

# Operators on Differential Forms

In the following

- $\{U, \zeta\}$  is a coordinate patch for a Riemann surface  $M$ ,
- $f(x, y), g(x, y), u(x, y), v(x, y)$  are functions on  $\zeta(U) \subset \mathbb{R}^2$
- $F$  is a 0-form on  $M$ ,
- $\omega$  is a 1-form on  $M$  with

$$\begin{aligned}\omega|_{\{U, \zeta\}} &= f(x, y) dx + g(x, y) dy \\ &= u(x, y) dz + v(x, y) d\bar{z}\end{aligned}$$

- $\Omega$  is a 2-form on  $M$

We shall also use the complex notation

$$\begin{aligned}dz &= dx + idy & d\bar{z} &= dx - idy \\ u_z(x, y) &= \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) u(x, y) & u_{\bar{z}}(x, y) &= \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) u(x, y)\end{aligned}$$

## The Differential Operator $d$

For  $C^1$  forms

$$\begin{aligned}dF|_{\{U, \zeta\}} &= \frac{\partial}{\partial x} (F \circ \zeta^{-1})(x, y) dx + \frac{\partial}{\partial y} (F \circ \zeta^{-1})(x, y) dy \\ &= (F \circ \zeta^{-1})_z(x, y) dz + (F \circ \zeta^{-1})_{\bar{z}}(x, y) d\bar{z} \\ d\omega|_{\{U, \zeta\}} &= \left[ \frac{\partial g}{\partial x}(x, y) - \frac{\partial f}{\partial y}(x, y) \right] dx \wedge dy \\ &= [v_z(x, y) - u_{\bar{z}}(x, y)] dz \wedge d\bar{z} \\ d\Omega &= 0\end{aligned}$$

## The Differential Operators $\partial, \bar{\partial}$

For  $C^1$  forms

$$\begin{aligned}\partial F|_{\{U, \zeta\}} &= \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (F \circ \zeta^{-1})(x, y) dz = (F \circ \zeta^{-1})_z dz \\ \bar{\partial} F|_{\{U, \zeta\}} &= \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (F \circ \zeta^{-1})(x, y) d\bar{z} = (F \circ \zeta^{-1})_{\bar{z}} d\bar{z} \\ \partial \omega|_{\{U, \zeta\}} &= v_z(x, y) dz \wedge d\bar{z} \\ \bar{\partial} \omega|_{\{U, \zeta\}} &= -u_{\bar{z}}(x, y) dz \wedge d\bar{z} \\ \partial \Omega &= 0 \\ \bar{\partial} \Omega &= 0\end{aligned}$$

## The Conjugation Operator \*

$$\begin{aligned} *\omega|_{\{U,\zeta\}} &= -g(x, y) dx + f(x, y) dy \\ &= -iu(x, y) dz + iv(x, y) d\bar{z} \end{aligned}$$

## Complex Conjugation

$$\begin{aligned} \bar{\omega}|_{\{U,\zeta\}} &= \overline{f(x, y)} dx + \overline{g(x, y)} dy \\ &= \overline{v(x, y)} dz + \overline{u(x, y)} d\bar{z} \end{aligned}$$

## Various Identities

$$(1) \quad dx = \frac{1}{2}(dz + d\bar{z})$$

$$(2) \quad dy = \frac{1}{2i}(dz - d\bar{z})$$

$$(3) \quad dz \wedge d\bar{z} = -2i dx \wedge dy$$

$$(4) \quad dz' = \frac{dz'}{dz} dz \quad d\bar{z}' = \frac{d\bar{z}'}{d\bar{z}} d\bar{z}$$

$$(5) \quad dz' \wedge d\bar{z}' = \left| \frac{dz'}{dz} \right|^2 dz \wedge d\bar{z}$$

$$(6) \quad \bar{u}_z(x, y) = \overline{u_{\bar{z}}(x, y)} \quad \bar{u}_{\bar{z}}(x, y) = \overline{u_z(x, y)}$$

$$(7) \quad \text{If } u(x, y) = w(x + iy) \text{ with } w \text{ analytic, } u_z(x, y) = w'(x + iy) \quad u_{\bar{z}}(x, y) = 0$$

$$(8) \quad \text{If } u(x, y) = w(x - iy) \text{ with } w \text{ analytic, } u_{\bar{z}}(x, y) = w'(x - iy) \quad u_z(x, y) = 0$$

$$(9) \quad d^2 = 0$$

$$(10) \quad \partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$$

$$(11) \quad d = \partial + \bar{\partial}$$

$$(12) \quad ** = -1$$

$$(13) \quad \overline{dF} = d\bar{F}$$

$$(14) \quad \overline{*\omega} = * \bar{\omega}$$

$$(15) \quad (*\omega) \wedge (*\omega') = \omega \wedge \omega'$$

$$(16) \quad \partial\omega = (d + d*)\omega \quad \bar{\partial}\omega = (d - d*)\omega$$

$$(17) \quad \Delta F = d * dF = -2i\bar{\partial}\partial F$$