Tempered Distributions and the Fourier Transform

The theory of tempered distributions allows us to give a rigorous meaning to the Dirac delta function. It is "defined", on a hand waving level, by the properties that

(i) $\delta(x) = 0$ except when $x = 0$
(ii) $\delta(0)$ is "so infinite" that
(iii) the area under its graph is one.

Still on a handwaving level, if f is any continuous function, then the functions $f(x)\delta(x)$ and $f(0)\delta(x)$ are the same since they are both zero for every $x \neq 0$. Consequently

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = \int_{-\infty}^{\infty} f(0)\delta(x)\,dx = f(0)\int_{-\infty}^{\infty} \delta(x)\,dx = f(0) \tag{1}$$

That $\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0)$ is by far the most important property of the Dirac delta function. But there is no "normal" function $\delta(x)$ that satisfies (1).

The basic idea which allows us to make make rigorous sense of (1) is to generalize the meaning of "a function on $\mathbb{R}$". We shall call the generalization a "tempered distribution on $\mathbb{R}$". Of course a function on $\mathbb{R}$, in the conventional sense, is a rule which assigns a number to each $x \in \mathbb{R}$. A tempered distribution will be a rule which assigns a number to each nice (to be made precise shortly) function on $\mathbb{R}$. We will associate to the conventional function $f : \mathbb{R} \to \mathbb{C}$ the tempered distribution which assigns to the nice function $\varphi(x)$ the number $\int_{-\infty}^{\infty} f(x)\varphi(x)\,dx$. The tempered distribution which corresponds to the Dirac delta function will assign to $\varphi(x)$ the number $\varphi(0)$.

Our first order of business is to make precise "nice function".

Definition 1 Schwartz space is the vector space

$$\mathcal{S}(\mathbb{R}) = \left\{ \varphi : \mathbb{R} \to \mathbb{C} \;\middle|\; \varphi \text{ is } C^\infty,\ \sup_{x\in\mathbb{R}} \left|x^m \varphi^{(n)}(x)\right| < \infty \text{ for all integers } m, n \geq 0 \right\}$$

Here $\varphi^{(n)}$ is the n^{th} derivative of φ.

Observe that

(1) $\mathcal{S}(\mathbb{R})$ is indeed a vector space. That is,

$$\varphi, \psi \in \mathcal{S}(\mathbb{R}),\ a, b \in \mathbb{C} \implies a\varphi + b\psi \in \mathcal{S}(\mathbb{R})$$

(2) If $f(x)$ is any continuous function on $\mathbb{R}$ which is bounded by a constant times $1 + |x|^p$ for some $p \in \mathbb{N}$ and if $\varphi \in \mathcal{S}(\mathbb{R})$, then $f(x)\varphi(x)$ is a continuous function that is

bounded by some constant times $\frac{1}{1+x^2}$ (take $m = p + 2$ and $n = 0$ in Definition 1) so that the integral $\int_{-\infty}^{\infty} f(x)\varphi(x)\ dx$ converges.

(3) Define, for each $m, n \in \mathbb{Z}$ with $m, n \geq 0$ and each $\varphi \in \mathcal{S}(\mathbb{R})$

$$\|\varphi\|_{m,n} = \sup_{x\in\mathbb{R}} \left|x^m \varphi^{(n)}(x)\right|$$

Then

(a) $\|\varphi\|_{m,n} \geq 0$

(b) $\|a\varphi\|_{m,n} = |a|\ \|\varphi\|_{m,n}$

(c) $\|\varphi + \psi\|_{m,n} \leq \|\varphi\|_{m,n} + \|\psi\|_{m,n}$

for all $\varphi, \psi \in \mathcal{S}(\mathbb{R})$ and $a \in \mathbb{C}$. These are precisely the defining conditions for $\|\ \cdot\ \|_{m,n}$ to be a semi–norm.

(4) In order for $\|\ \cdot\ \|_{m,n}$ to be a norm it must also obey $\|\varphi\|_{m,n} = 0 \iff \varphi = 0$. This is the case if and only if $n = 0$. If $n \neq 0$ the constant function $\varphi(x) = 1$ has $\|\varphi\|_{m,n} = 0$.

Example 2

(a) For any polynomial $P(x)$, the function $\varphi(x) = P(x)e^{-x^2}$ is in Schwartz space. This is because, firstly, for any $m, n \geq 0$, $x^m\varphi^{(n)}(x)$ is again a polynomial times e^{-x^2} and, secondly,

$$e^{-x^2} = \frac{1}{e^{x^2}} \leq \frac{1}{1 + x^2 + \frac{1}{2!}x^2 + \cdots + \frac{1}{p!}x^{2p}} \tag{2}$$

for every $p \in \mathbb{N}$. (The terms that we have dropped from the Taylor expansion of e^{x^2} are all positive.) Consequently, $x^m\varphi^{(n)}(x)$ is bounded.

(b) If φ is C^∞ and of compact support (which means that there is some $M > 0$ such that $\varphi(x) = 0$ for all $|x| > M$) then $\varphi \in \mathcal{S}(\mathbb{R})$. One such function is

$$\varphi(x) = \begin{cases} 0 & \text{if } |x| \geq 1 \\ e^{-\frac{1}{(x-1)^2}}\ e^{-\frac{1}{(x+1)^2}} & \text{if } -1 < x < 1 \end{cases}$$

The heart of the proof that this function really is C^∞ at $x = \pm 1$ is the observation that, for any $p \geq 0$, $\lim\limits_{y\to 0} \frac{1}{|y|^p} e^{-\frac{1}{y^2}} = 0$, which follows immediately from (2) with $x = \frac{1}{y}$.

Lemma 3 *Let $\varphi, \psi \in \mathcal{S}(\mathbb{R})$. Then*

(a) $a\varphi + b\psi \in \mathcal{S}(\mathbb{R})$ for all $a, b \in \mathbb{C}$ and

(b) $\frac{d^\ell}{dx^\ell}\varphi \in \mathcal{S}(\mathbb{R})$ for all $\ell \in \mathbb{N}$ and

(c) $\zeta\varphi \in \mathcal{S}(\mathbb{R})$ for all C^∞ functions ζ that are polynomially bounded and have polynomially bounded derivatives and

(d) the convolution $(\varphi * \psi)(x) = \int_{\mathbb{R}} \varphi(y)\psi(x - y)\ dy \in \mathcal{S}(\mathbb{R})$.

Proof: These are all pretty obvious. Parts (a) and (b) are immediate consequences of the bounds

$$\|a\varphi + b\psi\|_{m,n} \le |a|\,\|\varphi\|_{m,n} + |b|\,\|\psi\|_{m,n}$$
$$\big\|\tfrac{d^\ell}{dx^\ell}\varphi\|_{m,n} = \|\varphi\|_{m,n+\ell}$$

which are true for all integers $m, n, \ell \ge 0$ and $a, b \in \mathbb{C}$.

For part (c), let $m, n \in \mathbb{N}_0 = \big\{\, n \in \mathbb{Z} \bigm| n \ge 0 \,\big\}$. By hypothesis, there is an $L \in \mathbb{N}$ such that $\big[1 + |x|^L|\big]^{-1}\frac{d^{n'}}{dx^{n'}}\zeta$ is uniformly bounded for all $n' \le n$. By the product rule

$$\tfrac{d^n}{dx^n}(\zeta\varphi) = \sum_{n'=0}^{n} \tbinom{n}{n'}\ \tfrac{d^{n'}}{dx^{n'}}\zeta\ \tfrac{d^{n-n'}}{dx^{n-n'}}\varphi$$

where $\binom{n}{n'} = \frac{n!}{n'!(n-n')!}$, and part (c) follows from

$$\|\zeta\varphi\|_{m,n} \le \sum_{n'=0}^{n} \tbinom{n}{n'}\Big\|\big[1 + |x|^L|\big]^{-1}\tfrac{d^{n'}}{dx^{n'}}\zeta\Big\|_{L^\infty(\mathbb{R})}\ \Big\{\|\varphi\|_{m,n-n'} + \|\varphi\|_{m+L,n-n'}\Big\}$$

The proof of part (d) is similar to that of part (c) but uses that

- the function $\big[1 + y^2\big]^{-1} \in L^1(\mathbb{R})$ and
- $|x^m| \le \sum_{m'=0}^{m} \binom{m}{m'}\ \big|y^{m'}\big|\ \big|(x-y)^{m-m'}\big|$
- All derivatives of $\varphi(y)\psi(x-y)$ with respect to x are absolutely integrable with respect to y, so that we are allowed to move derivatives with respect to x inside the integral $\int_{\mathbb{R}} \varphi(y)\psi(x-y)\ dy$. ■

Next, we introduce a metric on $\mathcal{S}(\mathbb{R})$ which is chosen so that φ and ψ are close together if and only if $\|\varphi - \psi\|_{m,n}$ is small for every m, n. The details are given in the following

Theorem 4 *Define* $d : \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \to \mathbb{R}$ *by*

$$d(\varphi, \psi) = \sum_{m,n=0}^{\infty} 2^{-m-n}\frac{\|\varphi - \psi\|_{m,n}}{1 + \|\varphi - \psi\|_{m,n}}$$

Then

(a) $d(\varphi, \psi)$ is well–defined for all $\varphi, \psi \in \mathcal{S}(\mathbb{R})$ and is a metric.

(b) With this metric, $\mathcal{S}(\mathbb{R})$ is a complete metric space.

(c) In this metric, $\varphi = \lim_{k\to\infty} \varphi_k$ if and only if $\lim_{k\to\infty} \|\varphi_k - \varphi\|_{m,n} = 0$ for every $m, n \ge 0$.

Proof: (a) To prove that $\sum\limits_{m,n=0}^{\infty} 2^{-m-n} \frac{\|\varphi-\psi\|_{m,n}}{1+\|\varphi-\psi\|_{m,n}}$ is well–defined it suffices to observe, firstly, that $\frac{A}{1+A} \le 1$ for every $A \ge 0$ and, secondly, that $\sum\limits_{m,n=0}^{\infty} 2^{-m-n}$ converges because the geometric series $\sum\limits_{n=0}^{\infty} 2^{-n} = 2$.

- The metric axiom $d(\varphi,\psi) \ge 0$ is obvious.
- The metric axiom that $d(\varphi,\psi) = 0 \implies \varphi = \psi$ is obvious because $d(\varphi,\psi) = 0$ forces the $n = m = 0$ term in its definition, namely $\frac{\|\varphi-\psi\|_{0,0}}{1+\|\varphi-\psi\|_{0,0}}$, to vanish. And that first term is zero if and only if $\|\varphi - \psi\|_{0,0} = \sup_{x\in\mathbb{R}} |\varphi(x) - \psi(x)|$ is zero.
- The metric axiom $d(\varphi,\psi) = d(\psi,\varphi)$ is obvious.
- The triangle inequality follows from

$$\frac{\|\varphi-\psi\|_{m,n}}{1+\|\varphi-\psi\|_{m,n}} \le \frac{\|\varphi-\zeta\|_{m,n}}{1+\|\varphi-\zeta\|_{m,n}} + \frac{\|\zeta-\psi\|_{m,n}}{1+\|\zeta-\psi\|_{m,n}}$$

  which is proven as follows. We supress the subscripts m,n. Because $\frac{x}{1+x} = 1 - \frac{1}{1+x}$ is an increasing function of x

$$\begin{aligned} \tfrac{\|\varphi-\psi\|}{1+\|\varphi-\psi\|} &\le \tfrac{\|\varphi-\zeta\|+\|\zeta-\psi\|}{1+\|\varphi-\zeta\|+\|\zeta-\psi\|} = \tfrac{\|\varphi-\zeta\|}{1+\|\varphi-\zeta\|+\|\zeta-\psi\|} + \tfrac{\|\zeta-\psi\|}{1+\|\varphi-\zeta\|+\|\zeta-\psi\|} \\ &\le \tfrac{\|\varphi-\zeta\|}{1+\|\varphi-\zeta\|} + \tfrac{\|\zeta-\psi\|}{1+\|\zeta-\psi\|} \end{aligned}$$

(c) For the "only if" part, assume that $\varphi = \lim\limits_{k\to\infty} \varphi_k$ and let $m,n \ge 0$. Then

$$d(\varphi,\varphi_k) \ge 2^{-m-n} \frac{\|\varphi-\varphi_k\|_{m,n}}{1+\|\varphi-\varphi_k\|_{m,n}} \implies \lim_{k\to 0} \frac{\|\varphi-\varphi_k\|_{m,n}}{1+\|\varphi-\varphi_k\|_{m,n}} = 0$$

For any $0 < \varepsilon < \frac{1}{2}$ and $x > 0$,

$$\tfrac{x}{1+x} < \varepsilon \implies x < \varepsilon(1+x) \implies x - \varepsilon x < \varepsilon \implies x < \tfrac{\varepsilon}{1-\varepsilon} < 2\varepsilon$$

Hence $\lim\limits_{k\to 0} \|\varphi - \varphi_k\|_{m,n} = 0$ too.

For the "if" part assume that $\lim\limits_{k\to\infty} \|\varphi_k - \varphi\|_{m,n} = 0$ for every $m,n \ge 0$. We must prove that, as a consequence, $\varphi = \lim\limits_{k\to\infty} \varphi_k$. The idea is that, in the definition of $d(\varphi,\psi)$, the sum of all terms with m or n large is small, regardless of what φ and ψ are. Precisely, write

$\psi_k = \varphi - \varphi_k$ and note that, for every $M \in \mathbb{N}$

$$\begin{aligned}
d(\varphi_k, \varphi) &= \sum_{m,n=0}^{\infty} 2^{-m-n} \frac{\|\psi_k\|_{m,n}}{1+\|\psi_k\|_{m,n}} \\
&= \sum_{0\le m,n\le M} 2^{-m-n} \frac{\|\psi_k\|_{m,n}}{1+\|\psi_k\|_{m,n}} + \sum_{\substack{m,n=0\\ n \text{ or } m>M}}^{\infty} 2^{-m-n} \frac{\|\psi_k\|_{m,n}}{1+\|\psi_k\|_{m,n}} \\
&\le \sum_{0\le m,n\le M} 2^{-m-n} \frac{\|\psi_k\|_{m,n}}{1+\|\psi_k\|_{m,n}} + \sum_{\substack{m,n=0\\ n \text{ or } m>M}}^{\infty} 2^{-m-n} \\
&\le \sum_{0\le m,n\le M} 2^{-m-n} \frac{\|\psi_k\|_{m,n}}{1+\|\psi_k\|_{m,n}} + 2\Big\{\sum_{m=M+1}^{\infty} 2^{-m}\Big\}\Big\{\sum_{n=0}^{\infty} 2^{-n}\Big\} \\
&= \sum_{0\le m,n\le M} 2^{-m-n} \frac{\|\psi_k\|_{m,n}}{1+\|\psi_k\|_{m,n}} + 2\Big\{\frac{1}{2^M}\Big\}\{2\}
\end{aligned}$$

Let $\varepsilon > 0$ and choose M so that $\frac{1}{2^M} \le \frac{\varepsilon}{8}$. For each $m, n \ge 0$, $\lim_{k\to\infty} \|\psi\|_{m,n} = 0$ so that there is a $K_{m,n}$ for which $k \ge K_{m,n}$ implies $\|\psi\|_{m,n} < \frac{\varepsilon}{8}$. Set $K = \max\big\{ K_{m,n} \,\big|\, 0 \le m, n \le M \big\}$. If $k \ge K$, then

$$d(\varphi_k, \varphi) \le \sum_{0\le m,n\le M} 2^{-m-n} \frac{\|\psi_k\|_{m,n}}{1+\|\psi_k\|_{m,n}} + 2\Big\{\frac{1}{2^M}\Big\}\{2\} < \tfrac{\varepsilon}{2} + \sum_{m,n=0}^{\infty} 2^{-m-n}\tfrac{\varepsilon}{8} = \varepsilon$$

(b) Let $\{\varphi_k\}$ be a Cauchy sequence with respect to the metric d. Then, as in part (c), for each $m, n \ge 0$, $\lim_{k,k'\to\infty} \|\varphi_k - \varphi_{k'}\|_{m,n} = 0$. In particular, $\lim_{k,k'\to\infty} \|\varphi_k - \varphi_{k'}\|_{0,0} = 0$, so that the sequence $\{\varphi_k\}$ is Cauchy in the set, $\mathcal{C}(\mathbb{R})$, of all bounded, continuous functions on $\mathbb{R}$ equipped with the uniform metric. As $\mathcal{C}(\mathbb{R})$ is complete, there exists a continuous function φ such that $\{\varphi_k\}$ converges uniformly to φ. As well, $\lim_{k,k'\to\infty} \|\varphi_k - \varphi_{k'}\|_{0,1} = 0$ so that the sequence $\{\varphi_k'\}$ of first derivatives is Cauchy in $\mathcal{C}(\mathbb{R})$ and there exists a continuous function φ_1 such that $\{\varphi_k'\}$ converges uniformly to φ_1. This ensures that φ is differentiable with $\varphi' = \varphi_1$. Continuing in this way, we see that φ is C^∞ and that, for each $n \ge 0$, the sequence $\{\varphi_k^{(n)}\}$ of n^{th} derivatives converges uniformly to $\varphi^{(n)}$. Finally, we have that, for each $m, n \ge 0$, there is a $K_{m,n}$ such that $|x|^m \big|\varphi_k^{(n)}(x) - \varphi_{k'}^{(n)}(x)\big| < \varepsilon$ for all $k, k' \ge K_{m,n}$ and all $x \in \mathbb{R}$. Consequently, if $k \ge K_{m,n}$,

$$\begin{aligned}
\|\varphi_k - \varphi\|_{m,n} &= \sup_{x\in\mathbb{R}} |x|^m \big|\varphi_k^{(n)}(x) - \varphi^{(n)}(x)\big| = \sup_{x\in\mathbb{R}} \lim_{k'\to\infty} |x|^m \big|\varphi_k^{(n)}(x) - \varphi_{k'}^{(n)}(x)\big| \\
&\le \sup_{x\in\mathbb{R}} \varepsilon = \varepsilon
\end{aligned}$$

So, by part (c), $\{\varphi_k\}$ converges to φ with respect to the metric d. ■

Remark. In practice, it is rarely necessary to directly use the definition of the metric d of Theorem 4. One usually just uses part (c) of Theorem 4 instead.

We are now ready to give

Definition 5 (Tempered Distributions) The space of all tempered distributions on $\mathbb{R}$, denoted $\mathcal{S}'(\mathbb{R})$, is the dual space of $\mathcal{S}(\mathbb{R})$. That is, it is the set of all functions

$$f : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$$

that are linear and continuous. One usually denotes by $\langle f, \varphi \rangle$ the value in $\mathbb{C}$ that the distribution $f \in \mathcal{S}'(\mathbb{R})$ assigns to $\varphi \in \mathcal{S}(\mathbb{R})$. In this notation,

- ▷ that f is linear means that $\langle f, a\varphi + b\psi \rangle = a \langle f, \varphi \rangle + b \langle f, \psi \rangle$ for all $\varphi, \psi \in \mathcal{S}(\mathbb{R})$ and all $a, b \in \mathbb{C}$.
- ▷ that f is continuous means that if $\varphi = \lim_{n\to\infty} \varphi_n$ in $\mathcal{S}(\mathbb{R})$, then $\langle f, \varphi \rangle = \lim_{n\to\infty} \langle f, \varphi_n \rangle$.

Example 6

(a) Here is the motivating example for the whole subject. Let $f : \mathbb{R} \to \mathbb{C}$ be any measurable function that is polynomially bounded (that is, there is a polynomial $P(x)$ such that $|f(x)| \le P(x)$ for all $x \in \mathbb{R}$). Then

$$f : \varphi \in \mathcal{S}(\mathbb{R}) \mapsto \langle f, \varphi \rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)\, dx$$

is a tempered distribution. The integral converges because every $\varphi \in \mathcal{S}(\mathbb{R})$ decays faster at infinity than one over any polynomial. See Lemma 7, below. The linearity in φ of $\langle f, \varphi \rangle$ is obvious. The continuity in φ of $\langle f, \varphi \rangle$ follows easily from Lemma 7 and Theorem 8, below.

(b) The Dirac delta function, and more generally the Dirac delta function translated to $b \in \mathbb{R}$, are defined as tempered distributions by

$$\langle \delta, \varphi \rangle = \varphi(0) \qquad \langle \delta_b, \varphi \rangle = \varphi(b)$$

Once again, the linearity in φ is obvious and the continuity in φ is easily verified if one applies Theorem 8.

(c) The derivative of the Dirac delta function δ_b is defined by

$$\langle \delta_b', \varphi \rangle = -\varphi'(b)$$

The reason for the name "derivative of the Dirac delta function" will be given in the section on differentiation, later.

(d) The principal value of $\frac{1}{x}$ is defined by

$$\big\langle P\tfrac{1}{x},\varphi\big\rangle = \lim_{\varepsilon\to 0+}\int_{|x|>\varepsilon}\frac{\varphi(x)}{x}\,dx$$

The first thing that we have to do is verify that the limit above actually exists. This is not a trivial statement, because $\int_0^1 \frac{1}{x}dx$ and $\int_{-1}^0 \frac{1}{x}\,dx$ do not even exist as improper integrals:

$$\begin{aligned}\int_0^1 \tfrac{1}{x}\,dx &= \lim_{\varepsilon\to 0+}\int_\varepsilon^1 \tfrac{1}{x}\,dx &&= \lim_{\varepsilon\to 0+}\ln\tfrac{1}{\varepsilon} = \infty\\ \int_{-1}^0 \tfrac{1}{x}\,dx &= \lim_{\varepsilon\to 0+}\int_{-1}^{-\varepsilon} \tfrac{1}{x}\,dx &&= \lim_{\varepsilon\to 0+}\ln\varepsilon = -\infty\end{aligned}$$

Here is the verification that the limit defining $\big\langle P\frac{1}{x},\varphi\big\rangle$ exists

$$\begin{aligned}\lim_{\varepsilon\to 0+}\int_{|x|>\varepsilon}\frac{\varphi(x)}{x}\,dx &= \lim_{\substack{\varepsilon\to 0+\\ M,M'\to\infty}}\left\{\int_\varepsilon^M \frac{\varphi(x)}{x}dx + \int_{-M'}^{-\varepsilon}\frac{\varphi(x)}{x}\,dx\right\}\\ &= \lim_{\substack{\varepsilon\to 0+\\ M,M'\to\infty}}\left\{\int_\varepsilon^1 \frac{\varphi(x)}{x}dx + \int_1^M \frac{\varphi(x)}{x}dx + \int_{-M'}^{-1}\frac{\varphi(x)}{x}\,dx + \int_{-1}^{-\varepsilon}\frac{\varphi(x)}{x}\,dx\right\}\\ &= \lim_{\substack{\varepsilon\to 0+\\ M,M'\to\infty}}\left\{\int_\varepsilon^1 \frac{\varphi(x)-\varphi(-x)}{x}\,dx + \int_1^M \frac{\varphi(x)}{x}\,dx + \int_{-M'}^{-1}\frac{\varphi(x)}{x}\,dx\right\}\end{aligned}$$

The first integral converges because, by the mean value theorem, we have, for some ξ between x and $-x$,

$$\left|\frac{\varphi(x)-\varphi(-x)}{x}\right| = \left|\frac{\varphi'(\xi)\,2x}{x}\right| \le 2\|\varphi\|_{0,1}$$

The second and third integrals converge because, for $|x|\ge 1$

$$\left|\frac{\varphi(x)}{x}\right| \le \tfrac{1}{x^2}|x\varphi(x)| \le \tfrac{1}{x^2}\|\varphi\|_{1,0}$$

These bounds give both that $\big\langle P\frac{1}{x},\varphi\big\rangle$ is well–defined and

$$\begin{aligned}\big|\big\langle P\tfrac{1}{x},\varphi\big\rangle\big| &\le 2\|\varphi\|_{0,1}\int_0^1 dx + \|\varphi\|_{1,0}\int_1^\infty \tfrac{1}{x^2}\,dx + \|\varphi\|_{1,0}\int_{-\infty}^{-1}\tfrac{1}{x^2}\,dx\\ &= 2\|\varphi\|_{0,1} + 2\|\varphi\|_{1,0}\end{aligned}$$

Linearity is again obvious. Continuity again follows by Theorem 8, below.

Lemma 7 *Let $f : \mathbb{R} \to \mathbb{C}$ be measurable and obey the bound $|f(x)| \le P(x)$ for all $\mathbf{x} \in \mathbb{R}$, where $P(x)$ is the polynomial $P(x) = \sum_{n=N_-}^{N_+} a_n x^n$ and $N_\pm$ are nonnegative integers.*

(a) There is a constant $C > 0$ such that $|f(x)|(1+x^2) \le C\big(|x|^{N_-} + |x|^{N_+ + 2}\big)$ for all $x \in \mathbb{R}$.

(b) Let $\varphi \in \mathcal{S}(\mathbb{R})$. Then

$$\int_{-\infty}^{\infty} |f(x)\varphi(x)|\ dx \le \pi C\big(\|\varphi\|_{N_-,0} + \|\varphi\|_{N_+ + 2,0}\big)$$

Proof: (a) Observe that, for any natural numbers $m \le n \le N$, we have $|x|^n \le |x|^m + |x|^N$ for all $x \in \mathbb{R}$ just because $|x|^n \le |x|^m$ when $|x| \le 1$ and $|x|^n \le |x|^N$ when $|x| \ge 1$. Consequently, using $m = N_-$ and $N = N_+ + 2$,

$$|f(x)|(1+x^2) \le P(x)\,(1+x^2) \le \sum_{n=N_-}^{N_+} |a_n|\big(|x|^n + |x|^{n+2}\big) \le \sum_{n=N_-}^{N_+} 2|a_n|\big(|x|^{N_-} + |x|^{N_+ + 2}\big)$$

and $C = 2\sum_{n=N_-}^{N_+} |a_n|$ does the job.

(b) By part (a)

$$\begin{aligned} \left|\int_{-\infty}^{\infty} f(x)\varphi(x)\ dx\right| &\le \int_{-\infty}^{\infty} |f(x)|(1+x^2)|\varphi(x)|\tfrac{1}{1+x^2}\ dx \\ &\le \int_{-\infty}^{\infty} C\big(|x|^{N_-} + |x|^{N_+ + 2}\big)|\varphi(x)|\tfrac{1}{1+x^2}\ dx \\ &\le \int_{-\infty}^{\infty} C\big(\|\varphi\|_{N_-,0} + \|\varphi\|_{N_+ + 2,0}\big)\tfrac{1}{1+x^2}\ dx \\ &= \pi C\big(\|\varphi\|_{N_-,0} + \|\varphi\|_{N_+ + 2,0}\big) \end{aligned}$$

■

Theorem 8 (Continuity Test) *A linear map $f : \varphi \in \mathcal{S}(\mathbb{R}) \mapsto \langle f, \varphi\rangle \in \mathbb{C}$ is continuous if and only if there are constants $C > 0$ and $N \in \mathbb{N}$ such that*

$$\big|\langle f, \varphi\rangle\big| \le C \sum_{0 \le m,n \le N} \|\varphi\|_{m,n}$$

Proof: $\Leftarrow$: Assume that $\big|\langle f, \varphi\rangle\big| \le C\sum_{0\le m,n\le N} \|\varphi\|_{m,n}$ and that the sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ converges to φ in $\mathcal{S}(\mathbb{R})$. Then

$$\big|\langle f, \varphi\rangle - \langle f, \varphi_k\rangle\big| = \big|\langle f, \varphi - \varphi_k\rangle\big| \le C \sum_{0\le m,n\le N} \|\varphi - \varphi_k\|_{m,n}$$

converges to zero as $k \to \infty$. So f is continuous.

$\Rightarrow$: Assume that $f \in \mathcal{S}'(\mathbb{R})$. In particular f is continuous at $\varphi = 0$. Then there is a $\delta > 0$ such that

$$d(\psi, 0) < \delta \implies \big|\langle f, \psi\rangle\big| < 1$$

Choose N so that $\displaystyle\sum_{n \text{ or } m > N} 2^{-m-n} < \frac{\delta}{2}$. Then

$$\begin{aligned}\sum_{0\le m,n\le N} \|\psi\|_{m,n} \le \tfrac{\delta}{2} &\implies d(\psi,0) = \sum_{m,n\ge 0} 2^{-m-n}\tfrac{\|\psi\|_{m,n}}{1+\|\psi\|_{m,n}} \le \sum_{m,n\le N}\|\psi\|_{m,n} + \sum_{n \text{ or } m>N} 2^{-m-n} \\ &\implies d(\psi,0) < \delta \\ &\implies \big|\langle f,\psi\rangle\big| < 1\end{aligned}$$

Consequently, for any $0 \neq \varphi \in \mathcal{S}(\mathbb{R})$, setting

$$\psi = \tfrac{\delta}{2}\Big[\sum_{m,n\le N}\|\varphi\|_{m,n}\Big]^{-1}\varphi$$

we have

$$\sum_{0\le m,n\le N}\|\psi\|_{m,n} = \sum_{0\le m,n\le N}\tfrac{\delta}{2}\Big[\sum_{m,n\le N}\|\varphi\|_{m,n}\Big]^{-1}\|\varphi\|_{m,n} = \tfrac{\delta}{2}$$

and hence

$$\big|\langle f,\varphi\rangle\big| = \tfrac{2}{\delta}\Big[\sum_{m,n\le N}\|\varphi\|_{m,n}\Big]\,\big|\langle f,\psi\rangle\big| < \tfrac{2}{\delta}\sum_{m,n\le N}\|\varphi\|_{m,n}$$

as desired. ■

Operations on Tempered Distributions

We now define a number of operations like, for example, addition and differentiation, on tempered distributions. The motivation for all of these definitions come from Example 6.a with $f \in \mathcal{S}(\mathbb{R})$. Then we can view f both as a conventional function and as a tempered distribution. We will define each operation in such a way that when it is applied to $f \in \mathcal{S}(\mathbb{R})$, viewed as a distribution, it yields the same answer as when the operation is applied to f viewed as an ordinary function, with the result viewed as a distribution. As a trivial example, suppose that we wish to define multiplication by 7. If $f \in \mathcal{S}(\mathbb{R})$ is viewed as an ordinary function, applying the operation of multiplication by 7 to it gives the ordinary function $7f$. But $7f$ can again be viewed as the distribution $\langle 7f, \varphi\rangle = \int 7f(x)\,\varphi(x)\,dx = 7\,\langle f, \varphi\rangle$. So we would define the operation of multiplication by 7 applied to any distribution f as the distribution $7f$ defined by $\langle 7f, \varphi\rangle = 7\,\langle f, \varphi\rangle$.

Addition and Scalar Multiplication

Motivation. If $f, g \in \mathcal{S}(\mathrm{I\!R})$ and $a, b \in \mathbb{C}$, then

$$\int_{-\infty}^{\infty} \big[af(x)+bg(x)\big]\,\varphi(x)\;dx = a\int_{-\infty}^{\infty} f(x)\,\varphi(x)\;dx + b\int_{-\infty}^{\infty} g(x)\,\varphi(x)\;dx = a\,\langle f,\varphi\rangle + b\,\langle g,\varphi\rangle$$

Definition. If $f, g \in \mathcal{S}'(\mathrm{I\!R})$ and $a, b \in \mathbb{C}$, then we define $af + bg \in \mathcal{S}'(\mathrm{I\!R})$ by

$$\langle af+bg,\varphi\rangle = a\,\langle f,\varphi\rangle + b\,\langle g,\varphi\rangle$$

Theorem. *If $f, g \in \mathcal{S}'(\mathrm{I\!R})$ and $a, b \in \mathbb{C}$, then $af + bg$, defined above, is a well–defined element of $\mathcal{S}'(\mathrm{I\!R})$. The operations of addition and scalar multiplication so defined obey the usual vector space axioms.*

Proof: Trivial. ∎

Differentiation

Motivation. If $f \in \mathcal{S}(\mathrm{I\!R})$, then, by integration by parts,

$$\int_{-\infty}^{\infty} f'(x)\,\varphi(x)\;dx = -\int_{-\infty}^{\infty} f(x)\,\varphi'(x)\;dx \qquad \text{(the boundary terms vanish)}$$

Definition. We define the first derivative of $f \in \mathcal{S}'(\mathrm{I\!R})$ by

$$\langle f',\varphi\rangle = -\,\langle f,\varphi'\rangle$$

More generally, we define the p^{th} derivative of $f \in \mathcal{S}'(\mathrm{I\!R})$ by

$$\big\langle f^{(p)},\varphi\big\rangle = (-1)^p\big\langle f,\varphi^{(p)}\big\rangle$$

Since $\|\varphi^{(p)}\|_{m,n} = \|\varphi\|_{m,n+p}$ the right hand side gives a well–defined element of $\mathcal{S}'(\mathrm{I\!R})$.

Remark. Note that *every* derivative of *every* distribution *always* exists.

Example. The Heavyside unit function

$$H(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$$

may also be viewed as the tempered distribution

$$\langle H, \varphi \rangle = \int_0^\infty \varphi(x)\, dx$$

via Example 6.a. The derivative of this distribution is

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle = -\int_0^\infty \varphi'(x)\, dx = -\big[\varphi(x)\big]_0^\infty = \varphi(0) = \langle \delta, \varphi \rangle$$

Thus H' is the Dirac delta function.

The Fourier Transform

Definition 9 The Fourier transform $\hat{f}(k)$ of a function $f \in \mathcal{S}(\mathbb{R})$ is defined by

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx \tag{3}$$

Since $f(x)$, and hence $e^{-ikx}f(x)$, is a continuous function of x which is bounded by a contant times $\frac{1}{1+x^2}$, the integral exists and $\hat{f}(k)$ is a well–defined complex number for each $k \in \mathbb{R}$. We shall show in Theorem 12, below that the map $f \mapsto \hat{f}$ is a continuous, linear map from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$ and furthermore that this map is one–to–one and onto with the inverse map being the inverse Fourier transform given by

$$\check{g}(x) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} g(k)\, dk \tag{4}$$

The computational properties of the Fourier transform are given in

Theorem 10 *Let $f, g \in \mathcal{S}(\mathbb{R})$ and $a, b \in \mathbb{C}$. Then*

(a) The Fourier transform of $af(x)+bg(x)$ is $a\hat{f}(k) + b\hat{g}(k)$.

(b) If $n \in \mathbb{N}$, then the Fourier transform of $f^{(n)}(x)$ is $(ik)^n \hat{f}(k)$.

(c) The Fourier transform, $\hat{f}(k)$, of $f(x)$ is infinitely differentiable and, for each $n \in \mathbb{N}$, $\frac{d^n}{dk^n}\hat{f}(k)$ is the Fourier transform of $(-ix)^n f(x)$.

(d) Let $a \in \mathbb{R}$. The Fourier transform of the translated function $(T_a f)(x) = f(x-a)$ is $e^{-iak}\hat{f}(k)$.

(e) The Fourier transform of $f(x) = e^{-x^2/2}$ is $\hat{f}(k) = \sqrt{2\pi}e^{-k^2/2}$.

(f) The Fourier transform of the convolution $h = f * g$ is $\hat{h}(k) = \hat{f}(k)\hat{g}(k)$.

(g) Let $r(x) = f(-x)$ and $c(x) = \overline{f(x)}$. Then

$$\hat{r}(k) = \overline{\hat{c}(k)} \qquad \hat{c}(k) = \overline{\hat{f}(-k)}$$

(h) $\int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)\overline{\hat{g}(k)}\, dk$

Proof: (a) The Fourier transform of $af + bg$ is

$$\int_{-\infty}^{\infty} e^{-ikx}[af(x) + bg(x)]\, dx = a\int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx + b\int_{-\infty}^{\infty} e^{-ikx} g(x)\, dx = a\hat{f}(k) + b\hat{g}(k)$$

(b) By induction, it suffices to prove the case $n = 1$. By integration by parts, the Fourier transform of the first derivative $f'(x)$ is

$$\int_{-\infty}^{\infty} e^{-ikx} f'(x)\ dx = -\int_{-\infty}^{\infty} f(x)\big(\tfrac{d}{dx}e^{-ikx}\big)\ dx = ik\int_{-\infty}^{\infty} e^{-ikx} f(x)\ dx = ik\hat{f}(k)$$

The boundary terms vanished because $\lim\limits_{x\to\infty} e^{-ikx}f(x) = \lim\limits_{x\to-\infty} e^{-ikx}f(x) = 0$.

(c) Again, by induction, it suffices to prove the case $n = 1$.

$$\tfrac{d}{dk}\hat{f}(k) = \tfrac{d}{dk}\int_{-\infty}^{\infty} e^{-ikx} f(x)\ dx = \int_{-\infty}^{\infty} \tfrac{\partial}{\partial k}\big(e^{-ikx}f(x)\big)\ dx = \int_{-\infty}^{\infty} (-ix)e^{-ikx} f(x)\ dx$$

is indeed $-i$ times the Fourier transform of $xf(x)$. The second equality, which moved the derivative with respect to k past the integral sign is justified by the elementary Lemma 11.b.

(d) The Fourier transform of $T_a f$ is

$$\int_{-\infty}^{\infty} e^{-ikx} f(x-a)\ dx \overset{x'=x-a}{=} \int_{-\infty}^{\infty} e^{-ik(x'+a)} f(x')\ dx' = e^{-ika}\hat{f}(k)$$

(e) Denote by $\hat{f}(k)$ the Fourier transform of the function $f(x) = e^{-x^2/2}$. By part (c) of this Theorem, $\frac{d}{dk}\hat{f}(k)$ is the Fourier transform of $-ixf(x) = -ixe^{-x^2/2} = i\frac{d}{dx}e^{-x^2/2} = if'(x)$. Thus by parts (a) and (b) of this Theorem, $\frac{d}{dk}\hat{f}(k) = -k\hat{f}(k)$ and

$$\tfrac{d}{dk}\big(\hat{f}(k)e^{k^2/2}\big) = e^{k^2/2}\big(\tfrac{d}{dk}\hat{f}(k) + k\hat{f}(k)\big) = 0$$

for all $k \in \mathbb{R}$. Consequently $\hat{f}(k)e^{k^2/2}$ must be some constant, independent of k. Hence to determine $\hat{f}(k)$ we need only to determine the value of that constant, which we may do by computing $\hat{f}(k)e^{k^2/2}\big|_{k=0} = \hat{f}(0)$. Since $\hat{f}(0) = \int_{-\infty}^{\infty} e^{-x^2/2}\ dx > 0$, it is determined by

$$\hat{f}(0)^2 = \Big[\int_{-\infty}^{\infty} e^{-x^2/2}\ dx\Big]^2 = \Big[\int_{-\infty}^{\infty} e^{-x^2/2}\ dx\Big]\Big[\int_{-\infty}^{\infty} e^{-y^2/2}\ dx\Big] = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)/2} dx dy$$

Changing to polar coordinates,

$$\hat{f}(0)^2 = \int_0^{\infty} dr\ r\int_0^{2\pi} d\theta\ e^{-r^2/2} = 2\pi\int_0^{\infty} dr\ re^{-r^2/2} = 2\pi\Big[-e^{-r^2/2}\Big]_0^{\infty} = 2\pi$$

Thus $\hat{f}(0) = \sqrt{2\pi}$ which tells us that $\hat{f}(k)e^{k^2/2} = \sqrt{2\pi}$ and hence that $\hat{f}(k) = \sqrt{2\pi}e^{-k^2/2}$ for all k.

(f) By the definition in Lemma 3.d,

$$\begin{aligned}
\hat{h}(k) &= \int_{\mathbb{R}} dx\ e^{-ik\cdot x} h(x) = \int_{\mathbb{R}} dx\ \int_{\mathbb{R}} dy\ e^{-ik\cdot y} f(y)\ e^{-ik\cdot(x-y)} g(x-y) \\
&= \int_{\mathbb{R}} dy\ e^{-ik\cdot y} f(y) \int_{\mathbb{R}} dx\ e^{-ik\cdot(x-y)} g(x-y) \qquad \text{by Fubini} \\
&= \int_{\mathbb{R}} dy\ e^{-ik\cdot y} f(y) \int_{\mathbb{R}} dx'\ e^{-ik\cdot x'} g(x') \qquad \text{with } x' = x-y \\
&= \hat{f}(k)\,\hat{g}(k)
\end{aligned}$$

(g) By definition,

$$\begin{aligned}
\hat{r}(k) &= \int_{-\infty}^{\infty} e^{-ikx} f(-x)\ dx = \int_{-\infty}^{\infty} e^{iky} f(y)\ dy \qquad \text{with } y = -x \\
&= \overline{\int_{-\infty}^{\infty} e^{-iky} \overline{f(y)}\ dy} = \overline{\hat{c}(k)} \\
\hat{c}(k) &= \int_{-\infty}^{\infty} e^{-ikx} \overline{f(x)}\ dx = \overline{\int_{-\infty}^{\infty} e^{ikx} f(x)\ dx} \\
&= \overline{\hat{f}(-k)}
\end{aligned}$$

(h) In this proof, we will use Theorem 12, below. (The proof of Theorem 12 will use only parts (b), (c), (d) and (e) of this theorem, which we have already proven.) By Theorem 12, both $\hat{f}(k)$ and $\hat{g}(k)$ are in $\mathcal{S}(\mathbb{R})$. So

$$\tfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)\overline{\hat{g}(k)}\ dk = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dx\ e^{-ikx} f(x)\overline{\hat{g}(k)} = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dk\ e^{-ikx} f(x)\overline{\hat{g}(k)}$$

The exchange of order of integration is justified by Fubini. Continuing,

$$\tfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)\overline{\hat{g}(k)}\ dk = \int_{-\infty}^{\infty} dx\ f(x)\Big[\ \overline{\tfrac{1}{2\pi}\int_{-\infty}^{\infty} dk\ e^{ikx} \hat{g}(k)}\ \Big] = \int_{-\infty}^{\infty} dx\ f(x)\overline{g(x)}$$

by Theorem 12. ■

Lemma 11

(a) Let $f : [a,b]\times[c,d] \to \mathbb{R}$ be continuous with $\frac{\partial f}{\partial y}$ existing and continuous. Then $g(y) = \int_a^b f(x,y)\ dx$ is differentiable with $g'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\ dx$.

(b) Let $f : (-\infty,\infty)\times[c,d] \to \mathbb{C}$ be continuous. Assume that $\frac{\partial f}{\partial y}$ exists and is continuous and that there is a constant C such that

$$|f(x,y)|, \big|\tfrac{\partial f}{\partial y}(x,y)\big| \le \tfrac{C}{1+x^2} \qquad \text{and} \qquad \big|\tfrac{\partial f}{\partial y}(x,y) - \tfrac{\partial f}{\partial y}(x,y')\big| \le C\tfrac{|y-y'|}{1+x^2}$$

for all $-\infty < x < \infty$ and $c \le y, y' \le d$. Then $g(y) = \int_{-\infty}^{\infty} f(x,y)\ dx$ is differentiable with $g'(y) = \int_{-\infty}^{\infty} \frac{\partial f}{\partial y}(x,y)\ dx$.

Proof: The proofs of parts (a) and (b) are very similar. We'll only do part (b). The assumptions that, for each $c \le y \le d$, $f(x,y)$, $\frac{\partial f}{\partial y}(x,y)$ are continuous and are bounded in absolute value by $\frac{C}{1+x^2}$ ensure that the integrals $\int_{-\infty}^{\infty} f(x,y)\ dx$ and $\int_{-\infty}^{\infty} \frac{\partial f}{\partial y}(x,y)\ dx$ exist. By the Mean Value Theorem, there is for each $x \in \mathbb{R}$ and each pair $y, y' \in [c,d]$ with $y \ne y'$, a number y'' between y and y' such that

$$\tfrac{f(x,y')-f(x,y)}{y'-y} = \tfrac{\partial f}{\partial y}(x,y'')$$

so that

$$\big|\tfrac{f(x,y')-f(x,y)}{y'-y} - \tfrac{\partial f}{\partial y}(x,y)\big| = \big|\tfrac{\partial f}{\partial y}(x,y'') - \tfrac{\partial f}{\partial y}(x,y)\big| \le C\tfrac{|y-y''|}{1+x^2} \le C\tfrac{|y-y'|}{1+x^2}$$

Consequently, if $y \ne y'$,

$$\begin{aligned}\left|\tfrac{g(y')-g(y)}{y'-y} - \int_{-\infty}^{\infty} \tfrac{\partial f}{\partial y}(x,y)\ dx\right| &= \left|\int_{-\infty}^{\infty} \left\{\tfrac{f(x,y')-f(x,y)}{y'-y} - \tfrac{\partial f}{\partial y}(x,y)\right\}\ dx\right| \\ &\le \int_{-\infty}^{\infty} C\tfrac{|y-y'|}{1+x^2}\ dx = \pi C|y-y'|\end{aligned}$$

This converges to zero as $y' \to y$ and so verifies the definition that $\lim\limits_{y' \to y} \frac{g(y')-g(y)}{y'-y}$ exists and equals $\int_{-\infty}^{\infty} \frac{\partial f}{\partial y}(x,y)\ dx$. ■

Theorem 12 *The maps*

$$\begin{aligned} f(x) \in \mathcal{S}(\mathbb{R}) \mapsto \hat{f}(k) &= \int_{-\infty}^{\infty} e^{-ikx} f(x)\ dx \\ g(k) \in \mathcal{S}(\mathbb{R}) \mapsto \check{g}(x) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} g(k)\ dk \end{aligned}$$

are one–to–one, continuous, linear maps from $\mathcal{S}(\mathbb{R})$ onto $\mathcal{S}(\mathbb{R})$ and are inverses of each other.

Proof: That $\hat{f}$ is linear in f was Theorem 10.a.

We now assume that $f \in \mathcal{S}(\mathbb{R})$ and prove that $\hat{f}(k) \in \mathcal{S}(\mathbb{R})$. Let m, n be nonnegative integers. By parts (b) and (c) of Theorem 10 followed by the product rule, $k^m \frac{d^n}{dk^n}\hat{f}(k)$ is the Fourier transform of

$$\begin{aligned}(-i)^m \tfrac{d^m}{dx^m}\big((-ix)^n f(x)\big) &= (-i)^{m+n} \sum_{\ell=0}^{\min\{m,n\}} \tbinom{m}{\ell}\big(\tfrac{d^\ell}{dx^\ell}x^n\big)\big(\tfrac{d^{m-\ell}}{dx^{m-\ell}}f(x)\big) \\ &= (-i)^{m+n} \sum_{\ell=0}^{\min\{m,n\}} \tbinom{m}{\ell}\tfrac{n!}{(n-\ell)!}x^{n-\ell}f^{(m-\ell)}(x)\end{aligned}$$

Hence

$$
\begin{aligned}
\|\hat f(k)\|_{m,n} &= \sup_{k\in\mathbb{R}} \left|k^m \tfrac{d^n}{dk^n}\hat f(k)\right| \\
&= \sup_{k\in\mathbb{R}} \left|\int_{-\infty}^{\infty} e^{-ikx}\Big[\sum_{\ell=0}^{\min\{m,n\}} \tbinom{m}{\ell}\tfrac{n!}{(n-\ell)!}x^{n-\ell}f^{(m-\ell)}(x)\Big]dx\right| \\
&\le \sum_{\ell=0}^{\min\{m,n\}} \tbinom{m}{\ell}\tfrac{n!}{(n-\ell)!}\int_{-\infty}^{\infty}\left|x^{n-\ell}f^{(m-\ell)}(x)\right|dx \\
&= \sum_{\ell=0}^{\min\{m,n\}} \tbinom{m}{\ell}\tfrac{n!}{(n-\ell)!}\int_{-\infty}^{\infty}\tfrac{1}{1+x^2}\big\{|x|^{n-\ell}+|x|^{n-\ell+2}\big\}\left|f^{(m-\ell)}(x)\right|dx \\
&\le \sum_{\ell=0}^{\min\{m,n\}} \tbinom{m}{\ell}\tfrac{n!}{(n-\ell)!}\big\{\|f\|_{n-\ell,m-\ell}+\|f\|_{n-\ell+2,m-\ell}\big\}\int_{-\infty}^{\infty}\tfrac{1}{1+x^2}\ dx \\
&= \sum_{\ell=0}^{\min\{m,n\}} \pi\tbinom{m}{\ell}\tfrac{n!}{(n-\ell)!}\big\{\|f\|_{n-\ell,m-\ell}+\|f\|_{n-\ell+2,m-\ell}\big\}
\end{aligned}
$$

Since $f \in \mathcal{S}(\mathbb{R})$, the right hand side is finite. This proves that $\|\hat f\|_{m,n}$ is finite for all nonnegative integers m, n, so that $\hat f \in \mathcal{S}(\mathbb{R})$.

It also proves that the map $f \mapsto \hat f$ is continuous, since if the sequence $\{f_j\}_{j\in\mathbb{N}}$ converges to f in $\mathcal{S}(\mathbb{R})$, then

$$
\|\hat f - \hat f_j\|_{m,n} \le \sum_{\ell=0}^{\min\{m,n\}} \pi\tbinom{m}{\ell}\tfrac{n!}{(n-\ell)!}\big\{\|f-f_j\|_{n-\ell,m-\ell}+\|f-f_j\|_{n-\ell+2,m-\ell}\big\}
$$

converges to zero as $j \to \infty$, for all nonnegative integers m, n. So $\{\hat f_j\}_{j\in\mathbb{N}}$ converges to $\hat f$ in $\mathcal{S}(\mathbb{R})$ too.

The proof that the map $g(k) \mapsto \check g(x)$ is a continuous, linear map from $\mathcal{S}(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})$ is similar.

We now assume that $f(x) \in \mathcal{S}(\mathbb{R})$ and prove that the inverse Fourier transform of $\hat f(k)$ is $f(x)$. In symbols, that

$$
f(x) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat f(k)\ dk \tag{5}
$$

We first prove the $(x = 0)$ special case that

$$
f(0) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat f(k)\ dk \tag{6}
$$

Write

$$f(x) = f(0)e^{-x^2/2} + xh(x) \qquad \text{where } h(x) = \begin{cases} \frac{1}{x}\big(f(x) - f(0)e^{-x^2/2}\big) & \text{if } x \neq 0 \\ f'(0) & \text{if } x = 0 \end{cases}$$

By Lemma 13, below, the function $h \in \mathcal{S}(\mathbb{R})$. So, by parts (e) and (c) of Theorem 10,

$$\hat{f}(k) = \sqrt{2\pi} f(0) e^{-k^2/2} + i\tfrac{d}{dk}\hat{h}(k)$$

and

$$\tfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)\ dk = \tfrac{f(0)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-k^2/2}\ dk + \tfrac{i}{2\pi}\int_{-\infty}^{\infty} \tfrac{d}{dk}\hat{h}(k)\ dk$$

The first term

$$\tfrac{f(0)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-k^2/2}\ dk = f(0)$$

by the computation at the end of the proof of Theorem 10.e. The second term is $\frac{i}{2\pi}$ times

$$\int_{-\infty}^{\infty} \tfrac{d}{dk}\hat{h}(k)\ dk = \lim_{A,B\to\infty}\int_{-A}^{B} \tfrac{d}{dk}\hat{h}(k)\ dk = \lim_{A,B\to\infty}\big[\hat{h}(B) - \hat{h}(-A)\big] = 0$$

Here we have used the fundamental theorem of calculus and the decay at $\pm\infty$ which follows from the fact that $\hat{h} \in \mathcal{S}(\mathbb{R})$, which, in turn, follows from $h \in \mathcal{S}(\mathbb{R})$. This completes the proof of (6).

Replacing f by $T_{-x}f$ in (6) gives

$$\big(T_{-x}f\big)(0) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{T_{-x}f}(k)\ dk$$

Using $f(x) = \big(T_{-x}f\big)(0)$ and $\widehat{T_{-x}f}(k) = e^{ikx}\hat{f}(k)$ gives (5).

The proof that

$$g(k) = \int_{-\infty}^{\infty} e^{-ikx}\check{g}(x)\ dx \tag{7}$$

is similar. The formulae (5) and (7) show that the maps $f(x) \mapsto \hat{f}(k)$ and $g(k) \mapsto \check{g}(x)$ are onto $\mathcal{S}(\mathbb{R})$ and are inverses of each other. ∎

Lemma 13 *Let $f \in \mathcal{S}(\mathbb{R})$ and define*

$$h(x) = \begin{cases} \frac{1}{x}\big(f(x) - f(0)e^{-x^2/2}\big) & \text{if } x \neq 0 \\ f'(0) & \text{if } x = 0 \end{cases}$$

Then $h \in \mathcal{S}(\mathbb{R})$.

Proof: That h is C^∞ away from $x=0$ and decays faster than any polynomial at $\pm\infty$ is obvious, because the same is true for the functions $\frac{1}{x}$, $f(x)$ and $e^{-x^2/2}$. So we only need to prove that h is C^∞ at $x=0$. To do so, we write $h(x) = h_1(x) + f(0)h_2(x)$, where

$$h_1(x) = \begin{Bmatrix} \frac{1}{x}\big(f(x) - f(0)\big) & \text{if } x \ne 0 \\ f'(0) & \text{if } x = 0 \end{Bmatrix} \qquad h_2(x) = \begin{Bmatrix} \frac{1}{x}\big(1 - e^{-x^2/2}\big) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{Bmatrix}$$

and prove that both h_1 and h_2 are C^∞ at $x=0$. In the case of h_2, this follows from the representation

$$h_2(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \tfrac{1}{n!} \tfrac{1}{2^n} x^{2n-1}$$

This representation is valid both for $x=0$ and $x \ne 0$ (using the power series expansion of the exponential). As the power series has infinite radius of convergence, h_2 is C^∞ on all of $\mathbb{R}$. For h_1, we use the fundamental theorem of calculus followed by the change of variables $t = xs$ to write

$$f(x) - f(0) = \int_0^x f'(t)\ dt = x \int_0^1 f'(xs)\ ds$$

Thus

$$h_1(x) = \int_0^1 f'(xs)\ ds$$

and this representation is valid both for $x=0$ and $x \ne 0$. The right hand side is C^∞ for all $x \in \mathbb{R}$, by repeated applications of the elementary Lemma 11.a. ■

Theorem 14 *The Fourier transform (3) has a unique continuous extension to $L^2(\mathbb{R})$. The inverse Fourier transform (4) has unique continuous extension to $L^2(\mathbb{R})$. The two extensions are inverses of each other.*

Proof: This is an immediate consequence of the B.L.T. Theorem and Theorems 10.h and 12.

■

Lemma 15 (the Riemann–Lebesgue lemma) *The Fourier transform (3) extends uniquely to a bounded map from $L^1(\mathbb{R})$ to $C_\infty(\mathbb{R})$, the space of continuous functions on $\mathbb{R}$ that vanish at infinity, equipped with the sup norm. So, if $f \in L^1(\mathbb{R})$, then $\hat{f}(k)$ is continuous and*

$$\sup_k |\hat{f}(k)| \le \|f\|_{L^1(\mathbb{R})} \qquad \lim_{|k|\to\infty} \hat{f}(k) = 0$$

Proof: By Theorem 12, the Fourier transform maps $\mathcal{S}(\mathbb{R})$, which is dense in $L^1(\mathbb{R})$, into $\mathcal{S}(\mathbb{R}) \subset C_\infty(\mathbb{R})$. It now suffices to recall that $C_\infty(\mathbb{R})$ is a Banach Space, observe that

$$\|\hat{f}\|_{C_\infty(\mathbb{R})} = \|\hat{f}\|_{L^\infty(\mathbb{R})} \le \|f\|_{L^1(\mathbb{R})}$$

and apply the B.L.T. Theorem. ■

Definition 16 We define the Fourier transform of the tempered distribution $f \in \mathcal{S}'(\mathbb{R})$ to be the tempered distribution

$$\langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle$$

The motivation for this definition is the computation that, if f and φ are both in $\mathcal{S}(\mathbb{R})$, then, writing $\varphi(k) = \overline{\psi(k)}$

$$\begin{aligned}
\left\langle \hat{f}, \varphi \right\rangle &= \int_{-\infty}^{\infty} \hat{f}(k)\varphi(k)\ dk = \int_{-\infty}^{\infty} \hat{f}(k)\overline{\psi(k)}\ dk \\
&= 2\pi \int_{-\infty}^{\infty} f(x)\overline{\check{\psi}(x)}\ dx \qquad \text{(by Theorem 10.h and Theorem 12)} \\
&= \int_{-\infty}^{\infty} f(x)\hat{\varphi}(x)\ dx
\end{aligned}$$

since

$$\overline{\check{\psi}(x)} = \tfrac{1}{2\pi}\overline{\int_{-\infty}^{\infty} e^{ikx}\psi(k)\ dk} = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\overline{\psi(k)}\ dk = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\varphi(k)\ dk = \tfrac{1}{2\pi}\hat{\varphi}(x)$$

Example 17 The Fourier transform of the Dirac delta function is given by

$$\langle \hat{\delta}, \varphi \rangle = \langle \delta, \hat{\varphi} \rangle = \hat{\varphi}(0) = \int_{-\infty}^{\infty} \varphi(x)\ dx = \langle 1, \varphi \rangle$$

That is, $\hat{\delta}$ is the constant function 1.

Example 18 The Fourier transform of the constant function 1, viewed as a tempered distribution, is

$$\langle \hat{1}, \varphi \rangle = \langle 1, \hat{\varphi} \rangle = \int_{-\infty}^{\infty} \hat{\varphi}(k)\ dk = 2\pi\varphi(0)$$

by (6). That is, the Fourier transform of the constant function 1 is $2\pi\delta(k)$.

For a more extensive treatment of Fourier transforms, see, for example, [Reed and Simon, Volume 2, §IX.1]. For a more extensive treatment of tempered distributions see, for example, [Reed and Simon, Volume 1, §V.3].

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