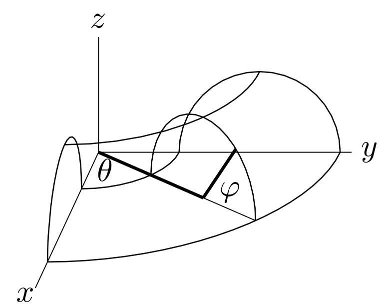
Torus Geodesics

Let $0 < \rho < R$ be constants. The surface in $\mathbb{R}^3$ whose equation in cylindrical coordinates is

$$(r-R)^2 + z^2 = \rho^2$$

is a torus, which we shall call M. Use as coordinates on M two angles θ and φ determined by

$$x = (R + \rho\cos\varphi)\cos\theta \quad y = (R + \rho\cos\varphi)\sin\theta \quad z = \rho\sin\varphi$$



By way of a check, observe that $r = R + \rho\cos\varphi$ and $z = \rho\sin\varphi$ obey the equation $(r-R)^2 + z^2 = \rho^2$. For any curve on M

$$\begin{aligned}
\dot{x}(t) &= -\dot{\varphi}(t)\ \rho\sin\varphi(t)\ \cos\theta(t) - \dot{\theta}(t)\ \big(R + \rho\cos\varphi(t)\big)\sin\theta(t) \\
\dot{y}(t) &= -\dot{\varphi}(t)\ \rho\sin\varphi(t)\ \sin\theta(t) + \dot{\theta}(t)\ \big(R + \rho\cos\varphi(t)\big)\cos\theta(t) \\
\dot{z}(t) &= \dot{\varphi}(t)\ \rho\cos\varphi(t)
\end{aligned}$$

Thus

$$\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2 = \rho^2\dot{\varphi}(t)^2 + \big(R + \rho\cos\varphi(t)\big)^2\dot{\theta}(t)^2$$

and the Lagrangian for free motion on M is $L(\theta,\varphi,v_\theta,v_\varphi) = \frac{1}{2}\rho^2 v_\varphi^2 + \frac{1}{2}(R+\rho\cos\varphi)^2 v_\theta^2$. The θ Euler–Lagrange equation is

$$\begin{aligned}
&\tfrac{d}{dt}\Big(\tfrac{\partial L}{\partial v_\theta}\big(\theta(t),\varphi(t),\dot{\theta}(t),\dot{\varphi}(t)\big)\Big) = \tfrac{\partial L}{\partial \theta}\big(\theta(t),\varphi(t),\dot{\theta}(t),\dot{\varphi}(t)\big) \\
&\quad\Longrightarrow\ \tfrac{d}{dt}\Big(\big(R + \rho\cos\varphi(t)\big)^2\dot{\theta}(t)\Big) = 0 \\
&\quad\Longrightarrow\ \big(R + \rho\cos\varphi(t)\big)^2\dot{\theta}(t) = p_\theta, \text{constant} \qquad \qquad (\mathrm{EL}_\theta)
\end{aligned}$$

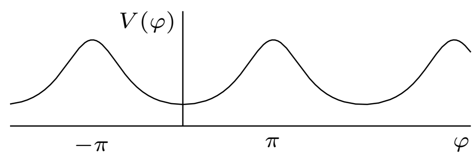
The φ Euler–Lagrange equation is

$$\begin{aligned}
&\tfrac{d}{dt}\Big(\tfrac{\partial L}{\partial v_\varphi}\big(\theta(t),\varphi(t),\dot{\theta}(t),\dot{\varphi}(t)\big)\Big) = \tfrac{\partial L}{\partial \varphi}\big(\theta(t),\varphi(t),\dot{\theta}(t),\dot{\varphi}(t)\big) \\
&\quad\Longrightarrow\ \rho^2\ddot{\varphi}(t) = -\rho\big(R + \rho\cos\varphi(t)\big)\sin\varphi(t)\ \dot{\theta}(t)^2 \\
&\quad\Longrightarrow\ \rho^2\ddot{\varphi}(t) = -\rho\big(R + \rho\cos\varphi(t)\big)^{-3}\sin\varphi(t)\ p_\theta^2 \qquad \qquad (\mathrm{EL}_\varphi)
\end{aligned}$$

By conservation of energy

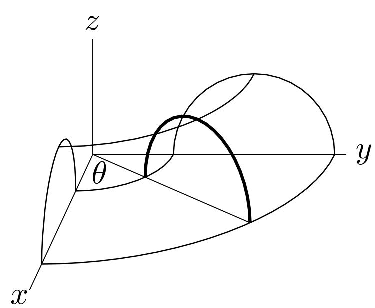
$$\begin{aligned} &\tfrac{1}{2}\rho^2\dot\varphi(t)^2 + \tfrac{1}{2}\big(R+\rho\cos\varphi(t)\big)^2\dot\theta(t)^2 = E, \text{ constant} \\ \implies &\tfrac{1}{2}\rho^2\dot\varphi(t)^2 = E - \tfrac{1}{2}(R+\rho\cos\varphi(t))^{-2}p_\theta^2 \end{aligned} \tag{E}$$

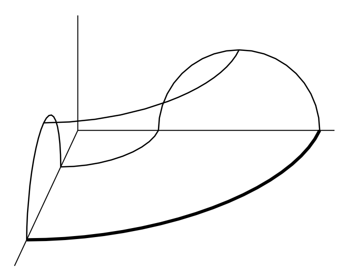
Observe that, by making appropriate choices of initial conditions, we may achieve any value of $p_\theta \in \mathbb{R}$. (For example, we could choose $\theta(0) = \varphi(0) = 0$, $\dot\varphi(0) = 0$ and $\dot\theta(0) = \frac{p_\theta}{(R+\rho)^2}$.) For any fixed p_θ, $E = \frac{1}{2}\rho^2\dot\varphi(t)^2 + \frac{1}{2}\big(R+\rho\cos\varphi(t)\big)^{-2}p_\theta^2 \ge \frac{1}{2}(R+\rho)^{-2}p_\theta^2$ since $\cos\varphi(t) \le 1$. By making appropriate choices of initial conditions, we may achieve any value of $E \ge \frac{1}{2}(R+\rho)^{-2}p_\theta^2$. Observe that $\frac{1}{2}\rho^2\dot\varphi(t)^2 + \frac{1}{2}\big(R+\rho\cos\varphi(t)\big)^{-2}p_\theta^2$ is exactly the sum of the kinetic and potential energies for a particle of mass ρ^2 moving in one dimension with potential energy $V(\varphi) = \frac{1}{2}\big(R+\rho\cos\varphi\big)^{-2}p_\theta^2$.



So you can develop some intuition about the behaviour of $\varphi(t)$ by imagining what happens to a particle moving on the surface in the figure above.

- If $p_\theta = 0$, then, from (EL_θ) and (EL_φ), $\dot\theta(t) = \ddot\varphi(t) = 0$ for all t and the constant speed geodesic sweeps out a circle with θ and $\dot\varphi$ constant. In the figure on the left below, the heavy line is the top half of the geodesic.





- If $p_\theta \ne 0$ and $E = \frac{1}{2}(R+\rho)^{-2}p_\theta^2$, then the condition

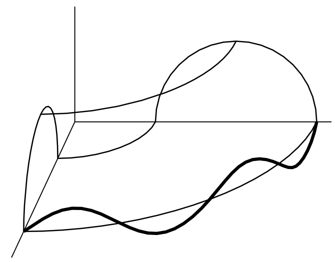
$$\tfrac{1}{2}\big(R+\rho\cos\varphi(t)\big)^{-2}p_\theta^2 \le E = \tfrac{1}{2}(R+\rho)^{-2}p_\theta^2 \iff \big(R+\rho\cos\varphi(t)\big)^{-2} \le (R+\rho)^{-2}$$

  forces $\cos\varphi(t) \ge 1$ and hence $\varphi(t) = 0$ for all t. The geodesic sweeps out the outside equator of the torus, $r = R+\rho$, $z = 0$, with $\dot\theta$ constant. In the figure on the right above, the heavy line is the one quarter of the geodesic.

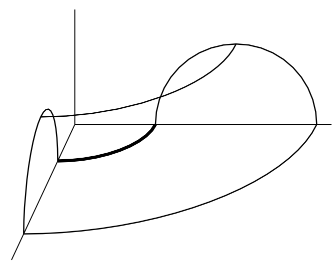
- If $p_\theta \ne 0$ and $\frac{1}{2}(R+\rho)^{-2}p_\theta^2 < E < \frac{1}{2}(R-\rho)^{-2}p_\theta^2$, then the condition

$$\tfrac{1}{2}\big(R+\rho\cos\varphi(t)\big)^{-2}p_\theta^2 \le E \iff \big(R+\rho\cos\varphi(t)\big)^{-2} \le \tfrac{2E}{p_\theta^2} \iff R+\rho\cos\varphi(t) \ge \tfrac{p_\theta}{\sqrt{2E}}$$

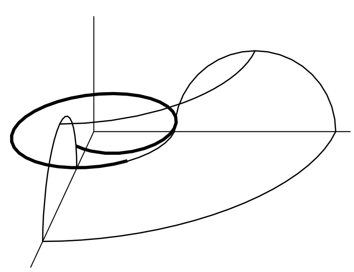
forces $\cos\varphi(t) \ge \frac{1}{\rho}\big(\frac{p_\theta}{\sqrt{2E}} - R\big) > -1$. Let $\cos\varphi_0 = \frac{1}{\rho}\big(\frac{p_\theta}{\sqrt{2E}} - R\big)$ with $0 < \varphi_0 < \pi$. The geodesic oscillates around the outside equator of the torus with φ oscillating between $\pm\varphi_0$ while $\dot\theta$ remains of fixed sign and bounded away from zero.



- If $p_\theta \ne 0$ and $E = \frac{1}{2}(R-\rho)^{-2}p_\theta^2$, then $\theta(t) = \frac{p_\theta}{(R-\rho)^2}t$, $\varphi(t) = \pi$ satisfies both (EL_θ) and (EL_φ) and has the desired values of p_θ and E. This geodesic sweeps out the inside equator of the torus, $r = R - \rho$, $z = 0$.



But we may also achieve the same values of p_θ and E by choosing some $-\pi < \varphi(0) < \pi$ (so that $\frac{1}{2}\big(R + \rho\cos\varphi(0)\big)^{-2}p_\theta^2 < \frac{1}{2}(R-\rho)^{-2}p_\theta^2$) and then choosing $\dot\varphi(0)$ to satisfy $E = \frac{1}{2}\rho^2\dot\varphi(0)^2 + \frac{1}{2}\big(R+\rho\cos\varphi(0)\big)^{-2}p_\theta^2$. If, for example, $\dot\varphi(0) > 0$, then $\varphi(t)$ increases towards π, but $\dot\varphi(t)$ decreases towards 0 at the same time in such a way that $\varphi(t)$ never actually achieves the value π (just as happened in Problem Set 2, #3). At the same time $\dot\theta$ remains of fixed sign and bounded away from zero. So the geodesic approachs the inner equator asymptotically.



- If $p_\theta \ne 0$ and $E > \frac{1}{2}(R-\rho)^{-2}p_\theta^2$, then

$$\tfrac{1}{2}\rho^2\dot\varphi(t)^2 = E - \tfrac{1}{2}(R+\rho\cos\varphi(t))^{-2}p_\theta^2 \ge E - \tfrac{1}{2}(R-\rho)^{-2}p_\theta^2 > 0$$

As $\dot{\varphi}(t)$ is continuous, it remains bounded away from zero and of constant sign. Since

$$\dot{\theta}(t) = \tfrac{p_\theta}{(R+\rho\cos\varphi(t))^2}$$

$\dot{\theta}(t)$ also remains bounded away from zero and of constant sign. So the geodesic wraps around the torus. The figure below shows part of such a geodesic. The heavy solid line is the portion with $r(t) \geq R$ and the heavy dashed line is the portion with $r(t) \leq R$.