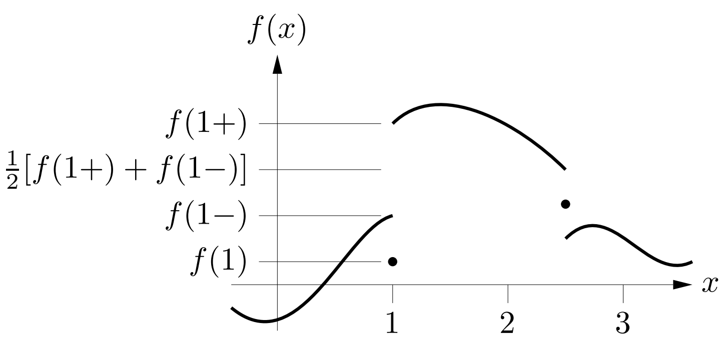
Fourier Series

Roughly speaking, a Fourier series expansion for a function is a representation of the function as sum of sin's and cosines. Expressing a musical tone as a sum of a fundamental tone and various harmonics is such a representation. So is a spectral decomposition of light waves. The main Fourier series expansions that we use in this course are stated in the next section. We shall never prove that these expansions are correct. But in the section "Validity of Fourier series" we give an elementary partial argument that, hopefully, will convince you that the expansions are indeed correct. In the section "Usefulness of Fourier Series" we introduce one of the many ways that Fourier series are used in applications.

The Main Fourier Series Expansions.

We shall shortly state three Fourier series expansions. They are applicable to functions that are piecewise continuous with piecewise continuous first derivative. In applications, most functions satisfy these regularity requirements. We start with the definition of "piecewise continuous".

A function $f(x)$ is said to be piecewise continuous if it is continuous except for isolated jump discontinuities. In the example below, $f(x)$ is continuous except for jump



discontinuities at $x = 1$ and $x = 2.5$. If a function $f(x)$ has a jump discontinuity at x_0, then the value of $f(x)$ as it enters and leaves the jump are

$$f(x_0-) = \lim_{\substack{x \to x_0 \\ x < x_0}} f(x) \qquad \text{and} \qquad f(x_0+) = \lim_{\substack{x \to x_0 \\ x > x_0}} f(x)$$

respectively. If f were continuous at x_0, we would have $f(x_0) = f(x_0+) = f(x_0-)$. At a jump, however, there is no a priori relation between $f(x_0)$ and $f(x_0\pm)$. In the example above, $f(1)$ is well below both $f(1-)$ and $f(1+)$. On the other hand, it is fairly common for the value of f at the jump x_0 to be precisely at the midpoint of the jump. That is $f(x_0) = \frac{1}{2}[f(x_0+) + f(x_0-)]$. In the example, this is the case at $x_0 = 2.5$.

Theorem (Fourier Series) *Let $f(x)$ be piecewise continuous with piecewise continuous first derivative.*

a) Let $f(0) = f(\ell) = 0$. Then

$$\sum_{k=1}^{\infty} b_k \sin\left(\tfrac{k\pi x}{\ell}\right) = \begin{cases} f(x) & \textit{if } f \textit{ is continuous at } x \\ \frac{f(x+)+f(x-)}{2} & \textit{otherwise} \end{cases}$$

for all $0 \le x \le \ell$ if and only if $b_k = \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{k\pi x}{\ell}\right)dx$.

b) Let $f'(0) = f'(\ell) = 0$. Then

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\tfrac{k\pi x}{\ell}\right) = \begin{cases} f(x) & \textit{if } f \textit{ is continuous at } x \\ \frac{f(x+)+f(x-)}{2} & \textit{otherwise} \end{cases}$$

for all $0 \le x \le \ell$ if and only if $a_k = \frac{2}{\ell}\int_0^\ell f(x)\cos\left(\frac{k\pi x}{\ell}\right)dx$.

c) Let f be periodic of period 2ℓ. Then

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos\left(\tfrac{k\pi x}{\ell}\right) + b_k \sin\left(\tfrac{k\pi x}{\ell}\right)\right] = \begin{cases} f(x) & \textit{if } f \textit{ is continuous at } x \\ \frac{f(x+)+f(x-)}{2} & \textit{otherwise} \end{cases}$$

for all x if and only if

$$a_k = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\tfrac{k\pi x}{\ell}\right)dx \qquad \text{and} \qquad b_k = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\tfrac{k\pi x}{\ell}\right)dx$$

Remark. One consequence of the above theorem is that if you are told that, for example,

$$\sum_{k=1}^{\infty} \beta_k \sin\left(\tfrac{k\pi x}{\ell}\right) = 0 \qquad \text{for all } 0 \le x \le \ell$$

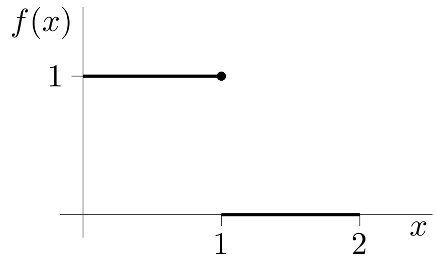
then, by part a with $f(x) = 0$, it is necessary that

$$\beta_k = \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\tfrac{k\pi x}{\ell}\right)dx = 0 \qquad \text{for all } k = 1, 2, \cdots$$

This is used repeatedly in using Fourier series to solve differential equations.

Example. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{if } 1 < x \le 2 \end{cases} =$$



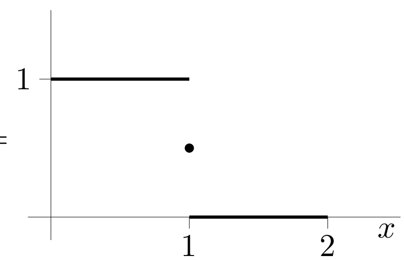
Apply part b of the Fourier series Theorem with this f and $\ell = 2$. Then

$$a_k = \tfrac{2}{\ell}\int_0^\ell f(x)\cos\left(\tfrac{k\pi x}{\ell}\right)dx = \int_0^1 \cos\left(\tfrac{k\pi x}{2}\right)dx = \begin{cases} 1 & \text{if } k=0 \\ \tfrac{2}{k\pi}\sin\left(\tfrac{k\pi x}{2}\right)\Big|_0^1 & \text{if } k \ge 1 \end{cases}$$

$$= \begin{cases} 1 & \text{if } k=0 \\ \tfrac{2}{k\pi}\sin\left(\tfrac{k\pi}{2}\right) & \text{if } k \ge 1 \end{cases} = \begin{cases} 1 & \text{if } k=0 \\ 0 & \text{if } k>1 \text{ and } k \text{ is even} \\ (-1)^{(k-1)/2}\tfrac{2}{k\pi} & \text{if } k \text{ is odd} \end{cases}$$

and by part b of the Theorem

$$1 + \sum_{p=0}^{\infty}(-1)^p \tfrac{2}{(2p+1)\pi}\cos\left(\tfrac{(2p+1)\pi x}{2}\right) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ \tfrac{1}{2} & \text{if } x = 1 \\ 0 & \text{if } 1 < x \le 2 \end{cases} =$$



We have summed over all odd k by setting $k = 2p+1$ and summing over $p = 0,\ 1,\ 2,\ \cdots$. There is a Java demo that shows the graphs of $1 + \sum_{p=0}^{N}(-1)^p \tfrac{2}{(2p+1)\pi}\cos\left(\tfrac{(2p+1)\pi x}{2}\right)$ for many values of N.

Usefulness of Fourier Series.

In this course, we use Fourier series as a tool for solving partial differential equations. This is just one of its many applications. To give you a preliminary taste of this application, we now briefly consider one typical partial differential equation problem. It arises in studying a vibrating string. Suppose that a vibrating string has its ends tied to nails at $x = y = 0$ and $x = \ell,\ y = 0$. Denote by $y(x,t)$ the amplitude of the string at position x and time t. If there are no external forces, no damping and the amplitude is small,

$$\begin{aligned}
&\tfrac{\partial^2 y}{\partial t^2}(x,t) = c^2\tfrac{\partial^2 y}{\partial x^2}(x,t) && \text{for all } 0\le x\le \ell,\ t\ge 0 && (1)\\
&y(0,t)=0 && \text{for all } t\ge 0 && (2)\\
&y(\ell,t)=0 && \text{for all } t\ge 0 && (3)\\
&y(x,0)=f(x) && \text{for all } 0\le x\le \ell && (4)\\
&\tfrac{\partial y}{\partial t}(x,0)=g(x) && \text{for all } 0\le x\le \ell && (5)
\end{aligned}$$

A detailed derivation of this system of equations is provided in another set of notes. The first equation is the Newton's law of motion appropriate for the current situation; the next two equations impose the requirements that the ends of the string are tied to nails; the final two equations specify the initial position and and speed of the string. We are assuming that the initial position and speed $f(x)$ and $g(x)$ are given functions. The unknown in the problem is the amplitude $y(x,t)$. For each fixed $t \ge 0$, $y(x,t)$ is a function of the one variable x. This function vanishes at $x = 0$ and $x = \ell$ and thus, by part a) of the Fourier series theorem has an expansion

$$y(x,t) = \sum_{k=1}^{\infty} b_k(t)\sin\left(\tfrac{k\pi x}{\ell}\right)$$

The solution $y(x,t)$ is completely determined by the, as yet unknown, coefficients $b_k(t)$. Furthermore these coefficients can be found by substituting $y(x,t) = \sum_{k=1}^{\infty} b_k(t)\sin\left(\frac{k\pi x}{\ell}\right)$ into the above five requirements on $y(x,t)$. First the PDE (1):

$$\begin{aligned}
0 = \tfrac{\partial^2 y}{\partial t^2}(x,t) - c^2\tfrac{\partial^2 y}{\partial x^2}(x,t) &= \sum_{k=1}^{\infty} b_k''(t)\sin\left(\tfrac{k\pi x}{\ell}\right) + \sum_{k=1}^{\infty}\tfrac{k^2\pi^2c^2}{\ell^2}b_k(t)\sin\left(\tfrac{k\pi x}{\ell}\right)\\
&= \sum_{k=1}^{\infty}\left[b_k''(t) + \tfrac{k^2\pi^2c^2}{\ell^2}b_k(t)\right]\sin\left(\tfrac{k\pi x}{\ell}\right)
\end{aligned}$$

This says that, for each fixed $t \ge 0$, the function 0, viewed as a function of x, has Fourier series expansion $\sum_{k=1}^{\infty}\left[b_k''(t) + \frac{k^2\pi^2c^2}{\ell^2}b_k(t)\right]\sin\left(\frac{k\pi x}{\ell}\right)$. Our Fourier series theorem then forces

$$b_k''(t) + \tfrac{k^2\pi^2c^2}{\ell^2}b_k(t) = 0 \qquad \text{for all } k, t \tag{1'}$$

Because $\sin 0 = \sin k\pi = 0$ for all integers k, conditions (2) and (3) are satisfied by $y(x,t) =$

$\sum_{k=1}^{\infty} b_k(t) \sin\left(\frac{k\pi x}{\ell}\right)$ regardless of what $b_k(t)$ are. Finally substituting into (4) and (5) gives

$$y(0,t) = \sum_{k=1}^{\infty} b_k(0) \sin\left(\tfrac{k\pi x}{\ell}\right) = f(x)$$

$$\tfrac{\partial y}{\partial t}(0,t) = \sum_{k=1}^{\infty} b_k'(0) \sin\left(\tfrac{k\pi x}{\ell}\right) = g(x)$$

By uniqueness of Fourier coefficients, once again,

$$b_k(0) = \tfrac{2}{\ell} \int_0^{\ell} f(x) \sin\left(\tfrac{k\pi x}{\ell}\right) dx \tag{4'}$$

$$b_k'(0) = \tfrac{2}{\ell} \int_0^{\ell} g(x) \sin\left(\tfrac{k\pi x}{\ell}\right) dx \tag{5'}$$

For each fixed k, equations (1'), (4') and (5') constitute one second order constant coefficient ordinary differential equation and two initial conditions for the unknown function $b_k(t)$. You already know how to solve constant coefficient ordinary differential equations. Denoting

$$\alpha_k = \tfrac{2}{\ell} \int_0^{\ell} f(x) \sin\left(\tfrac{k\pi x}{\ell}\right) dx \qquad\qquad \beta_k = \tfrac{2}{\ell} \int_0^{\ell} g(x) \sin\left(\tfrac{k\pi x}{\ell}\right) dx$$

the solution of (1'), (4') and (5') is

$$b_k(t) = \alpha_k \cos\left(\tfrac{k\pi ct}{\ell}\right) + \tfrac{\ell\beta_k}{k\pi c} \sin\left(\tfrac{k\pi ct}{\ell}\right)$$

and the solution of (1), (2), (3), (4), (5) is

$$y(x,t) = \sum_{k=1}^{\infty} \left[\alpha_k \cos\left(\tfrac{k\pi ct}{\ell}\right) + \tfrac{\ell\beta_k}{k\pi c} \sin\left(\tfrac{k\pi ct}{\ell}\right)\right] \sin\left(\tfrac{k\pi x}{\ell}\right)$$

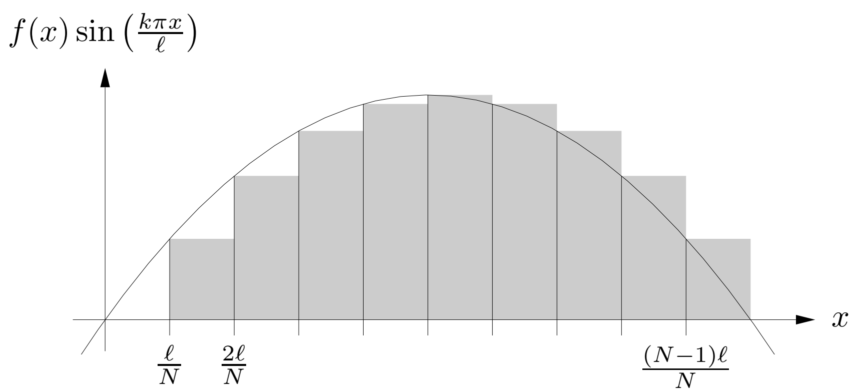
The interpretation of this formula is provided in another set of notes.

Validity of Fourier Series.

It is beyond the scope of this course to give a full justification for the Fourier series theorem. However, with just a little high school trigonometry, we can justify a formula that is useful in its own right and that also provides arbitrarily good approximations to the Fourier series expansions. For concreteness, we shall just consider Fourier sin series – that is part a) of the theorem. The other parts are similar. Imagine some application in which we have to measure some function $f(x)$ obeying $f(0) = f(\ell) = 0$. For example, $f(x)$ might be the amplitude at some fixed time of a vibrating string. Because we can only make finitely many measurements, we cannot determine $f(x)$ for all values of x. Suppose that we measure $f(x)$ for $x = \frac{\ell}{N},\ \frac{2\ell}{N},\ \cdots, \frac{(N-1)\ell}{N}$ where N is some fixed integer. Define

$$F_n^{(N)} = f\left(\tfrac{n\ell}{N}\right)$$

Because we do not know $f(x)$ for all x we cannot compute $b_k = \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{k\pi x}{\ell}\right)dx$. But we can get a Riemann sum approximation to it using only x's for which $f(x)$ is known. All we need to do is divide the domain of integration up into N intervals each of length ℓ/N. On the interval $\frac{n\ell}{N} \le x \le \frac{(n+1)\ell}{N}$ we approximate $f(x)\sin\left(\frac{k\pi x}{\ell}\right)$ by $f\left(\frac{n\ell}{N}\right)\sin\left(\frac{k\pi}{\ell}\frac{n\ell}{N}\right)$.



This gives

$$B_k^{(N)} = \tfrac{2}{\ell}\sum_{n=0}^{N-1}\tfrac{\ell}{N}f\left(\tfrac{n\ell}{N}\right)\sin\left(\tfrac{k\pi}{\ell}\tfrac{n\ell}{N}\right) = \tfrac{2}{N}\sum_{n=0}^{N-1}F_n^{(N)}\sin\left(\tfrac{nk\pi}{N}\right)$$

More generally, let $F_0,\ F_1,\ ,\cdots,\ F_N$ be $N+1$ numbers obeying $F_0 = F_N = 0$. Define

$$B_k = \tfrac{2}{N}\sum_{m=0}^{N-1}F_n\sin\left(\tfrac{mk\pi}{N}\right)$$

We shall show that the Fourier sin series representation for the F_n's

$$F_n = \sum_{k=0}^{N-1}B_k\sin\left(\tfrac{nk\pi}{N}\right) \tag{1}$$

is really true. To do so we just evaluate the right hand side by substituting in the definition of the B_k's.

$$\begin{aligned}\sum_{k=0}^{N-1} B_k \sin\left(\tfrac{nk\pi}{N}\right) &= \tfrac{2}{N}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1} F_m \sin\left(\tfrac{mk\pi}{N}\right)\sin\left(\tfrac{nk\pi}{N}\right)\\ &= \tfrac{2}{N}\sum_{m=0}^{N-1}\sum_{k=0}^{N-1} F_m \sin\left(\tfrac{mk\pi}{N}\right)\sin\left(\tfrac{nk\pi}{N}\right)\\ &= \sum_{m=0}^{N-1} F_m \left\{\tfrac{2}{N}\sum_{k=0}^{N-1}\sin\left(\tfrac{mk\pi}{N}\right)\sin\left(\tfrac{nk\pi}{N}\right)\right\}\end{aligned}$$

The validity of (1) is then an immediate consequence of part b) of

Lemma.

a)

$$\sum_{k=0}^{N-1}\cos(k\theta) = \begin{cases} N & \textit{if } \cos\theta = 1\\ \frac{1}{2}(1-\cos N\theta) + \frac{1}{2}\frac{\sin N\theta\sin\theta}{1-\cos\theta} & \textit{if } \cos\theta \ne 1\end{cases}$$

b) Let n and m be integers with $1 \le n, m \le N-1$. Then

$$\tfrac{2}{N}\sum_{k=0}^{N-1}\sin\left(\tfrac{mk\pi}{N}\right)\sin\left(\tfrac{nk\pi}{N}\right) = \begin{cases} 1 & \textit{if } n = m\\ 0 & \textit{if } n \ne m\end{cases}$$

Proof: a) If $\cos\theta = 1$ then θ is an integer multiple of 2π so that $\cos(k\theta) = 1$ for all integers k and $\sum_{k=0}^{N-1}\cos(k\theta) = \sum_{k=0}^{N-1} 1 = N$, as claimed.

If $\cos\theta \ne 0$, we must show

$$\sum_{k=0}^{N-1}\cos(k\theta) = \tfrac{1}{2}(1-\cos N\theta) + \tfrac{1}{2}\tfrac{\sin N\theta\sin\theta}{1-\cos\theta}$$

or equivalently (just moving the $\frac{1}{2}(1-\cos N\theta)$ to the left hand side)

$$\tfrac{1}{2} + \sum_{k=1}^{N-1}\cos(k\theta) + \tfrac{1}{2}\cos(N\theta) = \tfrac{1}{2}\tfrac{\sin N\theta\sin\theta}{1-\cos\theta}$$

or equivalently (just cross multiplying by $2(1-\cos\theta)$)

$$(1-\cos\theta) + \sum_{k=1}^{N-1} 2(1-\cos\theta)\cos(k\theta) + (1-\cos\theta)\cos(N\theta) = \sin N\theta\sin\theta$$

The left hand side is

$$
\begin{aligned}
&(1-\cos\theta)+\sum_{k=1}^{N-1}2(1-\cos\theta)\cos(k\theta)+(1-\cos\theta)\cos(N\theta)\\
&=1-\cos\theta+\sum_{k=1}^{N-1}2\cos(k\theta)-\sum_{k=1}^{N-1}2\cos\theta\cos(k\theta)+\cos(N\theta)-\cos\theta\cos(N\theta)\\
&=1-\cos\theta+\sum_{k=1}^{N-1}2\cos(k\theta)-\sum_{k=1}^{N-1}\cos\big((k+1)\theta\big)-\sum_{k=1}^{N-1}\cos\big((k-1)\theta\big)\\
&\qquad\qquad+\cos(N\theta)-\tfrac{1}{2}\cos\big((N-1)\theta\big)-\tfrac{1}{2}\cos\big((N+1)\theta\big)
\end{aligned}
$$

by the trig identity $\cos\theta\cos\phi=\frac{1}{2}\cos(\theta+\phi)+\frac{1}{2}\cos(\theta-\phi)$. Writing out the three sums on three separate rows

$$
\begin{aligned}
&(1-\cos\theta)+\sum_{k=1}^{N-1}2(1-\cos\theta)\cos(k\theta)+(1-\cos\theta)\cos(N\theta)\\
&=\ 1-\ \cos\theta\\
&\qquad+2\cos\theta+2\cos(2\theta)+\cdots+2\cos\big({\scriptstyle(N-2)\theta}\big)+2\cos\big({\scriptstyle(N-1)\theta}\big)\\
&\qquad\qquad\qquad-\ \cos(2\theta)-\cdots-\ \cos\big({\scriptstyle(N-2)\theta}\big)-\ \cos\big({\scriptstyle(N-1)\theta}\big)-\cos(N\theta)\\
&\quad-1-\ \cos\theta-\ \cos(2\theta)-\cdots-\ \cos\big({\scriptstyle(N-2)\theta}\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad-\tfrac{1}{2}\cos\big({\scriptstyle(N-1)\theta}\big)+\cos(N\theta)-\tfrac{1}{2}\cos\big({\scriptstyle(N+1)\theta}\big)\\
&=\tfrac{1}{2}\cos\big((N-1)\theta\big)-\tfrac{1}{2}\cos\big((N+1)\theta\big)\\
&=\sin(\theta)\sin(N\theta)
\end{aligned}
$$

which is exactly the desired identity.

b) Using $\sin\theta\sin\phi=\frac{1}{2}\cos(\theta-\phi)-\frac{1}{2}\cos(\theta+\phi)$

$$
2\sum_{k=0}^{N-1}\sin\big(\tfrac{mk\pi}{N}\big)\sin\big(\tfrac{nk\pi}{N}\big)=\sum_{k=0}^{N-1}\cos\big(\tfrac{(n-m)k\pi}{N}\big)-\sum_{k=0}^{N-1}\cos\big(\tfrac{(n+m)k\pi}{N}\big)
$$

We wish to apply the identity of part a twice, the first time with $\theta=\frac{(n-m)\pi}{N}$ and the second time with $\theta=\frac{(n+m)\pi}{N}$. Observe that with $1\le n,m\le N-1$ we have

$$
\begin{aligned}
-\pi<\tfrac{-(N-2)\pi}{N}\le\tfrac{(n-m)\pi}{N}\le\tfrac{(N-2)\pi}{N}<\pi &\implies\cos\big(\tfrac{(n-m)\pi}{N}\big)=1\quad\text{if and only if } n=m\\
0<\tfrac{2\pi}{N}\le\tfrac{(n+m)\pi}{N}\le\tfrac{2(N-1)\pi}{N}<2\pi &\implies\cos\big(\tfrac{(n-m)\pi}{N}\big)\ne 1
\end{aligned}
$$

and

$$\sin\left(N\tfrac{(n-m)\pi}{N}\right) = \sin\left(N\tfrac{(n+m)\pi}{N}\right) = 0$$

$$\cos\left(N\tfrac{(n-m)\pi}{N}\right) = \cos\left(N\tfrac{(n+m)\pi}{N}\right) = (-1)^{n-m}$$

since $(-1)^m = (-1)^{-m}$. Applying part a twice

$$\sum_{k=0}^{N-1} \cos\left(\tfrac{(n-m)k\pi}{N}\right) = \begin{cases} N & \text{if } n = m \\ \frac{1}{2} - \frac{1}{2}(-1)^{n-m} & \text{if } n \neq m \end{cases}$$

$$\sum_{k=0}^{N-1} \cos\left(\tfrac{(n+m)k\pi}{N}\right) = \tfrac{1}{2} - \tfrac{1}{2}(-1)^{n-m}$$

Subtracting gives

$$\sum_{k=0}^{N-1} \cos\left(\tfrac{(n-m)k\pi}{N}\right) - \sum_{k=0}^{N-1} \cos\left(\tfrac{(n+m)k\pi}{N}\right) = \begin{cases} N & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

which is the desired result. ■

We remark that trig identities are often cleaner looking and easier to derive using exponentials of complex numbers. This is indeed the case for Fourier series formulae, which are just big trig identities. See the notes on complex numbers and exponentials.