{x_1}{x_i}, \cdots, \tfrac{x_{i-1}}{x_i}, \tfrac{x_{i+1}}{x_i}, \cdots, \tfrac{x_{n+1}}{x_i}\big) \in \mathbb{R}^n$$

Observe that if φ_i is well–defined, because if $\vec{x}, \vec{y} \in \mathbb{R}^{n+1}$ are both nonzero and $L_{\vec{x}} = L_{\vec{y}}$, then, for each $1 \le i \le n+1$, either both x_i and y_i are zero or both x_i and y_i are nonzero and in the latter case

$$\big(\tfrac{x_1}{x_i}, \cdots, \tfrac{x_{i-1}}{x_i}, \tfrac{x_{i+1}}{x_i}, \cdots, \tfrac{x_{n+1}}{x_i}\big) = \big(\tfrac{y_1}{y_i}, \cdots, \tfrac{y_{i-1}}{y_i}, \tfrac{y_{i+1}}{y_i}, \cdots, \tfrac{y_{n+1}}{y_i}\big)$$

Each line through the origin in $\mathbb{R}^{n+1}$ intersects the unit sphere $S^n = \left\{ \vec{x} \in \mathbb{R}^{n+1} \ \middle|\ |\vec{x}| = 1 \right\}$ in exactly two points and the two points are antipodal (i.e. $\vec{x}$ and $-\vec{x}$). So you can think of $\mathbb{P}^n$ as S^n but with antipodal points identified:

$$\mathbb{P}^{n+1} = \left\{ \{\vec{x}, -\vec{x}\} \ \middle|\ \vec{x} \in S^n \right\}$$

Each line $L_{\vec{x}} \in \mathbb{P}^n$ that is not horizontal (i.e. with $x_{n+1} \neq 0$) intersects the northern hemisphere $\left\{ \vec{x} \in \mathbb{R}^{n+1} \ \middle|\ |\vec{x}| = 1,\ x_{n+1} \geq 0 \right\}$ in exactly one point. Each line $L_{\vec{x}} \in \mathbb{P}^n$ that is horizontal (i.e. with $x_{n+1} = 0$) intersects the northern hemisphere in exactly two points and the two points are antipodal. By ignoring x_{n+1}, you can think of the northern hemisphere as the closed unit disk $\left\{ \mathbf{x} \in \mathbb{R}^n \ \middle|\ |\mathbf{x}| \leq 1 \right\}$ in $\mathbb{R}^n$. So you can think of $\mathbb{P}^n$ as the closed unit ball in $\mathbb{R}^n$ but with antipodal points on the boundary $|\mathbf{x}| = 1$ identified.

In the case of three dimensions, you can also think of $SO(3)$ as being the closed unit disk $\left\{ \mathbf{x} \in \mathbb{R}^3 \ \middle|\ |\mathbf{x}| \leq 1 \right\} \subset \mathbb{R}^3$ but with antipodal points on the boundary $|\mathbf{x}| = 1$ identified. This is because, geometrically, each element of $SO(3)$ is a matrix which implements a rotation by some angle about some axis through the origin in $\mathbb{R}^3$. We can associate each $\omega\hat{\Omega} \in \mathbb{R}^3$, where $\hat{\Omega}$ is a unit vector and $\omega \in \mathbb{R}$, with the rotation by an angle $\pi\omega$ about the axis $\hat{\Omega}$. But then any two ω's that differ by an even integer give the same rotation. So the set of all rotations is associated with $\left\{ \omega\hat{\Omega} \ \middle|\ |\omega| \leq 1,\ \hat{\Omega} \in \mathbb{R}^3,\ |\hat{\Omega}| = 1 \right\}$ but with $1\hat{\Omega}$ and $-1\hat{\Omega}$ identified. Thus $SO(3)$ and $\mathbb{P}^3$ are diffeomorphic.

