# Families of Commuting Normal Matrices

**Definition M.1 (Notation)**

i) $\mathbb{C}^n \;=\; \big\{\ \mathbf{v} = (v_1, \cdots, v_n)\ \big|\ v_i \in \mathbb{C} \text{ for all } 1 \le i \le n\ \big\}$

ii) If $\lambda \in \mathbb{C}$ and $\mathbf{v} = (v_1, \cdots, v_n),\ \mathbf{w} = (w_1, \cdots, w_n) \in \mathbb{C}^n$, then

$$\lambda\mathbf{v} = (\lambda v_1,\ \cdots,\ \lambda v_n) \in \mathbb{C}^n$$
$$\mathbf{v} + \mathbf{w} = (v_1 + w_1,\ \cdots,\ v_n + w_n) \in \mathbb{C}^n$$
$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{j=1}^{n} \bar{v}_j w_j \in \mathbb{C}$$

The $^{-}$ means complex conjugate.

iii) Two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$ are said to be orthogonal (or perpendicular, denoted $\mathbf{v} \perp \mathbf{w}$) if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.

iv) If $\mathbf{v} \in \mathbb{C}^n$ and $A$ is the $m \times n$ matrix whose $(i, j)$ matrix element is $A_{i,j}$, then $A\mathbf{v}$ is the vector in $\mathbb{C}^m$ with

$$(A\mathbf{v})_i = \sum_{j=1}^{n} A_{i,j} v_j \qquad \text{for all } 1 \le i \le m$$

v) A linear subspace $V$ of $\mathbb{C}^n$ is a subset of $\mathbb{C}^n$ that is closed under addition and scalar multiplication. That is, if $\lambda \in \mathbb{C}$ and $\mathbf{v}, \mathbf{w} \in V$, then $\lambda\mathbf{v},\ \mathbf{v} + \mathbf{w} \in V$.

vi) If $V$ is a subset of $\mathbb{C}^n$, then its orthogonal complement is

$$V^\perp = \big\{\ \mathbf{v} \in \mathbb{C}^n\ \big|\ \mathbf{v} \perp \mathbf{w} \text{ for all } \mathbf{w} \in V\ \big\}$$

**Problem M.1** Let $V \subset \mathbb{C}^n$. Prove that $V^\perp$ is a linear subspace of $\mathbb{C}^n$.

**Lemma M.2** *Let $V$ be a linear subspace of $\mathbb{C}^n$ of dimension at least one. Let $A$ be an $n \times n$ matrix that maps $V$ into $V$. Then $A$ has an eigenvector in $V$.*

**Proof:** Let $\mathbf{e}_1,\ \cdots,\ \mathbf{e}_d$ be a basis for $V$. As $A$ maps $V$ into itself, there exist numbers $a_{i,j},\ 1 \le i, j \le d$ such that

$$A\mathbf{e}_j = \sum_{i=1}^{d} a_{i,j} \mathbf{e}_i \qquad \text{for all } 1 \le j \le d$$

Consequently, $A$ maps the vector $\mathbf{w} = \sum_{j=1}^{d} x_j \mathbf{e}_j \in V$ to

$$A\mathbf{w} = \sum_{i,j=1}^{d} a_{i,j} x_j \mathbf{e}_i$$

so that $\mathbf{w}$ is an eigenvector of $A$ of eigenvalue $\lambda$ if and only if (1) not all of the $x_i$'s are zero and (2)

$$\begin{aligned} \sum_{i,j=1}^{d} a_{i,j} x_j \mathbf{e}_i = \lambda \sum_{i=1}^{d} x_i \mathbf{e}_i \iff & \sum_{j=1}^{d} a_{i,j} x_j = \lambda x_i && \text{for all } 1 \le i \le d \\ \iff & \sum_{j=1}^{d} \big(a_{i,j} - \lambda\delta_{i,j}\big) x_j = 0 && \text{for all } 1 \le i \le d \end{aligned}$$

For any given $\lambda$, the linear system of equations "$\sum_{j=1}^{d} \big(a_{i,j} - \lambda\delta_{i,j}\big) x_j = 0$ for all $1 \le i \le d$" has a nontrivial solution $(x_1, \cdots, x_d)$ if and only if the $d \times d$ matrix $\big[a_{i,j} - \lambda\delta_{i,j}\big]_{1 \le i,j \le d}$ fails to be invertible and this, in turn, is the case if and only if $\det\big[a_{i,j} - \lambda\delta_{i,j}\big] = 0$. But $\det\big[a_{i,j} - \lambda\delta_{i,j}\big] = 0$ is a polynomial of degree $d$ in $\lambda$ and so always vanishes for at least one value of $\lambda$. ■

**Definition M.3 (Commuting)** Two $n \times n$ matrices $A$ and $B$ are said to commute if $AB = BA$.

**Lemma M.4** *Let $n \ge 1$ be an integer, $V$ be a linear subspace of $\mathbb{C}^n$ of dimension at least one and let $\mathcal{F}$ be a nonempty set of $n \times n$ mutually commuting matrices that map $V$ into $V$. That is, $A, B \in \mathcal{F} \Rightarrow AB = BA$ and $A \in \mathcal{F},\ \mathbf{w} \in V \Rightarrow A\mathbf{w} \in V$. Then there exists a nonzero vector $\mathbf{v} \in V$ that is an eigenvector for every matrix in $\mathcal{F}$.*

**Proof:** We shall show that

> "There is a linear subspace $W$ of $V$ of dimension at least one, such that each $A \in \mathcal{F}$ is a multiple of the identity matrix when restricted to $W$."

This suffices to prove the lemma. The proof will be by induction on the dimension $d$ of $V$. If $d = 1$, we may take $W = V$, since the restriction of any matrix to a one dimensional vector space is a multiple of the identity.

Suppose that the claim has been proven for all dimensions strictly less than $d$. If every $A \in \mathcal{F}$ is a multiple of the identity, when restricted to $V$, we may take $W = V$ and we are done. If not, pick any $A \in \mathcal{F}$ that is not a multiple of the identity when restricted to $V$.

By Lemma M.2, it has at least one eigenvector $\mathbf{v} \in V$. Let $\lambda$ be the corresponding eigenvalue and set

$$V' = V \cap \left\{ \mathbf{w} \in \mathbb{C}^n \;\middle|\; A\mathbf{w} = \lambda \mathbf{w} \right\}$$

Then $V'$ is a linear subspace of $V$ of dimension strictly less than $d$ (since $A$, restricted to $V$, is not $\lambda \mathbb{1}$). We claim that every $B \in \mathcal{F}$ maps $V'$ into $V'$. To see this, let $B \in \mathcal{F}$ and $\mathbf{w} \in V'$ and set $\mathbf{w}' = B\mathbf{w}$. We wish to show that $\mathbf{w}' \in V'$. But

$$\begin{aligned} A\mathbf{w}' = AB\mathbf{w} &= BA\mathbf{w} && (A \text{ and } B \text{ commute}) \\ &= B\lambda\mathbf{w} && (\text{Definition of } V') \\ &= \lambda B\mathbf{w} = \lambda \mathbf{w}' \end{aligned}$$

so $\mathbf{w}'$ is indeed in $V'$. We have verified that $V'$ has dimension at least one and strictly smaller than $d$ and that every $B \in \mathcal{F}$ maps $V'$ into $V'$. So we may apply the inductive hypothesis with $V$ replaced by $V'$. ∎

**Definition M.5 (Adjoint)** The adjoint of the $r \times c$ matrix $A$ is the $c \times r$ matrix

$$A^*_{i,j} = \overline{A_{j,i}}$$

**Problem M.2** Let $A$ and $B$ be any $n \times n$ matrices. Prove that $B = A^*$ if and only if $\langle B\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, A\mathbf{w} \rangle$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$.

**Problem M.3** Let $A$ be any $n \times n$ matrix. Let $V$ be any linear subspace of $\mathbb{C}^n$ and $V^\perp$ its orthogonal complement. Prove that if $AV \subset V$ (i.e. $\mathbf{w} \in V \Rightarrow A\mathbf{w} \in V$), then $A^* V^\perp \subset V^\perp$.

**Definition M.6 (Normal, Self–Adjoint, Unitary)**

i) An $n \times n$ matrix $A$ is normal if $AA^* = A^*A$. That is, if $A$ commutes with its adjoint.

ii) An $n \times n$ matrix $A$ is self–adjoint if $A = A^*$.

iii) An $n \times n$ matrix $U$ is unitary if $UU^* = \mathbb{1}$. Here $\mathbb{1}$ is the $n \times n$ identity matrix. Its $(i, j)$ matrix element is one if $i = j$ and zero otherwise.

**Problem M.4** Let $A$ be a normal matrix. Let $\lambda$ be an eigenvalue of $A$ and $V$ the eigenspace of $A$ of eigenvalue $\lambda$. Prove that $V$ is the eigenspace of $A^*$ of eigenvalue $\bar{\lambda}$.

**Problem M.5** Let $A$ be a normal matrix. Let $\mathbf{v}$ and $\mathbf{w}$ be eigenvectors of $A$ with different eigenvalues. Prove that $\mathbf{v} \perp \mathbf{w}$.

**Problem M.6** Let $A$ be a self-adjoint matrix. Prove that

a) $A$ is normal

b) Every eigenvalue of $A$ is real.

**Problem M.7** Let $U$ be a unitary matrix. Prove that

a) $U$ is normal

b) Every eigenvalue $\lambda$ of $U$ obeys $|\lambda| = 1$, i.e. is of modulus one.

**Theorem M.7** *Let $n \geq 1$ be an integer. Let $\mathcal{F}$ be a nonempty set of $n \times n$ mutually commuting normal matrices. That is, $A, B \in \mathcal{F} \Rightarrow AB = BA$ and $A \in \mathcal{F} \Rightarrow AA^* = A^*A$. Then there exists an orthonormal basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ of $\mathbb{C}^n$ such that $\mathbf{e}_j$ is an eigenvector of $A$ for every $A \in \mathcal{F}$ and $1 \leq j \leq n$.*

**Proof:** By Lemma M.4, with $V = \mathbb{C}^n$, there exists a nonzero vector $\mathbf{v}_1$ that is an eigenvector for every $A \in \mathcal{F}$. Set $\mathbf{e}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$ and $V_1 = \left\{ \lambda \mathbf{e}_1 \,\middle|\, \lambda \in \mathbb{C} \right\}$. By Problem M.4, $\mathbf{e}_1$ is also an eigenvector of $A^*$ for every $A \in \mathcal{F}$, so $A^* V_1 \subset V_1$ for all $A \in \mathcal{F}$. By Problem M.3, $AV_1^\perp \subset V_1^\perp$ for all $A \in \mathcal{F}$.

By Lemma M.4, with $V = V_1^\perp$, there exists a nonzero vector $\mathbf{v}_2 \in V_1^\perp$ that is an eigenvector for every $A \in \mathcal{F}$. Choose $\mathbf{e}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}$. As $\mathbf{e}_2 \in V_1^\perp$, $\mathbf{e}_2$ is orthogonal to $\mathbf{e}_1$. Define $V_2 = \left\{ \lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \,\middle|\, \lambda_1, \lambda_2 \in \mathbb{C} \right\}$. By Problem M.4, $\mathbf{e}_2$ is also an eigenvector of $A^*$ for every $A \in \mathcal{F}$, so $A^* V_2 \subset V_2$ for all $A \in \mathcal{F}$. By Problem M.3, $AV_2^\perp \subset V_2^\perp$ for all $A \in \mathcal{F}$.

By Lemma M.4, with $V = V_2^\perp$, there exists a nonzero vector $\mathbf{v}_3 \in V_2^\perp$ that is an eigenvector for every $A \in \mathcal{F}$. Choose $\mathbf{e}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|}$. As $\mathbf{e}_3 \in V_2^\perp$, $\mathbf{e}_3$ is orthogonal to both $\mathbf{e}_1$ and $e_2$. And so on. ■