Inverse Scattering

Suppose that we are interested in a system in which sound waves, for example, scatter off of some obstacle. Let $p(\mathbf{x}, t)$ be the pressure at position $\mathbf{x}$ and time t. In (a somewhat idealized) free space, p obeys the wave equation $\frac{\partial^2 p}{\partial t^2} = c^2 \Delta p$, where c is the speed of sound. We shall assume that in most of the world, c takes a constant value c_0. But we introduce an obstacle by allowing c to depend on position in some compact region. We further allow for some absorbtion in that region. Then p obeys

$$\frac{\partial^2 p}{\partial t^2} + \gamma(\mathbf{x}) \frac{\partial p}{\partial t} = c(\mathbf{x})^2 \Delta p$$

where $\gamma(\mathbf{x})$ is the damping coefficient of the medium at $\mathbf{x}$. For solutions of fixed (temporal) frequency, $p(\mathbf{x}, t) = \mathrm{Re}\left[u(\mathbf{x}) e^{-i\omega t}\right]$ with

$$\Delta u + \frac{\omega^2}{c(\mathbf{x})^2}\left[1 + i\frac{\gamma(\mathbf{x})}{\omega}\right] u = 0$$

Outside of some compact region

$$\frac{\omega^2}{c(\mathbf{x})^2}\left[1 + i\frac{\gamma(\mathbf{x})}{\omega}\right] = \frac{\omega^2}{c_0^2} = k^2 \qquad \text{where} \qquad k = \frac{\omega}{c_0} > 0$$

If we define the index of refraction by

$$n(\mathbf{x}) = \frac{c_0^2}{c(\mathbf{x})^2}\left[1 + i\frac{\gamma(\mathbf{x})}{\omega}\right]$$

then

$$\Delta u + k^2 n(\mathbf{x}) u = 0 \tag{1}$$

with $n(\mathbf{x}) = 1$ outside of some compact region. We first consider two special cases.

Example 1 (Free Space) In the absence of any obstacle $\Delta u + k^2 u = 0$ on all of $\mathbb{R}^3$. Then we can solve just by Fourier transforming. The general solution is a mixture of solutions of the form $u = e^{ik\hat{\boldsymbol{\theta}}\cdot\mathbf{x}}$ where $\hat{\boldsymbol{\theta}}$ is a unit vector. This represents a plane wave coming in from infinity in direction $\hat{\boldsymbol{\theta}}$.

Example 2 (Point Source) If we have free space everywhere except at the origin and we have a unit point source at the origin, then

$$\Delta u + k^2 u = \delta(\mathbf{x})$$

Except at the origin, where there is a singularity, we still have $\Delta u + k^2 u = 0$. The point source generates expanding spherical waves. So u should be a function of $r = |\mathbf{x}|$ only and obey

$$u''(r) + \tfrac{2}{r}u'(r) + k^2 u(r) = 0$$

This is easily solved by changing variables to $v(r) = ru(r)$, which obeys

$$v''(r) + k^2 v(r) = 0$$

So $v(r) = \alpha \sin(kr) + \beta \cos(kr)$ and $u(r) = \alpha \frac{\sin(kr)}{r} + \beta \frac{\cos(kr)}{r}$. To be an outgoing (rather than incoming) wave $u(r) = \alpha' \frac{e^{ikr}}{r}$. (Note that $e^{ikr}e^{-i\omega t}$ is constant on $r = \frac{\omega}{k}t$, which is a sphere that is expanding with speed c_0.) To give the Dirac delta function on the right hand side of $\Delta u + k^2 u = \delta(\mathbf{x})$ coefficient one, we need $u(\mathbf{x}) = -\frac{e^{ik|\mathbf{x}|}}{4\pi|\mathbf{x}|}$. (See, for example, the notes on Poisson's equation.)

Now let's return to the general case. We want to think of a physical situation in which we send a plane wave $u^i(\mathbf{x}) = e^{ik\hat{\boldsymbol{\theta}}\cdot\mathbf{x}}$ in from infinity. This plane wave shakes up the obstacle which then emits a bunch of expanding spherical waves $\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-y|}$ emanating from various points $\mathbf{y}$ in the obstacle. So the full solution is of the form

$$u(\mathbf{x}) = u^i(\mathbf{x}) + u^s(\mathbf{x})$$

where the scattered wave, u^s, obeys the "radiation condition"

$$\tfrac{\partial}{\partial r}u^s(\mathbf{x}) - iku^s(\mathbf{x}) = O\big(\tfrac{1}{|\mathbf{x}|^2}\big) \qquad \text{as} \qquad |\mathbf{x}| \to \infty \tag{2}$$

This condition is chosen to allow outgoing waves $\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}$ but not incoming waves $\frac{e^{-ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}$.

Define

$$\Phi(\mathbf{x}, \mathbf{y}) = \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}$$

Since $\delta(\mathbf{x} - \mathbf{y})$ is the kernel of the identity operator,

$$(\Delta_x + \mathbf{k}^2)\Phi(\mathbf{x}, \mathbf{y}) = -\delta(\mathbf{x} - \mathbf{y})$$

says, roughly, that $u(\mathbf{x}) \mapsto -\int \Phi(\mathbf{x}, \mathbf{y})u(\mathbf{y})\, d\mathbf{y}$ is the inverse of the map $u(\mathbf{x}) \mapsto (\Delta + k^2)u(\mathbf{x})$ for functions that obey the radiation condition. We can exploit this to convert (1), (2) into an equivalent integral equation

$$\begin{aligned} \Delta u + k^2 n(\mathbf{x})u = 0 \quad &\Longrightarrow \quad \Delta u + k^2 u = k^2\big(1 - n(\mathbf{x})\big)u \\ &\Longrightarrow \quad \Delta u^s + k^2 u^s = k^2\big(1 - n(\mathbf{x})\big)u \end{aligned}$$

since $\Delta u^i + k^2 u^i = 0$. As u^s obeys the radiation condition

$$u^s(\mathbf{x}) = -k^2 \int \Phi(\mathbf{x},\mathbf{y})\big(1-n(\mathbf{y})\big)u(\mathbf{y})\,d\mathbf{y}$$

so that

$$u(\mathbf{x}) = u^i(\mathbf{x}) - k^2 \int \big(1-n(\mathbf{y})\big)\Phi(\mathbf{x},\mathbf{y})u(\mathbf{y})\,d\mathbf{y} \tag{3}$$

This is called the Lippmann–Schwinger equation. Observe that it is of the form $u = u^i - Fu$ or $(\mathbb{1}-F)u = u^i$ where F is the linear operator $u(\mathbf{x}) \mapsto k^2 \int \Phi(\mathbf{x},\mathbf{y})\big(1-n(\mathbf{y})\big)u(\mathbf{y})\,d\mathbf{y}$. This operator is compact (if you impose the appropriate norms) and so behaves much like a finite dimensional matrix. If F has operator norm smaller than one, which is the case if $k^2(1-n)$ is small enough, then $\mathbb{1}-F$ is trivially invertible and the equation $(\mathbb{1}-F)u = u^i$ has a unique solution. Even if F has operator norm larger than or equal to one, $(\mathbb{1}-F)u = u^i$ fails to have a unique solution only if F has eigenvalue one. One can show that this is impossible in the present setting. Thus, one can prove

Theorem. *If $n \in C^2(\mathbb{R}^3)$, $n(\mathbf{x})-1$ has compact support and $\operatorname{Re} n(\mathbf{x}), \operatorname{Im} n(\mathbf{x}) \ge 0$, then (1), (2) has a unique solution.*

For large $|\mathbf{x}|$, Φ has the asymptotic behaviour

$$\Phi(\mathbf{x},\mathbf{y}) = \tfrac{e^{ik|\mathbf{x}|}}{4\pi|\mathbf{x}|}e^{-ik\hat{\mathbf{x}}\cdot\mathbf{y}} + O\big(\tfrac{1}{|\mathbf{x}|^2}\big)$$

so that, when the incoming plane wave is moving in direction $\hat{\boldsymbol{\theta}}$,

$$u(\mathbf{x};\hat{\boldsymbol{\theta}}) = u^i(\mathbf{x};\hat{\boldsymbol{\theta}}) + \tfrac{e^{ik|\mathbf{x}|}}{4\pi|\mathbf{x}|}u_\infty(\hat{\mathbf{x}};\hat{\boldsymbol{\theta}}) + O\big(\tfrac{1}{|\mathbf{x}|^2}\big)$$

where

$$u_\infty(\hat{\mathbf{x}};\hat{\boldsymbol{\theta}}) = -k^2 \int e^{-ik\hat{\mathbf{x}}\cdot\mathbf{y}}\big(1-n(\mathbf{y})\big)u(\mathbf{y};\hat{\boldsymbol{\theta}})\,d\mathbf{y}$$

If we are observing the scattered wave from vantage points far from the obstacle, we will only be able to measure $u_\infty(\hat{\mathbf{x}};\hat{\boldsymbol{\theta}})$. The inverse problem then is

Question: Given $u_\infty(\hat{\mathbf{x}};\hat{\boldsymbol{\theta}})$, for all $\hat{\mathbf{x}},\hat{\boldsymbol{\theta}} \in S^2$, can we determine n? The short answer is

Answer: Yes, because we have the

Theorem. *If $n_1, n_2 \in C^2(\mathbb{R}^3)$ with n_1-1, n_2-1 of compact support and $u_{1,\infty}(\hat{\mathbf{x}};\hat{\boldsymbol{\theta}}) = u_{2,\infty}(\hat{\mathbf{x}};\hat{\boldsymbol{\theta}})$, for all $\hat{\mathbf{x}},\hat{\boldsymbol{\theta}} \in S^2$, then $n_1 = n_2$.*

We can get a rough idea why this Theorem is true by looking at the Born approximation. In this approximation u^s is ignored in the computation of u_∞ so that

$$\begin{aligned} u_\infty(\hat{\mathbf{x}};\hat{\boldsymbol{\theta}}) &\approx -k^2 \int e^{-ik\hat{\mathbf{x}}\cdot\mathbf{y}}\big(1-n(\mathbf{y})\big)u^i(\mathbf{y};\hat{\boldsymbol{\theta}})\,d\mathbf{y} \\ &= -k^2 \int e^{-ik(\hat{\mathbf{x}}-\hat{\boldsymbol{\theta}})\cdot\mathbf{y}}\big(1-n(\mathbf{y})\big)\,d\mathbf{y} \end{aligned}$$

If we measure $u_\infty(\hat{\mathbf{x}};\hat{\boldsymbol{\theta}})$, then, in this approximation, we know the Fourier transform of $1-n(\mathbf{y})$ on the set $\left\{\ k(\hat{\mathbf{x}}-\hat{\boldsymbol{\theta}})\ \middle|\ \hat{\mathbf{x}},\hat{\boldsymbol{\theta}}\in S^2\ \right\}$ which is exactly the closed ball of radius $2k$ centered on the origin in $\mathbb{R}^3$. Since $1-n(\mathbf{y})$ is of compact support, its Fourier transform is analytic. So knowledge of the Fourier transform on any open ball uniquely determines it.

References

- Andreas Kirsch, **An Introduction to the Mathematical Theory of Inverse Problems**, Springer, 1996.

