Resumé - holomorphic/harmonic forms on compact Riemann surfaces

Notation.

$$\begin{aligned}
M = &\ \text{a compact Riemann surface of genus } g \\
&\qquad \text{with canonical homology basis } \{a_1,\cdots,a_g,b_1,\cdots,b_g\} \\
\aleph_j = &\begin{cases} a_j & \text{if } 1\le j\le g \\ b_{j-g} & \text{if } g+1\le j\le 2g \end{cases} \\
H = &\ \big\{\ \omega\in L^2(M)\ \big|\ \omega \text{ harmonic}\ \big\} \\
\mathcal{H} = &\ \big\{\ \omega\in L^2(M)\ \big|\ \omega \text{ holomorphic}\ \big\}
\end{aligned}$$

Theorem.

a) $\dim H = 2g$

b) There is a unique basis $\alpha_1,\cdots,\alpha_{2g}$ of H obeying

$$\int_{\aleph_j}\alpha_k = \delta_{j,k}$$

Furthermore, $\alpha_k=\overline{\alpha_k}$ for all $1\le k\le 2g$ and

$$\iint_M \alpha_j\wedge\alpha_k = \aleph_j\cdot\aleph_k = \begin{cases} 0 & \text{if } 1\le j,k\le g \\ 0 & \text{if } g+1\le j,k\le 2g \\ \delta_{j,k-g} & \text{if } 1\le j,k-g\le g \\ -\delta_{j-g,k} & \text{if } 1\le j-g,k\le g \end{cases}$$

for all $1\le j,k\le 2g$.

Proposition. *Let $\theta,\tilde\theta$ be a closed 1–forms on M. Then*

$$\iint_M \theta\wedge\tilde\theta = \sum_{j=1}^g\Big[\int_{a_j}\theta\int_{b_j}\tilde\theta - \int_{b_j}\theta\int_{a_j}\tilde\theta\ \Big]$$

Corollary. *If $\theta\in H$, then*

$$\|\theta\|^2 = \sum_{j=1}^g\Big[\int_{a_j}\theta\int_{b_j}*\bar\theta - \int_{b_j}\theta\int_{a_j}*\bar\theta\ \Big]$$

If $\theta\in\mathcal{H}$, then

$$\|\theta\|^2 = \sum_{j=1}^g i\Big[\int_{a_j}\theta\int_{b_j}\bar\theta - \int_{b_j}\theta\int_{a_j}\bar\theta\ \Big]$$

Theorem.

a) $\dim \mathcal{H} = g$

b) *There is a unique basis* $\omega_1, \cdots, \omega_g$ *of* $\mathcal{H}$ *obeying*

$$\int_{\mathbf{a}_j} \omega_k = \delta_{j,k}$$

Furthermore, $\Pi_{j,k} = \int_{b_j} \omega_k$ *obeys*

- $\Pi_{j,k} = \Pi_{k,j}$ *for all* $1 \le j, k \le g$ *(*Π *is symmetric) and*
- $\frac{1}{2i} \sum_{1 \le j,k \le g} \bar{\mu}_j \big(\Pi_{j,k} - \overline{\Pi_{j,k}}\big) \mu_k \ge 0$ *for all* $\vec{\mu} \in \mathbb{C}^g$*, with equality if and only if* $\vec{\mu} = 0$ *(*$\mathrm{Im}\, \Pi$ *is positive definite).*

c) *There is a unique* $\theta \in \mathcal{H}$ *with prescribed values of* $\int_{a_1} \theta, \cdots, \int_{a_g} \theta$.

c') *There is a unique* $\theta \in \mathcal{H}$ *with prescribed values of* $\mathrm{Re} \int_{a_1} \theta$, $\cdots$, $\mathrm{Re} \int_{a_g} \theta$, $\mathrm{Re} \int_{b_1} \theta$, $\cdots$, $\mathrm{Re} \int_{b_g} \theta$.