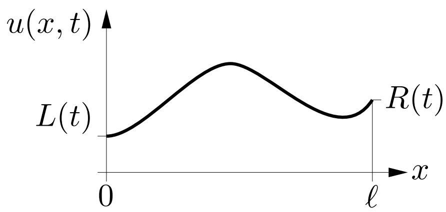
Numerical Solution of Partial Differential Equations

In these notes we develop a method for generating, numerically, approximate solutions to the vibrating string problem



$$u_{tt}(x,t) = c^2 u_{xx}(x,t) \quad 0 \le x \le \ell \quad t \ge 0 \qquad \text{(wave equation) (1)}$$

$$u(x,0) = f(x) \qquad 0 \le x \le \ell \qquad \text{(initial position) (2a)}$$

$$u_t(x,0) = g(x) \qquad 0 \le x \le \ell \qquad \text{(initial speed) (2b)}$$

$$u(0,t) = L(t) \qquad t \ge 0 \qquad \text{(left boundary) (3a)}$$

$$u(\ell,t) = R(t) \qquad t \ge 0 \qquad \text{(right boundary) (3b)}$$

The function $u(x,t)$ gives the amplitude of the string at position x and time t. Equation (1) is the wave equation. It is the equation of motion for the vibrating string and is a consequence of Newton's law, $F = ma$. Equations (2a,b) specify the initial position and speed of the string and equations (3a,b) specify the position of the two ends of the string for all time.

The method will be an extension of those (like Euler's method, for example) used for generating, numerically, approximate solutions to the initial value problem

$$y'(t) = f\big(t, y(t)\big) \qquad t \ge 0 \qquad \text{(ode) (4)}$$

$$y(0) = y_0$$

Recall that under Euler's method, rather that generating approximate values for $y(t)$ for all values of $t \ge 0$, we pick a step size Δt and consider only $t = 0,\ \Delta t,\ 2\Delta t,\ \cdots,\ t_n = n\Delta t,\ \cdots$. We approximate the ordinary differential equation (4) by an equation, that does not contain any derivatives and that involves only the times t_n, by approximating

$$y'(t_n) = \lim_{h \to 0} \frac{y(t_n + h) - y(t_n)}{h} \approx \frac{y(t_n + \Delta t) - y(t_n)}{\Delta t} = \frac{y(t_{n+1}) - y(t_n)}{\Delta t}$$

Denoting $y(t_n) = y_n$, this gives

$$\frac{y_{n+1} - y_n}{\Delta t} \approx y'(t_n) = f\big(t_n, y(t_n)\big) = f(t_n, y_n)$$

which simplifies to the Euler's method formula

$$y_{n+1} \approx y_n + \Delta t\, f(t_n, y_n)$$

We now apply the same strategy to the wave equation (1). As there are two independent variables, x and t, we pick two step sizes, Δx and Δt and set $x_m = m\Delta x,\ t_n = n\Delta t$. To ensure that the right hand boundary $x = \ell$ is one of the x_m's, we choose $\Delta x = \ell/M$ for some integer M. We shall generate approximate values for $u(x_m, t_n)$ for $0 \le m \le M$ and $n \ge 0$ by replacing the partial differential equation(1) by a difference equation. To do so we need approximations for u_{tt} and u_{xx} analogous to $y'(t_n) \approx \frac{y(t_{n+1})-y(t_n)}{\Delta t}$.

A Difference Approximation for $y''(t_n)$

We can get a symmetric looking approximation for $y''(t_n)$ by combining

$$y''(t_n) = \lim_{h\to 0} \frac{y'(t_n) - y'(t_n - h)}{h} \approx \frac{y'(t_n) - y'(t_n - \Delta t)}{\Delta t}$$

with

$$y'(t_n) = \lim_{h\to 0} \frac{y(t_n + h) - y(t_n)}{h} \approx \frac{y(t_n + \Delta t) - y(t_n)}{\Delta t}$$

$$y'(t_n - \Delta t) = \lim_{h\to 0} \frac{y(t_n - \Delta t + h) - y(t_n - \Delta t)}{h} \approx \frac{y(t_n) - y(t_n - \Delta t)}{\Delta t}$$

In all three cases we approximated a limit $\lim_{h\to 0}$ by choosing $h = \Delta t$. Substituting

$$\begin{aligned} y''(t_n) &\approx \frac{y'(t_n) - y'(t_n - \Delta t)}{\Delta t} \approx \frac{\frac{y(t_n+\Delta t) - y(t_n)}{\Delta t} - \frac{y(t_n) - y(t_n - \Delta t)}{\Delta t}}{\Delta t} \\ &= \frac{y(t_n + \Delta t) - 2y(t_n) + y(t_n - \Delta t)}{\Delta t^2} \qquad \text{(difference approximation) (5)} \end{aligned}$$

The Explicit Finite Difference Method for the Wave Equation

In terms of $u_{m,n} = u(m\Delta x, n\Delta t) = u(x_m, t_n)$ the analogs of the difference approximation (5) for $u_{tt}(x_m, t_n)$ and $u_{xx}(x_m, t_n)$ are

$$\begin{aligned} u_{tt}(x_m, t_n) &\approx \frac{u(x_m, t_n + \Delta t) - 2u(x_m, t_n) + u(x_m, t_n - \Delta t)}{\Delta t^2} \\ &= \frac{u_{m,n+1} - 2u_{m,n} + u_{m,n-1}}{\Delta t^2} \\ u_{xx}(x_m, t_n) &\approx \frac{u(x_m + \Delta x, t_n) - 2u(x_m, t_n) + u(x_m - \Delta x, t_n)}{\Delta x^2} \\ &= \frac{u_{m+1,n} - 2u_{m,n} + u_{m-1,n}}{\Delta x^2} \end{aligned}$$

Substituting into the wave equation (1)

$$\frac{u_{m,n+1} - 2u_{m,n} + u_{m,n-1}}{\Delta t^2} = c^2 \frac{u_{m+1,n} - 2u_{m,n} + u_{m-1,n}}{\Delta x^2}$$

and simplifying

$$\boxed{u_{m,n+1} = \frac{c^2 \Delta t^2}{\Delta x^2}\big(u_{m+1,n} + u_{m-1,n}\big) + 2\left(1 - \frac{c^2 \Delta t^2}{\Delta x^2}\right) u_{m,n} - u_{m,n-1}}$$

(finite difference wave equation) (6)

The finite difference wave equation (6) is used in much the same way as Euler's method. Before we can use it to generate approximate values of $u(x,t)$ at time t_{n+1} we must know approximate values at times t_n and t_{n-1}. We shall shortly see how to use the two initial conditions (2a,b) to get approximate values for $u(x,t)$ at times t_0 and t_1. So suppose that $u_{m,0}$ and $u_{m,1}$ are known for all $0 \le m \le M$. Then (6) with $n = 1$

$$u_{m,2} = \frac{c^2 \Delta t^2}{\Delta x^2}\big(u_{m+1,1} + u_{m-1,1}\big) + 2\left(1 - \frac{c^2 \Delta t^2}{\Delta x^2}\right) u_{m,1} - u_{m,0} \tag{7}$$

determines $u_{m,2}$ for all $1 \le m \le M - 1$. For example, when $M = 4$, setting, successively $m = 1, 2, 3$ in (7) gives

$$\begin{aligned}
u_{1,2} &= \frac{c^2 \Delta t^2}{\Delta x^2}\big(u_{2,1} + u_{0,1}\big) + 2\left(1 - \frac{c^2 \Delta t^2}{\Delta x^2}\right) u_{1,1} - u_{1,0} \\
u_{2,2} &= \frac{c^2 \Delta t^2}{\Delta x^2}\big(u_{3,1} + u_{1,1}\big) + 2\left(1 - \frac{c^2 \Delta t^2}{\Delta x^2}\right) u_{2,1} - u_{2,0} \\
u_{3,2} &= \frac{c^2 \Delta t^2}{\Delta x^2}\big(u_{4,1} + u_{2,1}\big) + 2\left(1 - \frac{c^2 \Delta t^2}{\Delta x^2}\right) u_{3,1} - u_{3,0}
\end{aligned}$$

Notice that every $u_{m,n}$ which appears on the right hand side has $n \in \{0,1\}$ and $m \in \{0,1,2,3,4\}$. All these $u_{m,n}$'s are known prior to the beginning of the $n = 1$ step. In general, when $1 \le m \le M - 1$, the subscripts $m + 1$ amd $m - 1$, which appear on the right hand side of (6), are both between 0 and M. The two boundary conditions (3a,b) determine $u_{0,2}$ and $u_{M,2}$ respectively. For $M = 4$

$$\begin{aligned}
&(3a) \quad \Longrightarrow \quad u_{0,2} = u(0, 2\Delta t) = L(2\Delta t) \\
&(3b) \quad \Longrightarrow \quad u_{4,2} = u(M\Delta x, 2\Delta t) = u(\ell, 2\Delta t) = R(2\Delta t)
\end{aligned}$$

At this point $u_{m,n}$ is known for every $0 \le m \le M$ and $n \le 2$. Then (6) with $n = 2$ yields $u_{m,3}$ for all $1 \le m \le M - 1$. Again, (3a) and (3b) determine $u_{0,3}$ and $u_{M,3}$. And so on.

The first step

To generate, using (6), approximate values of $u(x,t)$ at time $t_1 = \Delta t$ it is necessary to already know approximate values at times $t_0 = 0$ and $t_{-1} = -\Delta t$. The initial position condition (2a) tells us that

$$u_{m,0} = u(m\Delta x, 0) = f(m\Delta x) \tag{8}$$

and the initial speed condition (2b) tells us indirectly and approximately $u_{m,-1}$. Naively,

$$\begin{aligned} g(x) = u_t(x,0) &= \lim_{h\to 0} \frac{u(x,0) - u(x,-h)}{h} \approx \frac{u(x,0) - u(x,-\Delta t)}{\Delta t} \\ \implies \quad u_{m,-1} &= u(m\Delta x, -\Delta t) \approx u(m\Delta x, 0) - \Delta t\, g(m\Delta x) \\ &= u_{m,0} - \Delta t\, g(m\Delta x) \end{aligned}$$

We can get a more accurate approximation by using

$$\begin{aligned} g(x) = u_t(x,0) &= \lim_{h\to 0} \frac{u(x,h) - u(x,-h)}{2h} \approx \frac{u(x,\Delta t) - u(x,-\Delta t)}{2\Delta t} \\ \implies \quad u_{m,-1} &= u(m\Delta x, -\Delta t) \approx u(m\Delta x, \Delta t) - 2\Delta t\, g(m\Delta x) \\ &= u_{m,1} - 2\Delta t\, g(m\Delta x) \end{aligned} \tag{9}$$

instead. (To see that it is more accurate, compare the Taylor expansions of $\frac{u(x,\Delta t)-u(x,-\Delta t)}{2\Delta t}$ and $\frac{u(x,0)-u(x,-\Delta t)}{\Delta t}$ in powers of Δt.) Substituting (9) into the finite difference wave equation (6) with n set to 0 gives

$$\begin{aligned} u_{m,1} &= \frac{c^2\Delta t^2}{\Delta x^2}\big(u_{m+1,0} + u_{m-1,0}\big) + 2\left(1 - \frac{c^2\Delta t^2}{\Delta x^2}\right)u_{m,0} - u_{m,-1} \\ &= \frac{c^2\Delta t^2}{\Delta x^2}\big(u_{m+1,0} + u_{m-1,0}\big) + 2\left(1 - \frac{c^2\Delta t^2}{\Delta x^2}\right)u_{m,0} - u_{m,1} + 2\Delta t\, g(m\Delta x) \\ \implies \quad 2u_{m,1} &= \frac{c^2\Delta t^2}{\Delta x^2}\big(u_{m+1,0} + u_{m-1,0}\big) + 2\left(1 - \frac{c^2\Delta t^2}{\Delta x^2}\right)u_{m,0} + 2\Delta t\, g(m\Delta x) \\ \implies \quad u_{m,1} &= \frac{1}{2}\frac{c^2\Delta t^2}{\Delta x^2}\big(u_{m+1,0} + u_{m-1,0}\big) + \left(1 - \frac{c^2\Delta t^2}{\Delta x^2}\right)u_{m,0} + \Delta t\, g(m\Delta x) \end{aligned} \tag{10}$$

Equations (8) and (10) give $u_{m,n}$ for all $n = 0, 1$ and $1 \le m \le M-1$.

The final procedure

To start, set, for all $1 \le m \le M-1$,

$$\begin{aligned} u_{m,0} &= f(m\Delta x) \\ u_{m,1} &= \frac{1}{2}\frac{c^2\Delta t^2}{\Delta x^2}\big(u_{m+1,0}+u_{m-1,0}\big) + \left(1-\frac{c^2\Delta t^2}{\Delta x^2}\right)u_{m,0} + \Delta t\, g(m\Delta x) \end{aligned}$$

Then, for each successive step $n = 1, 2, 3, \cdots$, set

$$u_{m,n+1} = \frac{c^2\Delta t^2}{\Delta x^2}\big(u_{m+1,n}+u_{m-1,n}\big) + 2\left(1-\frac{c^2\Delta t^2}{\Delta x^2}\right)u_{m,n} - u_{m,n-1}$$

for all $1 \le m \le M-1$. Whenever $u_{0,k}$ or $u_{M,k}$ is encountered on the right hand sides of these formulae use

$$u_{0,k} = L(k\Delta t)$$

$$u_{M,k} = R(k\Delta t)$$

Further reading

The subject of numerical methods for partial differential equations is enormous. It is also a lot more subtle than suggested by the above discussion. You can start learning more about this subject by reading the partial differential equations chapter in the popular book

Numerical Recipes, W. H. Press, B. P. Flannery, S. A. Teukolsky and W. T. Vetterling, Cambridge University Press.

