The Riemann–Roch Theorem

Well, a Riemann surface is a certain kind of Hausdorf space. You know what a Hausdorf space is, don't you? Its also compact, ok. I guess it is also a manifold. Surely you know what a manifold is. Now let me tell you one non-trivial theorem, the Riemann–Roch Theorem

— Gian-Carlo Rota's recollection of Lefschetz lecturing in the 1940's, quoted in *A Beautiful Mind* by Sylvia Nasar.

Introduction

In this section M will always denote a compact Riemann surface of genus g. We introduce divisors on Riemann surfaces as a device for describing the zeros and poles of meromorphic functions and differentials on M. Associated to each divisor is are vector spaces of meromorphic functions and differentials. The Riemann–Roch theorem is a relation between the dimensions of these spaces.

Divisors

The zeros and poles of a meromorphic function or form on M can be described assoicating with each point in M an integer. To each zero we associate the order of the zero, to each pole we associate minus the order of the pole, and to points that are neither zeros nor poles we associate zero. This leads us to the following definition.

A *divisor* is a function

$$\alpha : M \to \mathbb{Z}$$

that takes on non-zero values for only finitely many $p \in M$.We will write divisors multiplicatively as

$$\mathfrak{A} = \prod_{p \in M} p^{\alpha(p)}, \quad \mathfrak{B} = \prod_{p \in M} p^{\beta(p)}$$

(although it is acutally more common to write them additively as $\sum_{p \in M} \alpha(p)p$.) The set of all divisors, denoted $\mathrm{Div}(M)$, has a natural abelian group structure. With our notation the product and inverse are given by

$$\mathfrak{A}\mathfrak{B} = \prod_{p \in M} p^{\alpha(p)+\beta(p)}$$

$$\mathfrak{A}^{-1} = \prod_{p \in M} p^{-\alpha(p)}$$

The identity element 1 corresponds to $\alpha(p) = 0$ for every $p \in M$. With these definitions, $\mathrm{Div}(M)$ forms an abelian group (in fact, the free abelian group generated by points of M).

Recall that if ω is a meromorphic q-differential then order of ω at $p \in M$, denoted $\mathrm{ord}_p(\omega)$ is power corresponding to the first non-zero term in the Laurent expansion of a local expression for ω. If ω is not identically zero, then $\operatorname{ord}_p(\omega) = 0$ for all but finitely many points (the zeros and poles of ω). Thus

$$\operatorname{div}(\omega) = \prod_{p \in M} p^{\operatorname{ord}_p(\omega)}$$

is an example of a divisor on M. Notice that

$$\operatorname{div}(\omega\eta) = \operatorname{div}(\omega) \cdot \operatorname{div}(\eta)$$

In particular, if we let $\mathcal{K}^*(M) = \mathcal{K}(M) \backslash \{0\}$ denote the multiplicative group of meromorphic functions $\mathcal{K}(M)$ (that is, 0-differentials). Then $\operatorname{div} : \mathcal{K}^* \to \operatorname{Div}(M)$ is a group homomorphism.

The *degree* of the divisor $\deg : \operatorname{Div}(M) \to \mathbb{Z}$ is defined as

$$\deg(\prod_{p \in M} p^{\alpha(p)}) = \sum_{p \in M} \alpha(p)$$

The degree is also a group homomorphism.

Proposition 1 $\deg(\operatorname{div}(f)) = 0$ *for any meromorphic function* f.

Proof: Think of f as a holomorphic map from M to $\mathbb{C}_\infty$. Recall that we defined the branching number $b_f(p)$ for such a map. We proved that for any $q \in \mathbb{C}^\infty$

$$\sum_{p \in f^{-1}(q)} b_f(p) + 1 = m$$

where m is the degree of f. (This is different from the degree of $\operatorname{div}(f)$!) In particular

$$\sum_{p \in f^{-1}(0)} b_f(p) + 1 - \sum_{p \in f^{-1}(\infty)} b_f(p) + 1 = 0$$

If p is a zero of f then $b_f(p) + 1 = \operatorname{ord}_p(f)$. On the other hand, if p is a pole of f then $b_f(p) + 1 = -\operatorname{ord}_p(f)$. Therefore the previous equality can be written

$$\operatorname{div}(\deg(f)) = \sum_{p \in M} \operatorname*{ord}_p(f) = 0$$

□

A *principal divisor* is a divisor of the form $\operatorname{div}(f)$ for a non-zero meromorphic function f. The principal divisors form a subgroup of the group of divisors of degree zero. A *(q-) canonical* divisor is a divisor of the form $\operatorname{div}(\omega)$ for a non-zero meromorphic (q-)differential ω.

The *divisor class group* is the quotient $\operatorname{Div}(M)/\text{principal divisors}$. Thus two divisors $\mathfrak{A}$ and $\mathfrak{B}$ are in the same equivalence class if $\mathfrak{A} = \operatorname{div}(f)\mathfrak{B}$ for some meromorphic function f. Note that $\deg$ is well defined on the divisor class group since if $\mathfrak{A} = \operatorname{div}(f)\mathfrak{B}$ then $\deg(\mathfrak{A}) = \deg(\operatorname{div}(f)\mathfrak{B}) = \deg(\operatorname{div}(f)) + \deg(\mathfrak{B}) = 0 + \deg(\mathfrak{B})$.

The *polar divisor* of f is defined by

$$f^{-1}(\infty) = \prod_{p \in M} p^{\max\{-\operatorname{ord}_p(f), 0\}}$$

while the *zero divisor* of f is defined by

$$f^{-1}(0) = \prod_{p \in M} p^{\max\{\operatorname{ord}_p(f), 0\}}$$

Since $\operatorname{div}(f) = f^{-1}(0)/f^{-1}(\infty)$ the polar divisors and the zero divisors define the same element in the divisor class group.

Any meromorphic differential defines the same element in the divisor class group. This element is called the canonical class. This follows from the fact that if ω and η are two meromorphic differentials then $f = \omega/\eta$ is a meromorphic function. Thus $\operatorname{div}(\omega) = \operatorname{div}(f)\operatorname{div}(\eta)$. Similarly we may define the q-canonical class (which is in fact just the q^{th} power of the canonical class.)

There is a natural partial ordering on divisors. Let $\mathfrak{A} = \prod_{p \in M} p^{\alpha(p)}$ and $\mathfrak{B} = \prod_{p \in M} p^{\beta(p)}$. We say $\mathfrak{A} \geq \mathfrak{B}$ if $\alpha(p) \geq \beta(p)$ for every $p \in M$ and $\mathfrak{A} > \mathfrak{B}$ if $\mathfrak{A} \geq \mathfrak{B}$ but $\mathfrak{A} \neq \mathfrak{B}$. A divisor $\mathfrak{A}$ is called *integral* (or *effective*) if $\mathfrak{A} \geq 1$ and *strictly integral* if $\mathfrak{A} > 1$.

A meromorphic function f is called a *multiple* if $\mathfrak{A}$ if $f = 0$ or $\operatorname{div}(f) \geq \mathfrak{A}$. If $\mathfrak{A} = \prod_{p \in M} p^{\alpha(p)}$ and f is non-zero, this means that f is has poles of order a most $-\operatorname{ord}_p(f)$ at points p where $\operatorname{ord}_p(f) < 0$. At all other points f is holomorphic, with zero of order at least $\operatorname{ord}_p(f)$ whenever $\operatorname{ord}_p(f) > 0$. Similarly, a meromorphic differential ω is called a multiple of $\mathfrak{A}$ if $\omega = 0$ or $\operatorname{div}(\omega) \geq \mathfrak{A}$

Define the vector spaces

$$L(\mathfrak{A}) = \{\text{meromorphic functions } f : \operatorname{div}(f) \text{ is a multiple of } \mathfrak{A}\}$$

$$\Omega(\mathfrak{A}) = \{\text{meromorphic differentials } \operatorname{div}(\omega) : \omega \text{ is a multiple of } \mathfrak{A}\}$$

Here are some elementary facts about these spaces.

Proposition 2 $L(1) = \mathbb{C}$.

Proof: If $\operatorname{div}(f) \geq 1$ then f is a holomorphic function on M, hence constant. $\square$

Proposition 3 *If* $\deg(\mathfrak{A}) > 0$ *then* $L(\mathfrak{A}) = \{0\}$.

Proof: If f is non-zero and contained in $L(\mathfrak{A})$ then $\deg(\operatorname{div}(f)) \geq \deg(\mathfrak{A}) > 0$. But this is impossible since the degree the principal divisor $\operatorname{div}(f)$ is zero. $\square$

Proposition 4 *If* $\mathfrak{A} \in \operatorname{Div}(M)$ *and* ω *is any non-zero meromorphic differential then*

$$\dim(\Omega(\mathfrak{A})) = \dim(L(\mathfrak{A}/\operatorname{div}(\omega)))$$

Proof: The linear map from $\Omega(\mathfrak{A}) \to L(\mathfrak{A}/\operatorname{div}(\omega))$ defined by $\eta \mapsto \eta/\omega$ is clearly one to one and onto. $\square$

Proposition 5 *For any divisor* $\mathfrak{A}$ *the dimensions* $\dim(L(\mathfrak{A}))$ *and* $\dim(\Omega(\mathfrak{A}))$ *only depend on the equivalence class of* $\mathfrak{A}$ *in the divisor class group.*

Proof: Suppose that $\mathfrak{B} = \operatorname{div}(f)\mathfrak{A}$ for some meromorphic function f. Then multiplication by f is a linear one to one and onto map from $L(\mathfrak{A}) \to L(\mathfrak{B})$. Thus $\dim(L(\mathfrak{A})) = \dim(L(\mathfrak{B}))$. Let ω be a non-zero meromorphic differential, whose exisitence was established in a previous section. Then $\dim(\Omega(\mathfrak{A})) = \dim(L(\mathfrak{A}/\operatorname{div}(\omega))) = \dim(L(\mathfrak{B}/\operatorname{div}(\omega))) = \dim(\Omega(\mathfrak{B}))$. $\square$

We now state the Riemann–Roch Theorem.

Theorem 6 *Let* M *be a compact Riemann surface of genus* g *and* $\mathfrak{A} \in \operatorname{Div}(M)$*. Then*

$$\dim(L(\mathfrak{A}^{-1}) = \deg(\mathfrak{A}) - g + 1 + \dim(\Omega(\mathfrak{A}))$$

Proof of Theorem 6 when $\mathfrak{A} = 1$*:* In this case $L(\mathfrak{A}^{-1}) = L(1)$ consists of holomorphic functions. These are precisely the constant functions. Thus $\dim(L(\mathfrak{A}^{-1})) = 1$. The space $\Omega(\mathfrak{A})$ consists of holomorphic differentials. This space is g dimensional. Finally, we have $\deg(1) = 0$. Since

$$1 = 0 - g + 1 + g$$

the Riemann-Roch theorem holds. $\square$

Proof of Theorem 6 when $\mathfrak{A} > 1$*:* Let $\mathfrak{A} = p_1^{n_1} \cdots p_m^{n_m}$ with $n_j > 0$. Then $\deg(\mathfrak{A}) = \sum n_j > 0$. Choose representatives $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ for a canonical homology basis. Define $\mathfrak{B} = p_1^{n_1+1} \cdots p_m^{n_m+1}$ and

$$\Omega_0(\mathfrak{B}^{-1}) = \{\omega \in \Omega(\mathfrak{B}^{-1}) : \operatorname{res}_{p_j}(\omega) = 0, \int_{a_j} \omega = 0, j = 1, \ldots, g\}$$

The point of introducing this space is that the exterior derivative d maps the space we are interested in, namely $L(\mathfrak{A}^{-1})$ into $\Omega_0(\mathfrak{B}^{-1})$. This follows by considering the action of d locally near the poles. If $f \in L(\mathfrak{A}^{-1})$, then the poles of *df* will have order one greater than the poles of f. Moreover, all the residues of *df* will be zero as will be the integrals of *df* over any closed cycle, in particular the a_j cycles. Thus

$$d : L(\mathfrak{A}^{-1}) \to \Omega_0(\mathfrak{B}^{-1}) \tag{1}$$

and we may compute the dimension of $L(\mathfrak{A}^{-1})$ using

$$\dim(L(\mathfrak{A}^{-1})) = \dim(\operatorname{Ker}(d)) + \dim(\operatorname{Im}(d)).$$

The dimension of $\operatorname{Ker}(d)$ is 1, since any function f with $df = 0$ is constant, and constants are contained in $L(\mathfrak{A}^{-1})$ since we are assuming that $\mathfrak{A}$ is integral. Thus

$$\dim(L(\mathfrak{A}^{-1})) = 1 + \dim(\operatorname{Im}(d)). \tag{2}$$

Next, we claim that

$$\dim(\Omega_0(\mathfrak{B}^{-1})) = \deg(\mathfrak{A}) \tag{3}$$

To prove this claim we recall that for each $p \in M$ and $n \geq 2$ there exist meromorphic one forms $\tau_p^{(n)}$ such that in a local co-ordinate system vanishing at p, $\tau_p^{(n)}$ is given by $z^{-n}dz$ and such that $\int_{a_j} \tau_p^{(n)} = 0$ for each j. For each $j = 1, \ldots, m$ we now choose local co-ordinates vanishing at p_j, and fixed for the rest of the proof, thus obtaining n_j elements $\tau_{p_j}^{(2)}, \ldots, \tau_{p_j}^{(n_j+1)}$ in $\Omega_0(\mathfrak{B}^{-1})$. These forms are clearly linearly independent. This implies that

$$\dim(\Omega_0(\mathfrak{B}^{-1})) \geq \sum_j n_j = \deg(\mathfrak{A})$$

To establish the opposite inequality, we expand an element $\omega \in \Omega_0(\mathfrak{B}^{-1})$ locally about each point p_j. Suppose in co-ordinates vanishing at p_j the form ω has the expansion $\sum_{k=-n_j-1}^{\infty} d_{j,k} z^k dz$. Define the map $S : \Omega_0(\mathfrak{B}^{-1}) \to \mathbb{C}^{\deg(\mathfrak{A})}$ by

$$S : \omega \mapsto (d_{1,-2}, \ldots, d_{1,-n_1-1}, \ldots \ldots d_{m,-2}, \ldots, d_{m,-n_m-1})$$

If $S\omega = 0$ then ω is a holomorphic one form whose integral over every a_j cycle vanishes, and hence is zero. Thus S is one to one and

$$\dim(\Omega_0(\mathfrak{B}^{-1}))) = \dim(\mathrm{Im}(S)) \leq \dim(\mathbb{C}^{\deg(\mathfrak{A})}) = \deg(\mathfrak{A}).$$

This establishes (3) and also shows that the forms $\tau_{p_j}^{(k)}$, for $k = 2, \ldots, n_j + 1$, $j = 1..m$ form a basis for $\Omega_0(\mathfrak{B}^{-1}))$.

Our final task is to compute the dimension of the image of d in (1). Notice that this image consists of precisely those $\omega \in \Omega_0(\mathfrak{B}^{-1}))$ with $\int_{b_j} \omega = 0$ for $j = 1, \ldots, g$. This is because the residues and the integrals over the a_j cycles are zero by assumption, so the extra conditions imply that integrals of ω are independent of path. Thus given such an ω we may define $f = \int_0^z \omega$, where 0 is any base point. Then $df = \omega$. The poles of f will have order one less than poles of ω so $f \in L(\mathfrak{A}^{-1})$, as required.

The observation that $\mathrm{Im}(d)$ is obtained from $\Omega_0(\mathfrak{B}^{-1}))$ by imposing g linear conditions shows that $\dim(\mathrm{Im}(d)) \geq \dim(\Omega_0(\mathfrak{B}^{-1}))) - g = \deg(\mathfrak{A}) - g$. This already gives the Riemann inequality

$$\dim(L(\mathfrak{A}^{-1})) = \dim(\mathrm{Ker}(d)) + \dim(\mathrm{Im}(d)) \geq 1 + \deg(\mathfrak{A}) - g.$$

However to obtain an equality we must do more. Start with the basis $\tau_{p_j}^k$, $k = 2, \ldots, n_j + 1$, $j = 1..m$ for $\Omega_0(\mathfrak{B}^{-1}))$. We proved the bilinear relation

$$\int_{b_l} \tau_{p_j}^k = 2\pi i \frac{\alpha_{j,k-2}^{(l)}}{k-1}$$

where $\alpha^{(l)}_{j,k-2}$ are coefficients for the expansion in the co-ordinate system about p_j for the basis $\{\omega_1, \ldots, \omega_g\}$ for $\Omega(1)$ (the holomorphic one forms) normalized so that $\int_{a_j} \omega_k = \delta_{j,k}$. Explicitly,

$$\omega_l = \sum_k \alpha^{(l)}_{j,k} z^k dz \quad \text{near } p_j$$

An element

$$\omega = \sum_{j=1}^{m} \sum_{k=2}^{n_j+1} d_{j,k} \tau^k_{p_j}$$

of $\Omega_0(\mathfrak{B}^{-1}))$ lies in $\mathrm{Im}(d)$ precisely when for every $l = 1, \ldots, g$,

$$0 = \int_{b_l} \omega = \sum_{j=1}^{m} \sum_{k=2}^{n_j+1} d_{j,k} \int_{b_l} \tau^k_{p_j} = 2\pi i \sum_{j=1}^{m} \sum_{k=2}^{n_j+1} d_{j,k} \frac{\alpha^{(l)}_{j,k-2}}{k-1}$$

This can be written as the matrix equation

$$TM \begin{bmatrix} d_{1,2} \\ \vdots \\ d_{1,n_1+1} \\ \vdots \\ \vdots \\ d_{m,2} \\ \vdots \\ d_{m,n_m+1} \end{bmatrix} = 0$$

where

$$T = \left[\begin{array}{ccc|cc|ccc} \alpha^{(1)}_{1,0} & \cdots & \alpha^{(1)}_{1,n_1+1} & \cdots & \cdots & \alpha^{(1)}_{m,0} & \cdots & \alpha^{(1)}_{m,n_m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha^{(g)}_{1,0} & \cdots & \alpha^{(g)}_{1,n_1+1} & \cdots & \cdots & \alpha^{(g)}_{m,0} & \cdots & \alpha^{(g)}_{m,n_m-1} \end{array}\right]$$

and M is the diagonal matrix

$$M = \mathrm{diag}[1, 1/2, \ldots, 1/n_1, \ldots \; \ldots, 1, 1/2, \ldots, 1/n_m]$$

Since M is invertitlbe, this shows that the dimension of $\mathrm{Im}(d)$ is equal to the dimension of the kernel of T.

The proof is completed by recognizing the transpose of matrix T Define a map $R : \Omega(1) \to \mathbb{C}^{\deg(\mathfrak{A})}$ as follows. Given $\omega \in \Omega(1)$ (that is, ω is holomorphic one-form) we exand about p_j in local co-ordinates:

$$\omega = \sum_{k=0}^{\infty} e_{j,k} z^k dz \quad \text{near } p_j$$

Then

$$R : \omega \mapsto (e_{1,0}, \ldots, e_{1,n_1-1}, \ldots \; \ldots, e_{m,0}, \ldots, e_{m,n_m-1})$$

Then the kernel of R is $\Omega(\mathfrak{A})$, since $R\omega = 0$ forces ω to have zeros of precisely the required orders. To compute the matrix for R with respect to the basis $\{\omega_1, \ldots, \omega_g\}$ for $\Omega(1)$ and the standard basis for $\mathbb{C}^g$ we must compute $R\omega_l$ and place the resulting vectors in the columns of the matrix. Since

$$R\omega_l = (\alpha_{1,0}^{(l)}, \ldots, \alpha_{1,n_1-1}^{(l)}, \ldots \ \ldots, \alpha_{m,0}^{(l)}, \ldots, \alpha_{m,n_m-1}^{(l)})$$

we see that the matrix for R is the transpose of T. We have

$$\dim(\mathrm{Ker}(T^t)) + \dim(\mathrm{Im}(T^t)) = \dim(\Omega(1)) = g$$

This implies that

$$\dim(\Omega(\mathfrak{A})) + \dim(\mathrm{Im}(T^t)) = g$$

On the other hand

$$\dim(\mathrm{Ker}(T)) = \dim(\mathbb{C}^{\deg(\mathfrak{A})}) - \dim(\mathrm{Im}(T^t))$$

This implies that

$$\dim(\mathrm{Im}(d)) = \deg(\mathfrak{A}) - \dim(\mathrm{Im}(T^t))$$

Combining these equations gives

$$\dim(\mathrm{Im}(d)) = \deg(\mathfrak{A}) - g + \dim(\Omega(\mathfrak{A}))$$

In view of (2), this completes the proof. □

Proof of Theorem 6 when $\mathfrak{A}$ is equivalent to an integral divisor: This is immediate since none of the quantities in the Riemann Roch equality change when $\mathfrak{A}$ is replaced with an equivalent divisor. □

To proceed we must compute the degree of the canonical class. To prepare for this we prove the easiest case of the uniformization theorem.

Lemma 7 *If $g = 0$ then M is biholomorphic to $\mathbb{C}_\infty$.*

Proof: Pick any $p \in M$. Then $\dim(L(p^{-1})) \geq \deg(p) - g + 1 = 2$. Since the holomorphic functions (that is, constants) only account for one dimension, there must be a function in $L(p^{-1})$ with a pole of order one at p. Thinking of f as a holmorphic map from $M \to \mathbb{C}_\infty$ we find that $\sum_{p \in f^{-1}(q)} b_f(p) + 1 = 1$ for every $q \in \mathbb{C}_\infty$. This shows that $b_f(p) = 0$ for all p and that $f^{-1}(q)$ always contains a single point. Hence f is a biholomorphic map. □

Lemma 8 *Let ω be a meromorphic differential. Then $\deg(\omega) = 2g - 2$.*

Proof: Since all meromorphic differentials are equivalent and therefore all have the same degree we may choose any meromorphic differential to do this computation.

When $g = 0$ and $M \cong C_\infty$ we choose $\omega = dz$ If $w = 1/z$ then $dz = w^{-2}dw$ so there is a single pole of order 2 at $z = \infty$. Thus $\deg(\omega) = -2 = 2g - 2$.

When $g > 0$ there exist holomorphic differentials. Let ω be a holomorphic differential. By Proposition 4

$$\dim(L(\operatorname{div}(\omega)^{-1})) = \dim(\Omega(\operatorname{div}(\omega)^{-1}\operatorname{div}(\omega))) = \dim(\Omega(1)) = g$$

and

$$\dim(\Omega(\operatorname{div}(\omega))) = \dim(L(\operatorname{div}(\omega)/\operatorname{div}(\omega))) = \dim(L(1)) = 1$$

Because ω is holomorphic, $\operatorname{div}(\omega)$ is integral. Hence we may use the portion of the Riemann-Roch theorem already proved to conclude

$$g = \deg(\operatorname{div}(\omega)) - g + 1 + 1,$$

which proves the lemma. □

Another way of saying this is that the degree of the canonical class in the divisor class group is $2g - 2$

Proof of Theorem 6 when $\operatorname{div}(\omega)/\mathfrak{A}$ *is equivalent to an integral divisor for some meromorphic differential* ω*:* We use the Riemann-Roch theorem for $\operatorname{div}(\omega)/\mathfrak{A}$ and Proposition 4 to write

$$\begin{aligned}
\dim(\Omega(\mathfrak{A})) &= \dim(L(\mathfrak{A}/\operatorname{div}(\omega))) \\
&= \deg(\operatorname{div}(\omega)/\mathfrak{A}) - g + 1 + \dim(\Omega(\operatorname{div}(\omega)/\mathfrak{A})) \\
&= \deg(\operatorname{div}(\omega)) - \deg(\mathfrak{A}) - g + 1 + \dim(\Omega(\mathfrak{A}^{-1})) \\
&= 2g - 2 - \deg(\mathfrak{A}) - g + 1 + \dim(\Omega(\mathfrak{A}^{-1})).
\end{aligned}$$

This completes the proof. □

We can now handle the remaining cases.

Proof of Theorem 6 when neither $\mathfrak{A}$ *nor* $\operatorname{div}(\omega)/\mathfrak{A}$ *for any meromorphic differential* ω *is equivalent to an integral divisor:* First we claim that

$$\dim(L(\mathfrak{A}^{-1})) = 0. \tag{4}$$

To see this suppose there is a nonzero $f \in L(\mathfrak{A}^{-1})$. Then $\operatorname{div}(f) \geq \mathfrak{A}^{-1}$. This implies that $\operatorname{div}(f)\mathfrak{A} \geq 1$, that is, $\mathfrak{A}$ is equivalent to an integral divisor. This contradicts our first assumption and proves the claim.

Similarly, we claim that

$$\dim(\Omega(\mathfrak{A})) = 0. \tag{5}$$

Suppose that there is a nonzero $\omega \in \Omega(\mathfrak{A})$. Then $\operatorname{div}(\omega) \geq \mathfrak{A}$ so that $\operatorname{div}(\omega)/\mathfrak{A} \geq 1$ contradicting our second assumption.

Therefore, to prove the Riemann-Roch theorem in this case, we must show that $\deg(\mathfrak{A}) = g - 1$. Write $\mathfrak{A} = \mathfrak{A}_1\mathfrak{A}_2$ where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are both integral and have no points in common. Then

$$\deg(\mathfrak{A}) = \deg(\mathfrak{A}_1) - \deg(\mathfrak{A}_2)$$

By the Riemann inequality for integral divisors

$$\dim(L(\mathfrak{A}_1^{-1})) \geq \deg(\mathfrak{A}_1) - g + 1 = \deg(\mathfrak{A}) + \deg(\mathfrak{A}_2) - g + 1$$

Suppose $\deg(\mathfrak{A}) \geq g$. Then $\dim(L(\mathfrak{A}_1^{-1})) \geq \deg(\mathfrak{A}_2) + 1$. Now the space $L(\mathfrak{A}^{-1}))$ is a subspace of $\dim(L(\mathfrak{A}_1^{-1}))$ obtained by imposing $\deg(\mathfrak{A}_2)$ linear conditions, namely that the functions vanish at the points and to the orders prescibed by $\mathfrak{A}_2$. This implies that $\dim(L(\mathfrak{A}^{-1})) \geq \dim(L(\mathfrak{A}_1^{-1})) - \deg(\mathfrak{A}_2) \geq 1$. This contradicts (4) and establishes that $\deg(\mathfrak{A}) \leq g - 1$

Now we repeat the argument above with for the divisor $\operatorname{div}(\omega)/\mathfrak{A}$, where ω is a meromorphic differential. Under the assumption $\deg(\operatorname{div}(\omega)/\mathfrak{A}) \geq g$ we derive $\dim(L(\mathfrak{A}/\operatorname{div}(\omega)) \geq 1$ But by (5)

$$\dim(L(\mathfrak{A}/\operatorname{div}(\omega))) = \dim(\Omega(\mathfrak{A})) = 0.$$

This implies $\deg(\operatorname{div}(\omega)/\mathfrak{A}) \leq g - 1$ so that $\deg(\operatorname{div}(\omega)) - \deg(\mathfrak{A}) \leq g - 1$ Since $\deg(\operatorname{div}(\omega)) = 2g - 2$ this shows that $\deg(\mathfrak{A}) \geq g - 1$.

Thus $\deg(\mathfrak{A}) = g - 1$ and the proof is complete. $\square$