Spectral Theory Examples

Example 1 (Spectrum of Multiplication Operators) Let
- $(X, \mathcal{M}, \mu)$ be a σ–finite (actually semifinite will do) measure space,
- $1 \le p \le \infty$ and
- $a : X \to \mathbb{C}$ be a bounded measurable function on X.

Define the bounded operator $A : L^p(X, \mathcal{M}, \mu) \to L^p(X, \mathcal{M}, \mu)$ by

$$(A\varphi)(x) = a(x)\varphi(x)$$

Point spectrum: Let $\lambda \in \mathbb{C}$. Then

$$\begin{aligned} \lambda\mathbb{1} - A \text{ is injective } &\iff \Big\{\varphi \in L^p(X, \mathcal{M}, \mu),\ (\lambda - a(x))\varphi(x) = 0 \text{ a.e.} \implies \varphi(x) = 0 \text{ a.e.}\Big\} \\ &\iff \lambda - a(x) \ne 0 \text{ a.e.} \\ &\iff \mu\big(\{x \in X \,|\, a(x) = \lambda\}\big) = 0 \end{aligned}$$

Hence

$$\sigma_p(A) = \big\{\ \lambda \in \mathbb{C}\ \big|\ \mu\big(\{x \in X \,|\, a(x) = \lambda\}\big) > 0\ \big\}$$

Resolvent Set: Let $\lambda \in \mathbb{C} \setminus \sigma_p(A)$. Then $\lambda\mathbb{1} - A$ has an inverse operator, which we now determine.

$$(\lambda\mathbb{1} - A)\varphi = \psi \iff \big(\lambda - a(x)\big)\varphi(x) = \psi(x) \text{ a.e.} \iff \varphi(x) = \tfrac{1}{\lambda - a(x)}\psi(x) \text{ a.e.}$$

Hence $(\lambda\mathbb{1} - A)^{-1}$ is the operator of multiplication by $\frac{1}{\lambda - a(x)}$ with the domain consisting of $\big\{\ \psi \in L^p(X, \mathcal{M}, \mu)\ \big|\ \frac{1}{\lambda - a(x)}\psi(x) \in L^p(X, \mathcal{M}, \mu)\ \big\}$. This is a bounded operator if and only if there is a $K > 0$ such that $\frac{1}{|\lambda - a(x)|} \le K$ almost everywhere. Thus

$$\rho(A) = \big\{\ \lambda \in \mathbb{C}\ \big|\ \exists\ \varepsilon > 0 \text{ such that } |\lambda - a(x)| \ge \varepsilon \text{ a.e.}\ \big\}$$

Residual spectrum: Let $\lambda \in \mathbb{C} \setminus \big(\sigma_p(A) \cup \rho(A)\big)$. So $\mu\big(\{x \in X \,|\, a(x) = \lambda\}\big) = 0$, but on the other hand $\mu\big(\{x \in X \,|\, |a(x) - \lambda| < \varepsilon\}\big) > 0$ for every $\varepsilon > 0$. We wish to determine if the range of $\lambda\mathbb{1} - A$, i.e. the domain of $(\lambda\mathbb{1} - A)^{-1}$, is dense.

- First consider $1 \le p < \infty$. I claim that in this case the range is dense. So pick any $\psi \in L^p$. We have to show that ψ is a limit of ψ_n's in the range. Set, for each $n \in \mathbb{N}$,

  $$E_n = \big\{\ x \in X\ \big|\ |a(x) - \lambda| \ge \tfrac{1}{n}\ \big\}$$

  The range of $\lambda\mathbb{1} - A$ contains $\big\{\ \chi_{E_n}\psi\ \big|\ \psi \in L^p(X, \mathcal{M}, \mu),\ n \in \mathbb{N}\ \big\}$ because, for every $\psi \in L^p(X, \mathcal{M}, \mu)$ and every $n \in \mathbb{N}$, $\chi_{E_n}\psi$ is the image under $\lambda\mathbb{1} - A$ of

$\frac{1}{\lambda - a(x)} \chi_{E_n}(x)\psi(x) \in L^p(X, \mathcal{M}, \mu)$. Furthermore, $\chi_{E_n}\psi$ converges pointwise almost everywhere to ψ as n tends to ∞, because $\mu\big(\{x \in X \mid |a(x)-\lambda| = 0\}\big)$ has measure zero. Furthermore $|\chi_{E_n}\psi|$ is bounded by $|\psi| \in L^p$ for all n. As $1 \le p < \infty$, the Lebesgue dominated convergence theorem implies that $\chi_{E_n}\psi$ converges in $L^p(X, \mathcal{M}, \mu)$ to ψ as $n \to \infty$. Thus the range of range of $\lambda\mathbb{1} - A$ is dense in $L^p(X, \mathcal{M}, \mu)$ if $1 \le p < \infty$.

- On the other hand, if $p = \infty$, the constant function 1 is not in the closure of the range of $\lambda\mathbb{1} - A$ because, for every $0 \ne \varphi \in L^p(X, \mathcal{M}, \mu)$, there is some set of nonzero measure on which $|\lambda - a(x)| \le \frac{1}{2\|\varphi\|_\infty}$ and hence on which $\big|1 - [\lambda - a(x)]\varphi]\big| \ge \frac{1}{2}$.

By way of summary,

$$\rho(A) = \big\{ \lambda \in \mathbb{C} \bigm| \exists\, \varepsilon > 0 \text{ such that } |\lambda - a(x)| \ge \varepsilon \text{ a.e.} \big\}$$

$$\sigma(A) = \big\{ \lambda \in \mathbb{C} \bigm| \nexists\, \varepsilon > 0 \text{ such that } |\lambda - a(x)| \ge \varepsilon \text{ a.e.} \big\}$$

$$\sigma_p(A) = \big\{ \lambda \in \mathbb{C} \bigm| \mu\big(\{x \in X \mid a(x) = \lambda\}\big) > 0 \big\}$$

$$\sigma_r(A) = \begin{cases} \emptyset & \text{if } 1 \le p < \infty \\ \big\{ \lambda \in \mathbb{C} \bigm| \nexists\, \varepsilon > 0 \text{ such that } |\lambda - a(x)| \ge \varepsilon \text{ a.e.} \big\} \setminus \sigma_p(A) & \text{if } p = \infty \end{cases}$$

$$\sigma_c(A) = \begin{cases} \big\{ \lambda \in \mathbb{C} \bigm| \nexists\, \varepsilon > 0 \text{ such that } |\lambda - a(x)| \ge \varepsilon \text{ a.e.} \big\} \setminus \sigma_p(A) & \text{if } 1 \le p < \infty \\ \emptyset & \text{if } p = \infty \end{cases}$$

Example 2 (Spectrum of Shift Operators) Define the right and left shift operators acting on ℓ^2 by

$$L(\alpha_1, \alpha_2, \alpha_3, \cdots) = (\alpha_2, \alpha_3, \cdots)$$
$$R(\alpha_1, \alpha_2, \alpha_3, \cdots) = (0, \alpha_1, \alpha_2, \alpha_3, \cdots)$$

First observe that $\|L\| = \|R\| = 1$ so that $\big\{ \lambda \in \mathbb{C} \bigm| |\lambda| > 1 \big\}$ is contained in the resolvent sets of both L and R.

Point spectrum of L: Since

$$\begin{aligned} & L(\alpha_1, \alpha_2, \alpha_3, \cdots) = \lambda(\alpha_1, \alpha_2, \alpha_3, \cdots) \\ \iff & (\alpha_2, \alpha_3, \cdots) \qquad = \lambda(\alpha_1, \alpha_2, \alpha_3, \cdots) \\ \iff & \alpha_{j+1} = \lambda\alpha_j \text{ for all } j \in \mathbb{N} \\ \iff & (\alpha_1, \alpha_2, \alpha_3, \cdots) \quad = \alpha_1(1, \lambda, \lambda^2, \lambda^3, \cdots) \end{aligned}$$

and since $(1, \lambda, \lambda^2, \lambda^3, \cdots) \in \ell^2$ if and only if $|\lambda| < 1$

$$\sigma_p(L) = \big\{ \lambda \in \mathbb{C} \bigm| |\lambda| < 1 \big\}$$

Point spectrum of R: Since

$$\begin{aligned} R(\alpha_1,\alpha_2,\alpha_3,\cdots) &= \lambda(\alpha_1,\alpha_2,\alpha_3,\cdots) \\ \iff (0,\alpha_1,\alpha_2,\alpha_3,\cdots) &= \lambda(\alpha_1,\alpha_2,\alpha_3,\cdots) \\ \iff (\alpha_1,\alpha_2,\alpha_3,\cdots) \quad &= (0,0,0,\cdots) \end{aligned}$$

we have

$$\sigma_p(R) = \emptyset$$

Other spectrum of L: Since the spectrum of any operator is closed and since $\|L\| = 1$, we must have $\rho(L) = \{\ \lambda \in \mathbb{C}\ \big|\ |\lambda| > 1\ \}$ and $\sigma(L) = \{\ \lambda \in \mathbb{C}\ \big|\ |\lambda| \le 1\ \}$. So the only remaining question is what part of the unit circle $|\lambda| = 1$ consists of residual spectrum. If λ were in the residual spectrum of L, $\overline{\lambda}$ would be in the point spectrum of $L^* = R$ and we know that R has no point spectrum. So $\sigma_r(L) = \emptyset$.

Other spectrum of R: If $|\lambda| < 1$, then $\lambda \in \sigma_p(L)$ and consequently, $\overline{\lambda} \in \sigma_p(L^*) \cup \sigma_r(L^*)$. As $L^* = R$ and $\sigma_p(R) = \emptyset$ we have that $\{\ \lambda \in \mathbb{C}\ \big|\ |\lambda| < 1\ \} \subset \sigma_r(R)$. Since the spectrum of any operator is closed and since $\|R\| = 1$, we must have $\rho(R) = \{\ \lambda \in \mathbb{C}\ \big|\ |\lambda| > 1\ \}$ and $\sigma(R) = \{\ \lambda \in \mathbb{C}\ \big|\ |\lambda| \le 1\ \}$. So the only remaining question is which part of the unit circle $|\lambda| = 1$ consists of residual spectrum. But if some λ with $|\lambda| = 1$ were in the residual spectrum of R, $\overline{\lambda}$ would be in the point spectrum of $R^* = L$ and we know that the point spectrum of L does not intersect the unit circle. So

$$\sigma_r(R) = \{\ \lambda \in \mathbb{C}\ \big|\ |\lambda| < 1\ \}$$

Example 3 (Translation Operators on a Torus) Think of $[0, 2\pi)$ as a circle by identifying 0 with 2π. Define, for each $\alpha \in \mathbb{R}$, the operator, T_α, of "translation by α" acting on $L^2\big([0, 2\pi)\big)$ by

$$(T_\alpha\varphi)(x) = \varphi\big(x - \alpha \bmod 2\pi\big)$$

Here $x - \alpha \bmod 2\pi$ is defined to be $x - \alpha + 2k\pi$ where k is the unique integer such that $0 \le x - \alpha + 2k\pi < 2\pi$. Observe that

$$T_\alpha T_\beta = T_{\alpha+\beta} \qquad T_0 = \mathbb{1} \qquad T_\alpha^* = T_\alpha^{-1} = T_{-\alpha}$$

for all $\alpha, \beta \in \mathbb{R}$.

Now $\big\{\mathbf{e}_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}\big\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2\big([0, 2\pi)\big)$. Hence the map $\mathcal{F} : L^2\big([0, 2\pi)\big) \to L^2(\mathbb{Z})$ (with the counting measure on $\mathbb{Z}$), defined by

$$(\mathcal{F}\varphi)(n) = \langle \mathbf{e}_n, \varphi \rangle_{L^2([0,2\pi))} \qquad \text{for all } n \in \mathbb{Z}$$

is unitary. Consequently $\mathcal{F}T_\alpha\mathcal{F}^* : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ has the same spectrum as T_α. For any $\mathbf{c} \in L^2(\mathbb{Z})$,

$$\big(\mathcal{F}^*\mathbf{c}\big)(x) = \sum_{\ell\in\mathbb{Z}} c(\ell)\mathbf{e}_\ell(x)$$

so that

$$\big(T_\alpha\mathcal{F}^*\mathbf{c}\big)(x) = \big(\mathcal{F}^*\mathbf{c}\big)(x-\alpha \bmod 2\pi) = \sum_{\ell\in\mathbb{Z}} c(\ell)\mathbf{e}_\ell(x-\alpha) = \sum_{\ell\in\mathbb{Z}} e^{-i\ell\alpha}c(\ell)\mathbf{e}_\ell(x)$$

and

$$\big(\mathcal{F}T_\alpha\mathcal{F}^*\mathbf{c}\big)(n) = \langle \mathbf{e}_n, T_\alpha\mathcal{F}^*\mathbf{c}\rangle_{L^2([0,2\pi))} = e^{-in\alpha}c(n)$$

Thus $\mathcal{F}T_\alpha\mathcal{F}^*$ is the operator of multiplication by $a(n) = e^{-in\alpha}$ and

$$\sigma_p(T_\alpha) = \sigma_p(\mathcal{F}T_\alpha\mathcal{F}^*) = \big\{\ e^{-in\alpha}\ \big|\ n \in \mathbb{Z}\ \big\}$$

If α is a rational multiple of 2π, then the range of $a(n)$, i.e. $\sigma_p(T_\alpha)$, consists just of a finite number of points on the unit circle in $\mathbb{C}$. In this case, every $\lambda \notin \sigma_p(\mathcal{F}T_\alpha\mathcal{F}^*)$ is a nonzero distance from the range of $a(n)$ and so is in $\rho(T)$. So, in this case, $\sigma(T) = \sigma(\mathcal{F}T_\alpha\mathcal{F}^*) = \sigma_p(T)$. If α is a not rational multiple of 2π, then the range of $a(n)$ is a dense subset of the unit circle in $\mathbb{C}$. In this case $\sigma(T) = \sigma(\mathcal{F}T_\alpha\mathcal{F}^*) = \big\{\ z \in \mathbb{Z}\ \big|\ |z| = 1\ \big\}$.

Example 4 (Rotation) Define rotation by α on $L^2(\mathbb{R}^2)$ by

$$(R_\alpha\varphi)(x,y) = \varphi\big(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha\big)$$

This action is simplified by going to polar coordinates. We implement the transition to polar coordinates by a unitary map

$$\begin{aligned}\mathcal{P} : L^2(\mathbb{R}^2) &\to L^2\big([0,\infty)\times[0,2\pi), r dr\, d\theta\big)\\ \varphi(x,y) &\mapsto (\mathcal{P}\varphi)(r,\theta) = \varphi\big(r\cos\theta, r\sin\theta\big)\end{aligned}$$

Since

$$\begin{aligned}(\mathcal{P}R_\alpha\varphi)\big(r,\theta\big) &= (R_\alpha\varphi)\big(r\cos\theta, r\sin\theta\big)\\ &= \varphi\big(r\cos\theta\cos\alpha + r\sin\theta\sin\alpha\,,\ r\sin\theta\cos\alpha - r\cos\theta\sin\alpha\big)\\ &= \varphi\big(r\cos(\theta-\alpha)\,,\ r\sin(\theta-\alpha)\big)\end{aligned}$$

we have that

$$(\mathcal{P}R_\alpha\mathcal{P}^{-1}\psi)(r,\theta) = \psi(r, \theta-\alpha \bmod 2\pi)$$

So, in polar coordinates, R_α just translates the θ argument, doing nothing to the r argument. We can "diagonalize" it just as we "diagonalized" T_α in Example 3. We define a unitary map

$$\begin{aligned}\mathcal{F} : L^2\big([0,\infty)\times[0,2\pi)\big) &\to L^2\big([0,\infty)\times\mathbb{Z}\big)\\ \varphi(r,\theta) \mapsto (\mathcal{F}\varphi)(r,n) &= \tfrac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{-in\theta}\varphi(r,\theta)\ d\theta\end{aligned}$$

We are still using the inner product $\int_0^{2\pi} d\theta \int_0^\infty dr\, r\ \overline{\varphi(r,\theta)}\,\psi(r,\theta)$ for $L^2\big([0,\infty)\times[0,2\pi)\big)$ and we are using the inner product $\sum_{n\in\mathbb{Z}}\int_0^\infty dr\, r\ \overline{\mathbf{c}(r,n)}\,\mathbf{d}(r,n)$ for $L^2\big([0,\infty)\times\mathbb{Z}\big)$. With these inner products, $\mathcal{F}$ is indeed unitary and

$$(\mathcal{F}^*\mathbf{c})(r,\theta) = \sum_{n\in\mathbb{Z}} \tfrac{1}{\sqrt{2\pi}} e^{in\theta}\mathbf{c}(r,n)$$

Once again

$$\begin{aligned}\big(\mathcal{F}\mathcal{P}R_\alpha\mathcal{P}^*\mathcal{F}^*\mathbf{c}\big)(r,n) &= \tfrac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{-in\theta}\big(\mathcal{P}R_\alpha\mathcal{P}^*\mathcal{F}^*\mathbf{c}\big)(r,\theta)\ d\theta\\ &= \tfrac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{-in\theta}\big(\mathcal{F}^*\mathbf{c}\big)(r,\theta-\alpha \bmod 2\pi)\ d\theta\\ &= \tfrac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_0^{2\pi} e^{-in\theta}e^{im(\theta-\alpha)}\mathbf{c}(r,m)\ d\theta\\ &= e^{-in\alpha}\mathbf{c}(r,n)\end{aligned}$$

and the operator $\mathcal{F}\mathcal{P}R_\alpha\mathcal{P}^*\mathcal{F}^*$ is multiplication by $e^{-in\alpha}$ so that

$$\sigma_p(R_\alpha) = \sigma_p(\mathcal{F}\mathcal{P}R_\alpha\mathcal{P}^*\mathcal{F}^*) = \big\{\ e^{-in\alpha}\ \big|\ n\in\mathbb{Z}\ \big\}$$

This time, for each $n\in\mathbb{N}$, the eigenspace corresponding to the eigenvalue $e^{-in\alpha}$ consists, in polar coordinates, of all functions of the form $e^{in\theta}f(r)$ with $f\in L^2\big([0,\infty), r\,dr\big)$. It is infinite dimensional.

As in Example 3, if α is a rational multiple of π, then $\sigma(R_\alpha)=\sigma_p(R_\alpha)$ and if α is not a rational multiple of π, then $\sigma(R_\alpha)$ is the full unit circle.

Example 5 (Translation Operators on $\mathbb{R}$) Define, for each $\alpha\in\mathbb{R}$, the operator, T_α, of "translation by α" acting on $L^2(\mathbb{R})$ by

$$(T_\alpha\varphi)(x) = \varphi(x-\alpha)$$

Observe that

$$T_\alpha T_\beta = T_{\alpha+\beta} \qquad T_\alpha^* = T_\alpha^{-1} = T_{-\alpha}$$

for all $\alpha, \beta \in \mathbb{R}$. If $\alpha = 0$, T_α is the identity operator and $\sigma(T_\alpha) = \sigma_p(T_\alpha) = \{1\}$. So fix any $\alpha \neq 0$. The Fourier transform

$$\mathcal{F} : L^2(\mathbb{R}, dx) \to L^2(\mathbb{R}, \tfrac{1}{2\pi} dk)$$
$$\varphi(x) \mapsto \hat{\varphi}(k) = \int e^{-ikx} \varphi(x)\ dx$$

is a unitary operator. So spectrum of T_α is the same as the spectrum of $\mathcal{F} T_\alpha \mathcal{F}^{-1}$. Since

$$(\mathcal{F} T_\alpha \varphi)(k) = e^{-i\alpha k} (\mathcal{F}\varphi)(k)$$

we have that $(\mathcal{F} T_\alpha \mathcal{F}^{-1} \psi)(k) = e^{-i\alpha k} \psi(k)$. Thus $\mathcal{F} T_\alpha \mathcal{F}^{-1}$ is the operator of multiplication by $e^{-ik\alpha}$. So, if $\alpha \neq 0$,

$$\sigma(T_\alpha) = \left\{\ e^{-ik\alpha}\ \middle|\ k \in \mathbb{R}\ \right\} = \left\{\ z \in \mathbb{C}\ \middle|\ |z| = 1\ \right\} \qquad \sigma_p(T_\alpha) = \sigma_r(T_\alpha) = \emptyset$$

Problem 1 Let A be a translation invariant, bounded operator on $L^2(\mathbb{R})$. "Translation invariant" means that $T_\alpha A T_{-\alpha} = A$ for all $\alpha \in \mathbb{R}$. For simplicity, assume that A is an integral operator

$$(A\varphi)(x) = \int a(x, y) \varphi(y)\ dx$$

Determine the spectrum of A. You may also make whatever regularity and decay assumptions on a that you need to justify your conclusions.

Example 6 (General Point Spectrum) Let $\mathcal{C}$ be any compact subset of $\mathbb{C}$. Let $\mathcal{C}_d = \big\{z_n\big\}_{n \in \mathbb{N}}$ be any countable dense subset of $\mathcal{C}$. We now construct a normal operator on a separable Hilbert space whose point spectrum is exactly $\mathcal{C}_d$ and whose spectrum is exactly $\mathcal{C}$. The Hilbert space is just ℓ^2 and the operator is

$$(A\mathbf{x})_n = z_n x_n \qquad \text{for all } n \in \mathbb{N} \text{ and } \mathbf{x} = \big(x_n\big)_{n \in \mathbb{N}} \in \ell^2$$

If we are willing to use a Hilbert space that is not separable, we can construct another normal operator whose point spectrum is $\mathcal{C}$ and whose spectrum is $\mathcal{C}$ too. Just write $\mathcal{C} = \big\{z_i\big\}_{i \in \mathcal{I}}$ with $\mathcal{I}$ an index set (for example we could take $\mathcal{I} = \mathcal{C}$.) Define

$$\mathcal{H} = \left\{\ \mathbf{x} = \big(x_i\big)_{i \in \mathcal{I}}\ \middle|\ \sum_{i \in \mathcal{I}} |x_i|^2 < \infty\ \right\}$$

with the inner product

$$\langle \mathbf{y}, \mathbf{x} \rangle = \sum_{i \in \mathcal{I}} \overline{y_i} x_i \quad \text{for all } \mathbf{x} = \big(x_i\big)_{i \in \mathcal{I}},\ \mathbf{y} = \big(y_i\big)_{i \in \mathcal{I}} \in \mathcal{H}$$

By definition, the condition $\sum_{i \in \mathcal{I}} |x_i|^2 < \infty$ can only be satisfied if only countable many components x_i of $\mathbf{x}$ are nonzero. It now suffices to take the operator

$$(B\mathbf{x})_i = z_i x_i \qquad \text{for all } i \in \mathcal{I} \text{ and } \mathbf{x} = \big(x_n\big)_{n \in \mathbb{N}} \in \ell^2$$