

# Complex Numbers and Exponentials

A complex number is nothing more than a point in the  $xy$ -plane. The sum and product of two complex numbers  $(x_1, y_1)$  and  $(x_2, y_2)$  is defined by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$$

respectively. It is conventional to use the notation  $x + iy$  (or in electrical engineering country  $x + jy$ ) to stand for the complex number  $(x, y)$ . In other words, it is conventional to write  $x$  in place of  $(x, 0)$  and  $i$  in place of  $(0, 1)$ . In this notation, The sum and product of two complex numbers  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$  is given by

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1z_2 = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$$

Addition and multiplication of complex numbers obey the familiar algebraic rules

$$z_1 + z_2 = z_2 + z_1$$

$$z_1z_2 = z_2z_1$$

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \quad z_1(z_2z_3) = (z_1z_2)z_3$$

$$0 + z_1 = z_1$$

$$1z_1 = z_1$$

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3 \quad (z_1 + z_2)z_3 = z_1z_3 + z_2z_3$$

The negative of any complex number  $z = x + iy$  is defined by  $-z = -x + (-y)i$ , and obeys  $z + (-z) = 0$ . The inverse of any complex number  $z = x + iy$ , other than 0, is defined by  $\frac{1}{z} = \frac{x}{x^2+y^2} + \frac{-y}{x^2+y^2}i$  and obeys  $\frac{1}{z}z = 1$ . The complex number  $i$  has the special property

$$i^2 = (0 + 1i)(0 + 1i) = (0 \times 0 - 1 \times 1) + i(0 \times 1 + 1 \times 0) = -1$$

The absolute value, or modulus,  $|z|$  of  $z = x + iy$  is given by

$$|z| = \sqrt{x^2 + y^2} = z\bar{z}$$

where  $\bar{z} = x - iy$  is called the complex conjugate of  $z$ . It is just the distance between  $z$  and

the origin. We have

$$\begin{aligned}
 |z_1 z_2| &= \sqrt{(x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2} \\
 &= \sqrt{x_1^2 x_2^2 - 2x_1 x_2 y_1 y_2 + y_1^2 y_2^2 + x_1^2 y_2^2 + 2x_1 y_2 x_2 y_1 + x_2^2 y_1^2} \\
 &= \sqrt{x_1^2 x_2^2 + y_1^2 y_2^2 + x_1^2 y_2^2 + x_2^2 y_1^2} \\
 &= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \\
 &= |z_1| |z_2|
 \end{aligned}$$

and

$$z^{-1} = \frac{z^*}{|z|^2}$$

for all complex numbers  $z_1, z_2$  and  $z \neq 0$ .

## The Complex Exponential

**Definition and Basic Properties.** For any complex number  $z = x + iy$  the exponential  $e^z$ , is defined by

$$e^{x+iy} = e^x \cos y + ie^x \sin y$$

For any two complex numbers  $z_1$  and  $z_2$

$$\begin{aligned}
 e^{z_1} e^{z_2} &= e^{x_1} (\cos y_1 + i \sin y_1) e^{x_2} (\cos y_2 + i \sin y_2) \\
 &= e^{x_1+x_2} (\cos y_1 + i \sin y_1) (\cos y_2 + i \sin y_2) \\
 &= e^{x_1+x_2} \{(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) + i(\cos y_1 \sin y_2 + \cos y_2 \sin y_1)\} \\
 &= e^{x_1+x_2} \{\cos(y_1 + y_2) + i \sin(y_1 + y_2)\} \\
 &= e^{(x_1+x_2)+i(y_1+y_2)} \\
 &= e^{z_1+z_2}
 \end{aligned}$$

so that the familiar multiplication formula also applies to complex exponentials. For any complex number  $a = \alpha + i\beta$  and real number  $t$

$$e^{at} = e^{\alpha t + i\beta t} = e^{\alpha t} [\cos(\beta t) + i \sin(\beta t)]$$

so that the derivative with respect to  $t$

$$\begin{aligned}\frac{d}{dt}e^{\alpha t} &= \alpha e^{\alpha t}[\cos(\beta t) + i \sin(\beta t)] + e^{\alpha t}[-\beta \sin(\beta t) + i\beta \cos(\beta t)] \\ &= (\alpha + i\beta)e^{\alpha t}[\cos(\beta t) + i \sin(\beta t)] \\ &= \alpha e^{\alpha t}\end{aligned}$$

is also the familiar one.

**Relationship with sin and cos.** When  $\theta$  is a real number

$$\begin{aligned}e^{i\theta} &= \cos \theta + i \sin \theta \\ e^{-i\theta} &= \cos \theta - i \sin \theta\end{aligned}$$

are complex numbers of modulus one. Solving for  $\cos \theta$  and  $\sin \theta$  (by adding and subtracting the two equations)

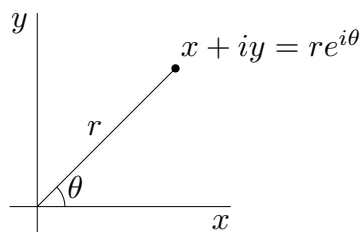
$$\begin{aligned}\cos \theta &= \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \\ \sin \theta &= \frac{1}{2i}(e^{i\theta} - e^{-i\theta})\end{aligned}$$

These formulae make it easy derive trig identities. For example

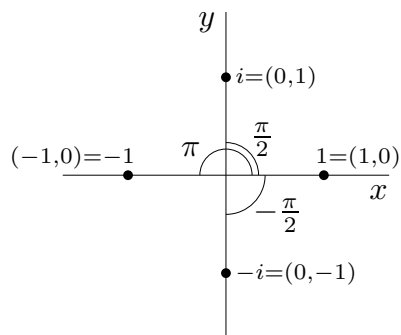
$$\begin{aligned}\cos \theta \cos \phi &= \frac{1}{4}(e^{i\theta} + e^{-i\theta})(e^{i\phi} + e^{-i\phi}) \\ &= \frac{1}{4}(e^{i(\theta+\phi)} + e^{i(\theta-\phi)} + e^{i(-\theta+\phi)} + e^{-i(\theta+\phi)}) \\ &= \frac{1}{4}(e^{i(\theta+\phi)} + e^{-i(\theta+\phi)} + e^{i(\theta-\phi)} + e^{i(-\theta+\phi)}) \\ &= \frac{1}{2}(\cos(\theta + \phi) + \cos(\theta - \phi))\end{aligned}$$

**Polar Coordinates.** Let  $z = x + iy$  be any complex number. Writing  $x$  and  $y$  in polar coordinates in the usual way gives

$$x + iy = r \cos \theta + ir \sin \theta = re^{i\theta}$$



In particular



$$\begin{aligned}
 1 &= e^{i0} = e^{2\pi i} = e^{2k\pi i} && \text{for } k = 0, \pm 1, \pm 2, \dots \\
 -1 &= e^{i\pi} = e^{3\pi i} = e^{(1+2k)\pi i} && \text{for } k = 0, \pm 1, \pm 2, \dots \\
 i &= e^{i\pi/2} = e^{5/2\pi i} = e^{(\frac{1}{2}+2k)\pi i} && \text{for } k = 0, \pm 1, \pm 2, \dots \\
 -i &= e^{-i\pi/2} = e^{3/2\pi i} = e^{(-\frac{1}{2}+2k)\pi i} && \text{for } k = 0, \pm 1, \pm 2, \dots
 \end{aligned}$$

The polar coordinate representation makes it easy to find square roots, third roots and so on. Fix any positive integer  $n$ . The  $n^{\text{th}}$  roots of unity are, by definition, all solutions  $z$  of

$$z^n = 1$$

Writing  $z = re^{i\theta}$

$$r^n e^{n\theta i} = 1e^{0i}$$

The polar coordinates  $(r, \theta)$  and  $(r', \theta')$  represent the same point in the  $xy$ -plane if and only if  $r = r'$  and  $\theta = \theta' + 2k\pi$  for some integer  $k$ . So  $z^n = 1$  if and only if  $r^n = 1$ , i.e.  $r = 1$ , and  $n\theta = 2k\pi$  for some integer  $k$ . The  $n^{\text{th}}$  roots of unity are all complex numbers  $e^{2\pi i \frac{k}{n}}$  with  $k$  integer. There are precisely  $n$  distinct  $n^{\text{th}}$  roots of unity because  $e^{2\pi i \frac{k}{n}} = e^{2\pi i \frac{k'}{n}}$  if and only if  $2\pi \frac{k}{n} - 2\pi i \frac{k'}{n} = 2\pi \frac{k-k'}{n}$  is an integer multiple of  $2\pi$ . That is, if and only if  $k - k'$  is an integer multiple of  $n$ . The are  $n$  distinct  $n^{\text{th}}$  roots of unity are

$$1, e^{2\pi i \frac{1}{n}}, e^{2\pi i \frac{2}{n}}, e^{2\pi i \frac{3}{n}}, \dots, e^{2\pi i \frac{n-1}{n}}$$

