Projective Curves

The n dimensional complex projective space is the set of all equivalence classes

$$\mathbb{CP}^n = \left\{ \ [z_1, \cdots, z_{n+1}] \ \middle| \ (z_1, \cdots, z_{n+1}) \in \mathbb{C}^{n+1} \setminus \{(0, \cdots, 0)\} \ \right\}$$

under the equivalence relation

$$(z_1, \cdots, z_{n+1}) \sim (z'_1, \cdots, z'_{n+1}) \iff \exists z \in \mathbb{C} \setminus \{0\} \text{ such that } (z'_1, \cdots, z'_{n+1}) = z(z_1, \cdots, z_{n+1})$$

We can think of $\mathbb{CP}^n$ as $\mathbb{C}^n$, which we identify with $\left\{ \ [z_1, \cdots, z_n, 1] \ \middle| \ (z_1, \cdots, z_n) \in \mathbb{C}^n \ \right\}$, with some points at infinity tacked on. Since $[z_1, \cdots, z_n, z] = [\frac{z_1}{z}, \cdots, \frac{z_n}{z}, 1]$ for all $z \neq 0$, the set of points in $\mathbb{CP}^n$ which we have not identified with points in $\mathbb{C}^n$ is $\left\{ \ [z_1, \cdots, z_n, 0] \ \middle| \ (z_1, \cdots, z_n) \in \mathbb{C}^n \setminus \{(0, \cdots, 0)\} \ \right\}$, which is just $\mathbb{CP}^{n-1}$. This is the set of points at infinity. Each complex line in $\mathbb{C}^n$ that passes through the origin is of the form $\left\{ \ z(z_1, \cdots, z_n) \ \middle| \ z \ \in \ \mathbb{C} \ \right\}$ for some $(z_1, \cdots, z_n) \in \mathbb{C}^n \setminus \{(0, \cdots, 0)\}$. (It has real dimensional two, but complex geometers still call it a line because it has complex dimension one.) There is one point at infinity $\mathbb{CP}^n$ for each complex line in $\mathbb{C}^n$. Since

$$[z_1, \cdots, z_n, 0] = \lim_{z \to 0} [z_1, \cdots, z_n, z] = \lim_{z \to 0} [\tfrac{z_1}{z}, \cdots, \tfrac{z_n}{z}, 1]$$

and $[\frac{z_1}{z}, \cdots, \frac{z_n}{z}, 1]$ is identified with the point $\frac{1}{z}(z_1, \cdots, z_n) \in \mathbb{C}^n$, you can get to the point $[z_1, \cdots, z_n, 0]$ at infinity in $\mathbb{CP}^n$ by "going to infinity" along the complex line in $\mathbb{C}^n$ that is associated with $[z_1, \cdots, z_n, 0]$.

In general, a function $F(z_1, \cdots, z_{n+1})$ on $\mathbb{C}^{n+1}$ does not make sense as a function on $\mathbb{CP}^n$ because F can take different values at equivalent points $(z_1, \cdots, z_{n+1}) \sim (z'_1, \cdots, z'_{n+1})$. But if F is a homogeneous polynomial of degree d, then $F(zz_1, \cdots, zz_{n+1}) = z^d F(z_1, \cdots, z_{n+1})$ so that at least

$$F(z_1, \cdots, z_{n+1}) = 0 \iff F(z'_1, \cdots, z'_{n+1}) = 0 \text{ for all } (z'_1, \cdots, z'_{n+1}) \sim (z_1, \cdots, z_{n+1})$$

Thus the zero set

$$M_F = \left\{ \ [z_1, \cdots, z_{n+1}] \in \mathbb{CP}^n \ \middle| \ F(z_1, \cdots, z_{n+1}) = 0 \ \right\}$$

is a well defined subset of $\mathbb{CP}^n$. If F is nonsingular, meaning that there are no solutions to the system of equations

$$F = \tfrac{\partial F}{\partial z_1} = \cdots \tfrac{\partial F}{\partial z_{n+1}} = 0$$

then M_F defines a smooth $n-1$ (complex) dimensional manifold in $\mathbb{CP}^n$. If $n = 2$ then M_F is a Riemann surface. (It turns out that connectedness is automatic in this case. Disconnectedness in $\mathbb{C}^2$ gives a singularity at infinity in $\mathbb{PC}^2$. For example: $f(z_1, z_2) = z_1(z_1 - 1)$, $F(z_1, z_2, z_3) = z_1(z_1 - z_3)$.) If $n > 2$, we can also get Riemann surfaces by taking the intersection $M_{F_1} \cap \cdots \cap M_{F_{n-1}}$ of $n-1$ such surfaces. The intersection is smooth if the $(n-1) \times (n+1)$ matrix $\left(\frac{\partial F_i}{\partial z_j}\right)$ of partial derivatives has maximal rank $n-1$. Again, it turns out that smoothness implies connectedness.

If f is any polynomial on $\mathbb{C}^n$, we can always find a homogeneous polynomial F on $\mathbb{C}^{n+1}$ with the same degree as f, such that the zero set of f in $\mathbb{C}^n$ and the part of M_F with $z_{n+1} = 1$ (i.e. excluding the part at infinity) coincide under the identification we discussed above. For example, if $f(x, y) = y^2 - x^3 + x$ (whose zero set is the elliptic curve we saw in class), then $F(x, y, z) = y^2 z - x^3 + xz^2$. The advantage of M_F is that it is always compact, since $\mathbb{CP}^n$ is compact.