Integration on Manifolds

This is intended as a lightning fast introduction to integration on manifolds. For a more thorough, but still elementary discussion see

B. O'Neill, Elementary Differential Geometry, Chapter 4.

W. Rudin, Principles of Mathematical Analysis, Chapter 10.

M. Spivak, Calculus on Manifolds

We shall define integrals over 0–, 1– and 2– (real) dimensional regions of a 2 (real) dimensional manifold. The same ideas also work for higher dimensions. Let M be a 2 (real) dimensional C^∞ manifold with maximal atlas $\mathcal{A}$. For an n–dimensional integral, the domain of integration will be called an n–chain and the object integrated will be called an n–form. The definitions are chosen so that (a) we can use coordinate patchs to express our integrals in terms of ordinary first and second year Calculus integrals for evaluation, but at the same time (b) the answer to the integral so obtained does not to depend on which coordinate patchs are used.

0–dimensional Integrals

Definition.

a) A 0–form is a (complex valued) continuous function F on M.

b) A 0–chain is an expression of the form $n_1P_1 + \cdots + n_kP_k$ with $P_1, \cdots, P_k$ distinct points of M and $n_1, \cdots n_k \in \mathbb{Z}$.

c) If F is a 0–form and $n_1P_1 + \cdots + n_kP_k$ is a 0–chain, then

$$\int_{n_1P_1+\cdots+n_kP_k} F = n_1F(P_1) + \cdots + n_kF(P_k)$$

The definition of a chain given in part (b) is somewhat intuitive. Under a more formal definition, a 0–chain is a function $\sigma : M \to \mathbb{Z}$ for which $\sigma(P)$ is zero for all but finitely many $P \in M$. The $\sigma : M \to \mathbb{Z}$ which corresponds to $n_1P_1 + \cdots + n_kP_k$ has $\sigma(P) = n_i$ when $P = P_i$ for some $1 \le i \le k$ and $\sigma(P) = 0$ if $P \notin \{P_1, \cdots, P_k\}$. Addition of 0–chains and multiplication of a 0–chain by an integer are defined by

$$(\sigma + \sigma')(P) = \sigma(P) + \sigma'(P) \qquad\qquad (n\sigma)(P) = n\sigma(P)$$

1–dimensional Integrals

Definition.

a) A 1–form ω is a rule which assigns to each coordinate patch $\{U\ ,\ \zeta=(x,y):M\to\mathbb{R}^2\}$ a pair (f,g) of (complex valued) continuous functions on $\zeta(U)$ such that

$$\omega\Big|_{\{U,\zeta\}}=f\,dx+g\,dy$$

is invariant under coordinate transformations. This means that

- if $\{U,\zeta\}$ and $\{\tilde U,\tilde\zeta\}$ are two patchs with $U\cap\tilde U\neq\emptyset$ and
- if ω assigns $\{U,\zeta\}$ the pair of functions (f,g) and assigns $\{\tilde U,\tilde\zeta\}$ the pair of functions $(\tilde f,\tilde g)$ and
- if the transition function $\tilde\zeta\circ\zeta^{-1}$ (from $\zeta(U\cap\tilde U)\subset\mathbb{R}^2$ to $\tilde\zeta(U\cap\tilde U)\subset\mathbb{R}^2$) is $\big(\tilde x(x,y),\tilde y(x,y)\big)$,

then

$$\begin{aligned}f(x,y)&=\tilde f\big(\tilde x(x,y),\tilde y(x,y)\big)\tfrac{\partial\tilde x}{\partial x}(x,y)+\tilde g\big(\tilde x(x,y),\tilde y(x,y)\big)\tfrac{\partial\tilde y}{\partial x}(x,y)\\ g(x,y)&=\tilde f\big(\tilde x(x,y),\tilde y(x,y)\big)\tfrac{\partial\tilde x}{\partial y}(x,y)+\tilde g\big(\tilde x(x,y),\tilde y(x,y)\big)\tfrac{\partial\tilde y}{\partial y}(x,y)\end{aligned}$$

b) The standard 1–simplex is $Q^1=[0,1]$. A path is a C^1 map $C:[0,1]\to M$.

c) Let $\{U,\zeta=(x,y)\}$ be a patch for M and let $\omega\big|_{U,\zeta}=f\,dx+g\,dy$. If $c(t):[0,1]\to U\subset M$ is a path with range in U, then

$$\int_c\omega=\int_0^1\Big[\underbrace{f\big(\overbrace{\zeta(\underbrace{c(t)}_{\in M})}^{\in\mathbb{R}^2}\big)}_{\in\mathbb{C}}\tfrac{dx(c(t))}{dt}+g\big(\zeta(c(t))\big)\tfrac{dy(c(t))}{dt}\Big]\,dt$$

If c does not have range in a single patch, split it up into a finite number of pieces, each with range in a single patch. This can always be done, since the range of c is always compact. The answer is independent of choice of patch(s).

d) A 1–chain is an expression of the form $n_1C_1+\cdots n_kC_k$ with $C_1,\cdots,C_k$ distinct paths and $n_1,\cdots n_k\in\mathbb{Z}$.

e) If ω is a 1–form and $n_1C_1+\cdots n_kC_k$ is a 1–chain, then

$$\int_{n_1C_1+\cdots n_kC_k}\omega=n_1\int_{C_1}\omega+\cdots+n_k\int_{C_k}\omega$$

Remark.

a) For now think of $f\,dx+g\,dy$ just as a pice of notation which specifies the two functions (f,g) that ω assigns to the patch $\{U,\zeta=(x,y)\}$. We will later define an operator d that maps n–forms to $(n+1)$–forms. In particular, it will map the coordinate function x, which is a zero form (but which is only defined on part of the manifold) to the 1–form $1dx+0dy$.

b) The motivation for the definition of a 1–form is the ordinary change of variables rule

$$\begin{aligned}\int \tilde f(\tilde x,\tilde y)d\tilde x+\tilde g(\tilde x,\tilde y)d\tilde y &= \int \tilde f\big(\tilde x(x,y),\tilde y(x,y)\big)\big[\tfrac{\partial\tilde x}{\partial x}(x,y)dx+\tfrac{\partial\tilde x}{\partial y}(x,y)dy\big]\\ &\qquad + \tilde g\big(\tilde x(x,y),\tilde y(x,y)\big)\big[\tfrac{\partial\tilde y}{\partial x}(x,y)dx+\tfrac{\partial\tilde y}{\partial y}(x,y)dy\big]\\ &= \int \big[\underbrace{\tilde f\tfrac{\partial\tilde x}{\partial x}+\tilde g\tfrac{\partial\tilde y}{\partial x}}_{f(x,y)}\big]dx+\big[\underbrace{\tilde f\tfrac{\partial\tilde x}{\partial y}+\tilde g\tfrac{\partial\tilde y}{\partial y}}_{g(x,y)}\big]dy\end{aligned}$$

for an integral along a curve.

c) The integral of part (c) is a generalization of the second year calculus definition of an integral along a parametrized line.

2–dimensional Integrals

Definition.

a) A 2–form Ω is a rule which assigns to each patch $\{U,\zeta\}$ a continuous function f on $\zeta(U)$ such that $\Omega\big|_{\{U,\zeta\}}=f\,dx\wedge dy$ is invariant under coordinate transformations. This means that
   - if $\{U,\zeta\}$ and $\{\tilde U,\tilde\zeta\}$ are two patchs with $U\cap\tilde U\neq\emptyset$ and
   - if Ω assigns $\{U,\zeta\}$ the function f and assigns $\{\tilde U,\tilde\zeta\}$ the function $\tilde f$ and
   - if the transition function $\tilde\zeta\circ\zeta^{-1}$ (from $\zeta(U\cap\tilde U)\subset\mathbb{R}^2$ to $\tilde\zeta(U\cap\tilde U)\subset\mathbb{R}^2$) is $\big(\tilde x(x,y),\tilde y(x,y)\big)$,

   then

$$f(x,y)=\tilde f\big(\tilde x(x,y),\tilde y(x,y)\big)\Big[\tfrac{\partial\tilde x}{\partial x}(x,y)\tfrac{\partial\tilde y}{\partial y}(x,y)-\tfrac{\partial\tilde x}{\partial y}(x,y)\tfrac{\partial\tilde y}{\partial x}(x,y)\Big]$$

b) The standard 2–simplex is

$$Q^2=\big\{\ (x,y)\in\mathbb{R}^2\ \big|\ x,y\ge 0, x+y\le 1\ \big\}$$

   A surface is a C^1 map $D:Q^2\to M$.

c) Let $\{U,\zeta=(x,y)\}$ be a patch and let $\Omega\big|_{U,\zeta}=f(x,y)dx\wedge dy$. If $D:Q^2\to U\subset M$ is a surface with range in U, then

$$\begin{aligned}\int_D\Omega=\iint_{Q^2} f\big(\zeta(D(s,t))\big)\big[\tfrac{\partial}{\partial s}x\big(D(s,t)\big)\tfrac{\partial}{\partial t}y\big(D(s,t)\big)\\ -\tfrac{\partial}{\partial t}x\big(D(s,t)\big)\tfrac{\partial}{\partial s}y\big(D(s,t)\big)\big]\ dsdt\end{aligned}$$

   If D does not have range in a single patch, split it up into a finite number of pieces, each with range in a single patch. This can always be done, since the range of D is always compact. The answer is independent of choice of patch(s).

d) A 2–chain is an expression of the form $n_1D_1+\cdots n_kD_k$ with $D_1,\cdots,D_k$ surfaces and $n_1,\cdots n_k\in\mathbb{Z}$.

e) If Ω is a 2–form and $n_1 D_1 + \cdots n_k D_k$ is a 2–chain, then

$$\int_{n_1 D_1 + \cdots n_k D_k} \Omega = n_1 \int_{D_1} \Omega + \cdots + n_k \int_{D_k} \Omega$$

The motivation for the definition of a 2–form is the ordinary change of variables rule

$$\begin{aligned}\int \tilde f(\tilde x, \tilde y) d\tilde x d\tilde y &= \int \tilde f\big(\tilde x(x,y), \tilde y(x,y)\big) \left| \det \begin{bmatrix} \frac{\partial \tilde x}{\partial x}(x,y) & \frac{\partial \tilde y}{\partial x}(x,y) \\ \frac{\partial \tilde x}{\partial y}(x,y) & \frac{\partial \tilde y}{\partial y}(x,y) \end{bmatrix} \right| dxdy \\ &= \int \tilde f \big| \tfrac{\partial \tilde x}{\partial x} \tfrac{\partial \tilde y}{\partial y} - \tfrac{\partial \tilde x}{\partial y} \tfrac{\partial \tilde y}{\partial x} \big| dxdy\end{aligned}$$

for an integral on a region in $\mathbb{R}^2$, except for the absolute value signs. So we are dealing with oriented (i.e. signed) areas.

The Boundary Operator δ

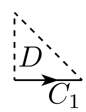
Definition.

a) For any 0–chain $\delta\big(n_1 P_1 + \cdots n_k P_k\big) = 0$.

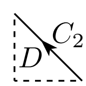
b) For a path $C : [0,1] \to M$, δC is the 0–chain $C(1) - C(0)$.
For a 1–chain $\delta\big(n_1 C_1 + \cdots n_k C_k\big) = n_1 \delta(C_1) + \cdots n_k \delta(C_k)$.

c) For a surface $D : Q^2 \to M$, δC is the 1–chain $C_1 + C_2 + C_3$ where, for $0 \le t \le 1$,

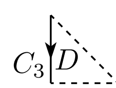
$$C_1(t) = D(t, 0)$$



$$C_2(t) = D(1-t, t)$$



$$C_3(t) = D(0, 1-t)$$



For a 2–chain $\delta\big(n_1 D_1 + \cdots n_k D_k\big) = n_1 \delta(D_1) + \cdots n_k \delta(D_k)$.

The boundary operator maps n–chains to $n-1$ chains and obeys

$$\delta^2 = 0$$

Proof: For a surface D,

$$\begin{aligned}\delta^2 D &= \delta\big(C_1 + C_2 + C_3\big) \\ &= [C_1(1) - C_1(0)] + [C_2(1) - C_2(0)] + [C_3(1) - C_3(0)] \\ &= [D(1,0) - D(0,0)] + [D(0,1) - D(1,0)] + [D(0,0) - D(0,1)] = 0\end{aligned}$$

The case $n = 2$ follows from this. The cases $n = 0, 1$ are trivial. ■

The Wedge Product

Definition. If ω is a k–form and ω' is a k'–form then $\omega\wedge\omega'$ is the $(k+k')$–form that is determined by $\omega\wedge\omega' = (-1)^{kk'}\omega'\wedge\omega$ (that is $\omega\wedge\omega'=\omega'\wedge\omega$ if at least one of k and k' is even and $\omega\wedge\omega' = -\omega'\wedge\omega$ if both k and k' are odd) and

a) if $k=k'=0$ and $(\omega\wedge\omega')(P) = \omega(P)\omega'(P)$.

b) if $k=0$ and $\omega'\big|_{U,\zeta} = f\,dx + g\,dy$ then

$$\omega\wedge\omega'\big|_{U,\zeta} = (\omega\circ\zeta^{-1})f\,dx + (\omega\circ\zeta^{-1})g\,dy$$

c) if $k=0$ and $\omega'\big|_{U,\zeta} = f\,dx\wedge dy$ then

$$\omega\wedge\omega'\big|_{U,\zeta} = (\omega\circ\zeta^{-1})f\,dx\wedge dy$$

d) if $k=k'=1$ and $\omega\big|_{U,\zeta} = f\,dx + g\,dy$ then $\omega'\big|_{U,\zeta} = f'\,dx + g\,dy'$ then

$$\omega\wedge\omega'\big|_{U,\zeta} = [fg' - gf']\,dx\wedge dy$$

In particular $dx\wedge dx = dy\wedge dy = 0$ and $dx\wedge dy = -dy\wedge dx$.

e) If $k+k'>2$, $\omega\wedge\omega' = 0$.

The Differential Operator d

Definition. Let M be a two real dimensional manifold. If $\{U,\zeta\}$ is a coordinate patch on M and

a) if $F : M\to\mathbb{C}$ is a C^1 0–form, then

$$dF\big|_{\{U,\zeta\}} = \tfrac{\partial}{\partial x}\big(F\circ\zeta^{-1}\big)(x,y)\,dx + \tfrac{\partial}{\partial y}\big(F\circ\zeta^{-1}\big)(x,y)\,dy$$

b) if ω is a C^1 1–form with $\omega\big|_{\{U,\zeta\}} = f(x,y)\,dx + g(x,y)\,dy$, then

$$d\omega\big|_{\{U,\zeta\}} = \big[\tfrac{\partial g}{\partial x}(x,y) - \tfrac{\partial f}{\partial y}(x,y)\big]\,dx\wedge dy$$

c) if Ω is a C^1 2–form, then $d\Omega = 0$

The differential operator d maps n–forms to $n+1$ forms and obeys

$$d^2 = 0$$

since, in the case $n=0$, (writing $f = F\circ\zeta^{-1}$)

$$d^2F = d\big(\tfrac{\partial f}{\partial x}\,dx + \tfrac{\partial f}{\partial y}\,dy\big) = \big[\tfrac{\partial^2 f}{\partial x\partial y} - \tfrac{\partial^2 f}{\partial y\partial x}\big]dx\wedge dy = 0$$

The cases $n=1,2$ are trivial. There is also a product rule. If ω is a k–form and ω' is a k'–form, then

$$d(\omega\wedge\omega') = (d\omega)\wedge\omega' + (-1)^k\omega\wedge(d\omega')$$

Stoke's Theorem

If ω is a C^1 k–form and D is a $(k+1)$–chain, then

$$\int_{\delta D} \omega = \int_D d\omega$$

"**Proof:**"

For $k = 0$ this is the fundamental theorem of calculus.

$$F(C(1)) - F(C(0)) = \int_C dF$$

For $k = 1$, this is Green's Theorem.

$$\int_{\delta D} f\,dx + g\,dy = \iint_D \left[\tfrac{\partial g}{\partial x} - \tfrac{\partial f}{\partial y}\right] dx \wedge dy$$

■

If ω is a compactly supported 1–form, $\iint_M d\omega = 0$. If ω is a closed 1–form (meaning that $d\omega = 0$) and if C_1 and C_2 are two 1–chains with $C_1 - C_2 = \delta D$ for some 2–chain D, then $\int_{C_1} \omega = \int_{C_2} \omega$.