An Elliptic Function – The Weierstrass Function

Definition W.1 An elliptic function $f(z)$ is a non constant meromorphic function on $\mathbb{C}$ that is doubly periodic. That is, there are two nonzero complex numbers $\omega_1,\ \omega_2$ whose ratio is not real, such that $f(z+\omega_1)=f(z)$ and $f(z+\omega_2)=f(z)$.

Fix two real numbers $\beta,\gamma>0$. The Weierstrass function with primitive periods γ and $i\beta$ is the function $\wp:\mathbb{C}\to\mathbb{C}$ defined by

$$\wp(z)=\frac{1}{z^2}+\sum_{\substack{\omega\in\gamma\mathbb{Z}\oplus i\beta\mathbb{Z}\\ \omega\neq 0}}\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}$$

It is an important example of an elliptic function. Its elementary properties are given in

Problem W.1 Prove that

a) For each fixed $z\in\mathbb{C}\setminus(\gamma\mathbb{Z}\oplus i\beta\mathbb{Z})$, the series $\sum_{\substack{\omega\in\gamma\mathbb{Z}\oplus i\beta\mathbb{Z}\\ \omega\neq 0}}\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}$ converges absolutely. The convergence is uniform on compact subsets of $\mathbb{C}\setminus(\gamma\mathbb{Z}\oplus i\beta\mathbb{Z})$.

b) $\wp(z)$ is analytic on $\mathbb{C}\setminus(\gamma\mathbb{Z}\oplus i\beta\mathbb{Z})$.

c) $\wp(z+\zeta)=\wp(z)$ for all $\zeta\in\gamma\mathbb{Z}\oplus i\beta\mathbb{Z}$.

d) $\wp(-z)=\wp(z)$.

e) $\overline{\wp(z)}=\wp(\bar z)$ for all $\mathbb{C}\setminus(\gamma\mathbb{Z}\oplus i\beta\mathbb{Z})$.

f) $\wp(x)$ and $\wp(x+i\frac{\beta}{2})$ are real for all $x\in\mathbb{R}$ and $\wp(iy)$ and $\wp(iy+\frac{\gamma}{2})$ are real for all $y\in\mathbb{R}$.

The following Lemma is one of the main properties of elliptic functions. It proves that an elliptic function takes each value the same number of times and that number is just the sum of the degrees of its poles.

Theorem W.2 *Let $f(z)$ be an elliptic function with periods $\omega_1,\ \omega_2$. Set $\Omega=\omega_1\mathbb{Z}+\omega_2\mathbb{Z}$. Suppose that $f(z)$ has poles of order $n_1,\ \cdots,\ n_k$ at $p_1+\Omega,\ \cdots,p_k+\Omega$ and is analytic elsewhere. Let c be any complex number. Suppose that $f(z)-c$ has zeroes of order $m_1,\ \cdots,\ m_h$ at*[1] *$z_1+\Omega,\ \cdots,z_h+\Omega$ and is nonzero elsewhere. Then*

$$\sum_{i=1}^{h}m_i=\sum_{i=1}^{k}n_k$$

[1] *Of course, $p_i-p_j\notin\Omega$ and $z_i-z_j\notin\Omega$ for all $i\neq j$.*

Proof: The function $\frac{f'(z)}{f(z)-c}$ is meromorphic, has a simple pole with residue $-n_j$ at $p_j+\Omega$, for $1\le j\le k$, a simple pole with residue m_i at $z_i+\Omega$, for $1\le i\le h$ and no other poles.

For each complex number ζ, let C_ζ be the contour which consists of the four line segments from ζ to $\zeta+\omega_1$ to $\zeta+\omega_1+\omega_2$ to $\zeta+\omega_2$ and back to ζ. Pick a ζ so that $f(z)$ has no pole on C_ζ and does not take the value c on C_ζ. This is always possible because any non constant meromorphic function only has finitely many poles in any compact region and takes the value c at only finitely many points in any compact region. Then, integrating by residues,

$$\int_{C_\zeta} \frac{f'(z)}{f(z)-c}dz = \pm 2\pi i\Big[\sum_{i=1}^h m_i - \sum_{i=1}^k n_k\Big]$$

with a plus sign if C_ζ is positively oriented and a negative sign otherwise. On the other hand, by periodicity,

$$\begin{aligned}\int_{C_\zeta} \frac{f'(z)}{f(z)-c}dz &= \int_\zeta^{\zeta+\omega_1}\Big[\frac{f'(z)}{f(z)-c} - \frac{f'(z+\omega_2)}{f(z+\omega_2)-c}\Big]dz + \int_\zeta^{\zeta+\omega_2}\Big[\frac{f'(z+\omega_1)}{f(z+\omega_1)-c} - \frac{f'(z)}{f(z)-c}\Big]dz\\ &= 0\end{aligned}$$

■

Corollary W.3 *$\wp(z)=\wp(z')$ if and only if $z-z'\in\gamma\mathbb{Z}\oplus i\beta\mathbb{Z}$ or $z+z'\in\gamma\mathbb{Z}\oplus i\beta\mathbb{Z}$. If $z\notin\gamma\mathbb{Z}\oplus i\beta\mathbb{Z}$ but $2z\in\gamma\mathbb{Z}\oplus i\beta\mathbb{Z}$, $\wp'(z)=0$.*

Proof: That

$$z-z'\in\gamma\mathbb{Z}\oplus i\beta\mathbb{Z}\Rightarrow\ \wp(z)=\wp(z') \qquad\qquad z+z'\in\gamma\mathbb{Z}\oplus i\beta\mathbb{Z}\Rightarrow\ \wp(z)=\wp(z')$$

is an immediate consequence of Problem W.1, parts (c) and (d).

The Weierstrass function $\wp$ has a pole of order two at each point of $\gamma\mathbb{Z}\oplus i\beta\mathbb{Z}$ and is analytic elsewhere, so the $\sum_{i=1}^k n_k$ of Theorem W.2 is two. Set $c=\wp(z')$ and $\Omega=\gamma\mathbb{Z}\oplus i\beta\mathbb{Z}$. Then $\wp-c$ has a zero at $z'+\Omega$ and at $-z'+\Omega$. If $z'-(-z')=2z'\notin\Omega$, then by Theorem W.2, these zeroes must be simple and there are no others. Furthermore, z' cannot be in Ω, because $\wp$ has a pole at each point of Ω.

Assume that $z'\notin\Omega$ but $2z'\in\Omega$. This is the only remaining possibility. By Problem W.1, $\wp$ is even and periodic with respect to Ω. Consequently its derivative is odd and periodic with respect to Ω, so that

$$\wp'(z')=-\wp'(-z')=-\wp'(-z'+2z')=-\wp'(z')\Rightarrow\ \wp'(z')=0$$

Thus $\wp-c$ has a zero of order at least two at each point of $z'+\Omega$. By Theorem W.2, these zeroes must be exactly double and there are no others. ■

Define

$$\sigma(z) = z \prod_{\substack{\omega\in\gamma\mathbb{Z}\oplus i\beta\mathbb{Z}\\ \omega\neq 0}} \left(1-\tfrac{z}{\omega}\right) e^{\frac{z}{\omega}+\frac{1}{2}\frac{z^2}{\omega^2}}$$

and

$$\zeta(z) = \tfrac{\sigma'(z)}{\sigma(z)} = \tfrac{1}{z} + \sum_{\substack{\omega\in\gamma\mathbb{Z}\oplus i\beta\mathbb{Z}\\ \omega\neq 0}} \tfrac{1}{z-\omega} + \tfrac{1}{\omega} + \tfrac{z}{\omega^2}$$

Problem W.2 Let $a_1,\ a_2,\ a_3,\ \cdots$ be a sequence of **nonzero** complex numbers. The infinite product $\prod_{n=1}^{\infty} a_n$ is said to converge if $\lim_{N\to\infty}\prod_{n=1}^{N} a_n$ exists **and is nonzero**. In this case, $\prod_{n=1}^{\infty} a_n$ is defined to be $\lim_{N\to\infty}\prod_{n=1}^{N} a_n$.

a) Prove that, if $\prod_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 1$.

b) Suppose that $\sum\limits_{n=1}^{\infty} |a_n - 1| < \infty$. Prove that $\prod_{n=1}^{\infty} a_n$ converges. Prove that if π is any permutation of $1, 2, 3, \cdots$, then $\prod_{n=1}^{\infty} a_{\pi(n)}$ also converges and $\prod_{n=1}^{\infty} a_{\pi(n)} = \prod_{n=1}^{\infty} a_n$.

Problem W.3 Prove that

a) $\left|(1-z)e^{z+\frac{1}{2}z^2} - 1\right| \le 3e^{3/2}|z|^3$ for all complex numbers z with $|z| < 1$.

b) For each fixed $z \in \mathbb{C}\setminus(\gamma\mathbb{Z}\oplus i\beta\mathbb{Z})$, the infinite product $z\prod_{\substack{\omega\in\gamma\mathbb{Z}\oplus i\beta\mathbb{Z}\\ \omega\neq 0}} \left(1-\tfrac{z}{\omega}\right) e^{\frac{z}{\omega}+\frac{1}{2}\frac{z^2}{\omega^2}}$ converges. The convergence is uniform on compact subsets of $\mathbb{C}\setminus(\gamma\mathbb{Z}\oplus i\beta\mathbb{Z})$.

c) $\sigma(z)$ is analytic on $\mathbb{C}$.

d) $\sigma(-z) = -\sigma(z)$.

Problem W.4 Prove that

a) For each fixed $z\in\mathbb{C}\setminus(\gamma\mathbb{Z}\oplus i\beta\mathbb{Z})$, the series $\sum_{\substack{\omega\in\gamma\mathbb{Z}\oplus i\beta\mathbb{Z}\\ \omega\neq 0}} \frac{1}{z-\omega}+\frac{1}{\omega}+\frac{z}{\omega^2}$ converges absolutely. The convergence is uniform on compact subsets of $\mathbb{C}\setminus(\gamma\mathbb{Z}\oplus i\beta\mathbb{Z})$.

b) $\zeta(z)$ is analytic on $\mathbb{C}\setminus(\gamma\mathbb{Z}\oplus i\beta\mathbb{Z})$.

c) $\zeta'(z) = -\wp(z)$.

d) $\zeta(-z) = -\zeta(z)$.

e) $\overline{\zeta(z)} = \zeta(\bar{z})$.

f) $\zeta(x)$ is real for all $x\in\mathbb{R}$ and $\zeta(iy)$ is pure imaginary for all $y\in\mathbb{R}$.

Lemma W.4 *There are constants* $\eta_1\in\mathbb{R}$ *and* $\eta_2\in i\mathbb{R}$ *satisfying*

$$\eta_1 i\beta - \eta_2\gamma \;=\; 2\pi i$$

such that

$$\zeta(z+\gamma) = \zeta(z)+\eta_1 \qquad \zeta(z+i\beta) = \zeta(z)+\eta_2$$
$$\sigma(z+\gamma) = -\sigma(z)\ e^{\eta_1(z+\frac{\gamma}{2})} \qquad \sigma(z+i\beta) = -\sigma(z)\ e^{\eta_2(z+i\frac{\beta}{2})}$$

Proof: For all $\omega \in \gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$,

$$\tfrac{d}{dz}\big[\zeta(z+\omega)-\zeta(z)\big] = \wp(z) - \wp(z+\omega) = 0$$

Hence there exist constants $C_\omega,\ \omega \in \gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$ such that

$$\zeta(z+\omega) = \zeta(z) + C_\omega$$

As $\frac{\sigma(z+\omega)}{\sigma(z)}$ solves the differential equation

$$\tfrac{d}{dz}\tfrac{\sigma(z+\omega)}{\sigma(z)} = \tfrac{\sigma'(z+\omega)}{\sigma(z)} - \tfrac{\sigma(z+\omega)\sigma'(z)}{\sigma^2(z)} = \big[\zeta(z+\omega)-\zeta(z)\big]\tfrac{\sigma(z+\omega)}{\sigma(z)} = C_\omega \tfrac{\sigma(z+\omega)}{\sigma(z)}$$

there exist constants $D_\omega,\ \omega \in \gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$ such that

$$\tfrac{\sigma(z+\omega)}{\sigma(z)} = D_\omega e^{C_\omega z}$$

By Problem W.3.d, if $\frac{\omega}{2} \notin \gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$, $\sigma(\omega/2) \neq 0$ and

$$-1 = \tfrac{\sigma(\omega/2)}{\sigma(-\omega/2)} = \tfrac{\sigma(z+\omega)}{\sigma(z)}\Big|_{z=-\omega/2} = D_\omega e^{-C_\omega \omega/2}$$

so that $D_\omega = -e^{C_\omega\omega/2}$. Set $\eta_1 = C_\gamma = \zeta\big(\frac{\gamma}{2}\big) - \zeta\big(-\frac{\gamma}{2}\big)$ and $\eta_2 = C_{i\beta} = \zeta\big(i\frac{\beta}{2}\big) - \zeta\big(-i\frac{\beta}{2}\big) \in i\mathbb{R}$. By Problem W.4.f, $\eta_1 \in \mathbb{R}$ and $\eta_2 \in i\mathbb{R}$. Then $D_\gamma = -e^{\eta_1\gamma/2}$, $D_{i\beta} = -e^{i\eta_2\beta/2}$ so that

$$\begin{aligned} &\zeta(z+\gamma) = \zeta(z) + \eta_1 && \zeta(z+i\beta) = \zeta(z) + \eta_2 \\ &\sigma(z+\gamma) = -\sigma(z)\ e^{\eta_1(z+\frac{\gamma}{2})} && \sigma(z+i\beta) = -\sigma(z)\ e^{\eta_2(z+i\frac{\beta}{2})} \end{aligned}$$

It remains only to prove that $\eta_1 i\beta - \eta_2\gamma = 2\pi i$. Let C be the contour in $\mathbb{C}$ consisting of the four line segments from $-\frac{\gamma}{2} - i\frac{\beta}{2}$ to $\frac{\gamma}{2} - i\frac{\beta}{2}$ to $\frac{\gamma}{2} + i\frac{\beta}{2}$ to $-\frac{\gamma}{2} + i\frac{\beta}{2}$ and back to $-\frac{\gamma}{2} - i\frac{\beta}{2}$. Then

$$\begin{aligned} \int_C \zeta(z)\,dz &= \int_{-\frac{\gamma}{2}-i\frac{\beta}{2}}^{\frac{\gamma}{2}-i\frac{\beta}{2}} \big[\zeta(z) - \zeta(z+i\beta)\big]\,dz - \int_{-\frac{\gamma}{2}-i\frac{\beta}{2}}^{-\frac{\gamma}{2}+i\frac{\beta}{2}} \big[\zeta(z) - \zeta(z+\gamma)\big]\,dz \\ &= \int_{-\frac{\gamma}{2}-i\frac{\beta}{2}}^{\frac{\gamma}{2}-i\frac{\beta}{2}} \big[-\eta_2\big]\,dz - \int_{-\frac{\gamma}{2}-i\frac{\beta}{2}}^{-\frac{\gamma}{2}+i\frac{\beta}{2}} \big[-\eta_1\big]\,dz \\ &= -\eta_2\gamma + i\eta_1\beta \end{aligned}$$

Inside C, $\zeta(z)$ has only one simple pole with residue 1, so

$$\int_C \zeta(z)\,dz = 2\pi i$$

as desired. ■

Problem W.5 Set

$$k(z) = -i\big(\zeta(z) - z\tfrac{\eta_1}{\gamma}\big)$$

Prove that

a) $k(-z) = -k(z)$.

b) $\overline{k(z)} = -k(\bar{z})$.

c) $k(z+\gamma) = k(z)$ and $k(z+i\beta) = k(z) - \frac{2\pi}{\gamma}$.

d) $k(iy)$ and $k\big(iy + \frac{\gamma}{2}\big)$ are real for all $y \in \mathbb{R}$.

e) $k(x)$ is pure imaginary and $k(x + i\frac{\beta}{2})$ has real part $\frac{\pi}{\gamma}$ for all $x \in \mathbb{R}$.

Lemma W.5

$$\wp(u+v) + \wp(u) + \wp(v) = \big[\zeta(u+v) - \zeta(u) - \zeta(v)\big]^2$$

for all $u, v \in \mathbb{C}$ *such that none of* u, v, $u+v$ *are in* $\gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$.

Proof: Fix any $v \in \mathbb{C} \setminus (\gamma\mathbb{Z} \oplus i\beta\mathbb{Z})$ and set

$$f(u) = \wp(u+v) + \wp(u) + \wp(v) - \big[\zeta(u+v) - \zeta(u) - \zeta(v)\big]^2$$

Then $f(u)$ is analytic except at $u \in \gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$ and $u \in -v + \gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$. I claim that $f(u)$ has an analytic extension to all of $\mathbb{C}$. Set

$$\begin{aligned}
g(u) &= \wp(u+v) + \wp(v) \\
h(u) &= \zeta(u+v) - \zeta(v) \\
r(u) &= \sum_{\substack{\omega\in\gamma\mathbb{Z}\oplus i\beta\mathbb{Z} \\ \omega\neq 0}} \frac{1}{(u-\omega)^2} - \frac{1}{\omega^2} \\
s(u) &= \sum_{\substack{\omega\in\gamma\mathbb{Z}\oplus i\beta\mathbb{Z} \\ \omega\neq 0}} \tfrac{1}{u-\omega} + \tfrac{1}{\omega} + \tfrac{u}{\omega^2}
\end{aligned}$$

All are analytic for u in a neighbourhood of 0 and $h(0) = s(0) = 0$. Furthermore

$$\begin{aligned}
f(u) &= \tfrac{1}{u^2} + r(u) + g(u) - \big[h(u) - \tfrac{1}{u} - s(u)\big]^2 \\
&= r(u) + g(u) - \big[h(u) - s(u)\big]^2 + \tfrac{2}{u}\big[h(u) - s(u)\big]
\end{aligned} \tag{W.1}$$

Because $h(0) - s(0) = 0$, $\frac{2}{u}\big[h(u) - s(u)\big]$ has an analytic extension to a neighbourhood of zero. Consequently, $f(u)$ has an analytic extension to a neighbourhood of zero. The other points of $\gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$ and $-v + \gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$ are dealt with similarly. Thus $f(u)$ has an analytic extension to all of $\mathbb{C}$. This analytic extension is periodic with respect to $\gamma\mathbb{Z} \oplus i\beta\mathbb{Z}$ and consequently is bounded on $\mathbb{C}$. But any function which is analytic and bounded on $\mathbb{C}$ must be constant. We may determine that constant by setting $u = 0$, or rather taking the limit as $u \to 0$, in (W.1). As $r(0) = h(0) = s(0) = s'(0) = 0$,

$$\begin{aligned}\lim_{u\to 0} f(u) &= r(0) + g(0) - \big[h(0) - s(0)\big]^2 + 2\big[h'(0) - s(0)\big] \\ &= 2\wp(v) + 2\zeta'(v) = 0\end{aligned}$$

So, for all allowed v,

$$\wp(u+v) + \wp(u) + \wp(v) - \big[\zeta(u+v) - \zeta(u) - \zeta(v)\big]^2$$

is independent of u and in fact takes the value zero. ■

For more information on elliptic functions in general and the Weierstrass function in particular, see

K. Chandasekharan, **Elliptic Functions**, Springer–Verlag, 1985.

Patrick Du Val, **Elliptic Functions and Elliptic Curves**, London Mathematical Society, Lecture Note Series **9**, Cambridge University Press.

Harry Rauch, **Elliptic Functions, Theta Functions and Riemann Surfaces**, Abaltimore Williams and Wilkins, 1973.

