

# Simple ODE Solvers — Derivation

These notes provide derivations of some simple algorithms for generating, numerically, approximate solutions to the initial value problem

$$\begin{aligned}y'(t) &= f(t, y(t)) \\ y(t_0) &= y_0\end{aligned}$$

Here  $f(t, y)$  is a given function,  $t_0$  is a given initial time and  $y_0$  is a given initial value for  $y$ . The unknown in the problem is the function  $y(t)$ . We start with

## Euler's Method

Our goal is to determine (approximately) the unknown function  $y(t)$  for  $t \geq t_0$ . We are told explicitly the value of  $y(t_0)$ , namely  $y_0$ . Using the given differential equation, we can also determine exactly the instantaneous rate of change of  $y$  at time  $t_0$ .

$$y'(t_0) = f(t_0, y(t_0)) = f(t_0, y_0)$$

If the rate of change of  $y(t)$  were to remain  $f(t_0, y_0)$  for all time, then  $y(t)$  would be exactly  $y_0 + f(t_0, y_0)(t - t_0)$ . The rate of change of  $y(t)$  does not remain  $f(t_0, y_0)$  for all time, but it is reasonable to expect that it remains close to  $f(t_0, y_0)$  for  $t$  close to  $t_0$ . If this is the case, then the value of  $y(t)$  will remain close to  $y_0 + f(t_0, y_0)(t - t_0)$  for  $t$  close to  $t_0$ . So pick a small number  $h$  and define

$$\begin{aligned}t_1 &= t_0 + h \\ y_1 &= y_0 + f(t_0, y_0)(t_1 - t_0) = y_0 + f(t_0, y_0)h\end{aligned}$$

By the above argument

$$y(t_1) \approx y_1$$

Now we start over. We now know the approximate value of  $y$  at time  $t_1$ . If  $y(t_1)$  were exactly  $y_1$ , then the instantaneous rate of change of  $y$  at time  $t_1$  would be exactly  $f(t_1, y_1)$ . If this rate of change were to persist for all future time,  $y(t)$  would be exactly  $y_1 + f(t_1, y_1)(t - t_1)$ . As  $y(t_1)$  is only approximately  $y_1$  and as the rate of change of  $y(t)$  varies with  $t$ , the rate of change of  $y(t)$  is only approximately  $f(t_1, y_1)$  and only for  $t$  near  $t_1$ . So we approximate  $y(t)$  by  $y_1 + f(t_1, y_1)(t - t_1)$  for  $t$  bigger than, but close to,  $t_1$ . Defining

$$\begin{aligned}t_2 &= t_1 + h = t_0 + 2h \\ y_2 &= y_1 + f(t_1, y_1)(t_2 - t_1) = y_1 + f(t_1, y_1)h\end{aligned}$$

we have

$$y(t_2) \approx y_2$$

We just repeat this argument ad infinitum. Define, for  $n = 0, 1, 2, 3, \dots$

$$t_n = t_0 + nh$$

Suppose that, for some value of  $n$ , we have already computed an approximate value  $y_n$  for  $y(t_n)$ . Then the rate of change of  $y(t)$  for  $t$  close to  $t_n$  is  $f(t, y(t)) \approx f(t_n, y(t_n)) \approx f(t_n, y_n)$  and, again for  $t$  close to  $t_n$ ,  $y(t) \approx y_n + f(t_n, y_n)(t - t_n)$ . Hence

$$\boxed{y(t_{n+1}) \approx y_{n+1} = y_n + f(t_n, y_n)h} \quad (\text{Eul})$$

This algorithm is called **Euler's Method**. The parameter  $h$  is called the **step size**.

Here is a table applying a few steps of Euler's method to the initial value problem

$$\begin{aligned} y' &= -2t + y \\ y(0) &= 3 \end{aligned}$$

with step size  $h = 0.1$ . For this initial value problem

$$\begin{aligned} f(t, y) &= -2t + y \\ t_0 &= 0 \\ y_0 &= 3 \end{aligned}$$

Of course this initial value problem has been chosen for illustrative purposes only. The exact solution is, easily,  $y(t) = 2 + 2t + e^t$ .

$n$	$t_n$	$y_n$	$f(t_n, y_n) = -2t_n + y_n$	$y_{n+1} = y_n + f(t_n, y_n) * h$
0	0.0	3.000	$-2 * 0.0 + 3.000 = 3.000$	$3.000 + 3.000 * 0.1 = 3.300$
1	0.1	3.300	$-2 * 0.1 + 3.300 = 3.100$	$3.300 + 3.100 * 0.1 = 3.610$
2	0.2	3.610	$-2 * 0.2 + 3.610 = 3.210$	$3.610 + 3.210 * 0.1 = 3.931$
3	0.3	3.931	$-2 * 0.3 + 3.931 = 3.331$	$3.931 + 3.331 * 0.1 = 4.264$
4	0.4	4.264	$-2 * 0.4 + 4.264 = 3.464$	$4.264 + 3.464 * 0.1 = 4.611$
5	0.5	4.611		

## The Improved Euler's Method

Euler's method is one algorithm which generates approximate solutions to the initial value problem

$$\begin{aligned}y'(t) &= f(t, y(t)) \\ y(t_0) &= y_0\end{aligned}$$

In applications,  $f(t, y)$  is a given function and  $t_0$  and  $y_0$  are given numbers. The function  $y(t)$  is unknown. Denote by  $\varphi(t)$  the exact solution for this initial value problem. In other words  $\varphi(t)$  is the function that obeys

$$\begin{aligned}\varphi'(t) &= f(t, \varphi(t)) \\ \varphi(t_0) &= y_0\end{aligned}$$

exactly.

Fix a step size  $h$  and define  $t_n = t_0 + nh$ . We now derive another algorithm that generates approximate values for  $\varphi$  at the sequence of equally spaced time values  $t_0, t_1, t_2, \dots$ . We shall denote the approximate values  $y_n$  with

$$y_n \approx \varphi(t_n)$$

By the fundamental theorem of calculus and the differential equation, the exact solution obeys

$$\begin{aligned}\varphi(t_{n+1}) &= \varphi(t_n) + \int_{t_n}^{t_{n+1}} \varphi'(t) dt \\ &= \varphi(t_n) + \int_{t_n}^{t_{n+1}} f(t, \varphi(t)) dt\end{aligned}$$

Fix any  $n$  and suppose that we have already found  $y_0, y_1, \dots, y_n$ . Our algorithm for computing  $y_{n+1}$  will be of the form

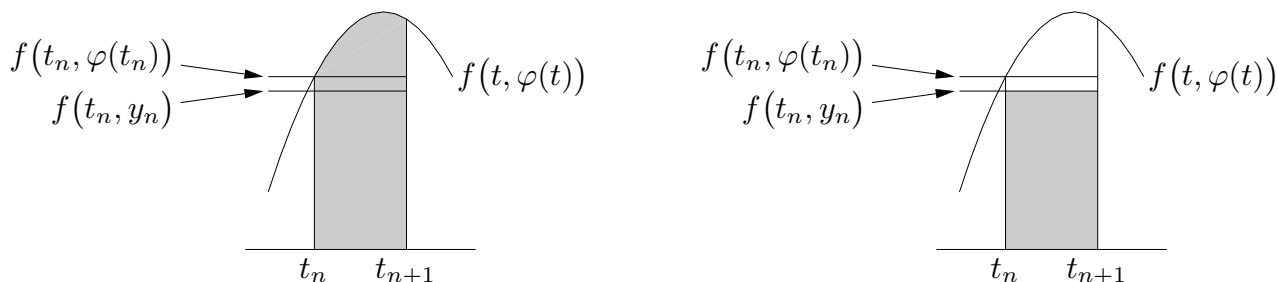
$$y_{n+1} = y_n + \text{approximate value for } \int_{t_n}^{t_{n+1}} f(t, \varphi(t)) dt$$

In fact Euler's method is of precisely this form. In Euler's method, we approximate  $f(t, \varphi(t))$  for  $t_n \leq t \leq t_{n+1}$  by the constant  $f(t_n, y_n)$ . Thus

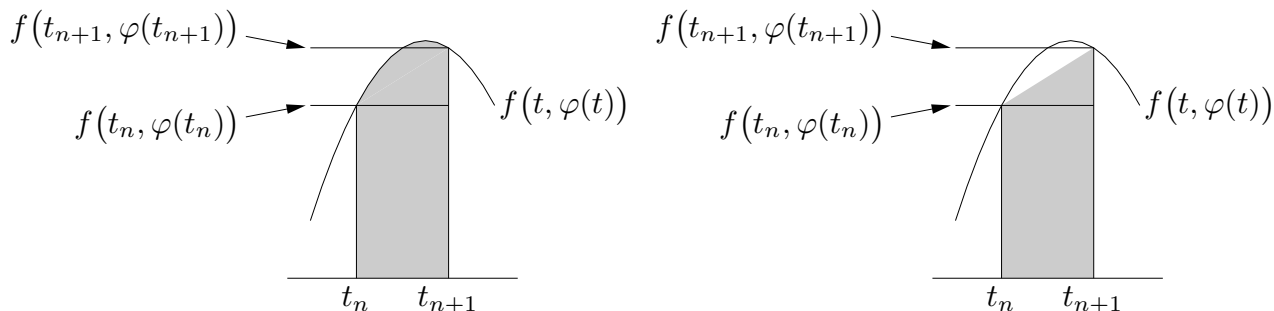
$$\text{Euler's approximate value for } \int_{t_n}^{t_{n+1}} f(t, \varphi(t)) dt = \int_{t_n}^{t_{n+1}} f(t_n, y_n) dt = f(t_n, y_n)h$$

The area of the complicated region  $0 \leq y \leq f(t, \varphi(t))$ ,  $t_n \leq t \leq t_{n+1}$  (represented by the shaded region under the parabola in the left half of the figure below) is approximated by

the area of the rectangle  $0 \leq y \leq f(t_n, y_n)$ ,  $t_n \leq t \leq t_{n+1}$  (the shaded rectangle in the right half of the figure below).



Our second algorithm, the improved Euler's method, gets a better approximation by attempting to approximate by the trapezoid on the right below rather than the rectangle on the right above. The exact area of this trapezoid is the length  $h$  of the base multiplied



by the average,  $\frac{1}{2}[f(t_n, \varphi(t_n)) + f(t_{n+1}, \varphi(t_{n+1}))]$ , of the heights of the two sides. Of course we do not know  $\varphi(t_n)$  or  $\varphi(t_{n+1})$  exactly. Recall that we have already found  $y_0, \dots, y_n$  and are in the process of finding  $y_{n+1}$ . So we already have an approximation for  $\varphi(t_n)$ , namely  $y_n$ , but not for  $\varphi(t_{n+1})$ . Improved Euler uses

$$\varphi(t_{n+1}) \approx \varphi(t_n) + \varphi'(t_n)h \approx y_n + f(t_n, y_n)h$$

in approximating  $\frac{1}{2}[f(t_n, \varphi(t_n)) + f(t_{n+1}, \varphi(t_{n+1}))]$ . Altogether

$$\begin{aligned} \text{Improved Euler's approximate value for } \int_{t_n}^{t_{n+1}} f(t, \varphi(t)) dt \\ = \frac{1}{2} \left[ f(t_n, y_n) + f\left(t_{n+1}, y_n + f(t_n, y_n)h\right) \right] h \end{aligned}$$

so that the improved Euler's method algorithm is

$$\boxed{y(t_{n+1}) \approx y_{n+1} = y_n + \frac{1}{2} \left[ f(t_n, y_n) + f\left(t_{n+1}, y_n + f(t_n, y_n)h\right) \right] h} \quad (\text{ImpEul})$$

Here are the first two steps of the improved Euler's method applied to

$$y' = -2t + y \quad y(0) = 3$$

with  $h = 0.1$ . In each step we compute  $f(t_n, y_n)$ , followed by  $y_n + f(t_n, y_n)h$ , which we denote  $\tilde{y}_{n+1}$ , followed by  $f(t_{n+1}, \tilde{y}_{n+1})$ , followed by  $y_{n+1} = y_n + \frac{1}{2}[f(t_n, y_n) + f(t_{n+1}, \tilde{y}_{n+1})]h$ .

$$\begin{aligned}
 t_0 = 0 \quad y_0 = 3 &\implies f(t_0, y_0) = -2 * 0 + 3 = 3 \\
 &\implies \tilde{y}_1 = 3 + 3 * 0.1 = 3.3 \\
 &\implies f(t_1, \tilde{y}_1) = -2 * 0.1 + 3.3 = 3.1 \\
 &\implies y_1 = 3 + \frac{1}{2}[3 + 3.1] * 0.1 = 3.305 \\
 t_1 = 0.1 \quad y_1 = 3.305 &\implies f(t_1, y_1) = -2 * 0.1 + 3.305 = 3.105 \\
 &\implies \tilde{y}_2 = 3.305 + 3.105 * 0.1 = 3.6155 \\
 &\implies f(t_2, \tilde{y}_2) = -2 * 0.2 + 3.6155 = 3.2155 \\
 &\implies y_2 = 3.305 + \frac{1}{2}[3.105 + 3.2155] * 0.1 = 3.621025
 \end{aligned}$$

Here is a table which gives the first five steps.

$n$	$t_n$	$y_n$	$f(t_n, y_n)$	$\tilde{y}_{n+1}$	$f(t_{n+1}, \tilde{y}_{n+1})$	$y_{n+1}$
0	0.0	3.000	3.000	3.300	3.100	3.305
1	0.1	3.305	3.105	3.616	3.216	3.621
2	0.2	3.621	3.221	3.943	3.343	3.949
3	0.3	3.949	3.349	4.284	3.484	4.291
4	0.4	4.291	3.491	4.640	3.640	4.647
5	0.5	4.647				

## The Runge-Kutta Method

The Runge-Kutta algorithm is similar to the Euler and improved Euler methods in that it also uses, in the notation of the last section,

$$y_{n+1} = y_n + \text{approximate value for } \int_{t_n}^{t_{n+1}} f(t, \varphi(t)) dt$$

But rather than approximating  $\int_{t_n}^{t_{n+1}} f(t, \varphi(t)) dt$  by the area of a rectangle, as does Euler, or by the area of a trapezoid, as does improved Euler, it approximates by the area under a parabola. That is, it uses Simpson's rule. According to Simpson's rule (if you don't know Simpson's rule, just take my word for it)

$$\int_{t_n}^{t_n+h} f(t, \varphi(t)) dt \approx \frac{h}{6} \left[ f(t_n, \varphi(t_n)) + 4f\left(t_n + \frac{h}{2}, \varphi\left(t_n + \frac{h}{2}\right)\right) + f(t_n + h, \varphi(t_n + h)) \right]$$

As we don't know  $\varphi(t_n)$ ,  $\varphi(t_n + \frac{h}{2})$  or  $\varphi(t_n + h)$ , we have to approximate them as well. The Runge-Kutta algorithm, incorporating all these approximations, is

$k_{n,1} = f(t_n, y_n)$ $k_{n,2} = f(t_n + \frac{1}{2}h, y_n + \frac{h}{2}k_{n,1})$ $k_{n,3} = f(t_n + \frac{1}{2}h, y_n + \frac{h}{2}k_{n,2})$ $k_{n,4} = f(t_n + h, y_n + hk_{n,3})$ $y_{n+1} = y_n + \frac{h}{6} [k_{n,1} + 2k_{n,2} + 2k_{n,3} + k_{n,4}]$	(RK)
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Here are the first two steps of the Runge-Kutta algorithm applied to

$$y' = -2t + y \quad y(0) = 3$$

with  $h = 0.1$ .

$t_0 = 0$	$y_0 = 3$	
	$\implies$	$k_{0,1} = f(0, 3) = -2 * 0 + 3 = 3$
	$\implies$	$y_0 + \frac{h}{2}k_{0,1} = 3 + 0.05 * 3 = 3.15$
	$\implies$	$k_{0,2} = f(0.05, 3.15) = -2 * 0.05 + 3.15 = 3.05$
	$\implies$	$y_0 + \frac{h}{2}k_{0,2} = 3 + 0.05 * 3.05 = 3.1525$
	$\implies$	$k_{0,3} = f(0.05, 3.1525) = -2 * 0.05 + 3.1525 = 3.0525$
	$\implies$	$y_0 + hk_{0,3} = 3 + 0.1 * 3.0525 = 3.30525$
	$\implies$	$k_{0,4} = f(0.1, 3.30525) = -2 * 0.1 + 3.30525 = 3.10525$
	$\implies$	$y_1 = 3 + \frac{0.1}{6} [3 + 2 * 3.05 + 2 * 3.0525 + 3.10525] = 3.3051708$
$t_1 = 0.1$		$y_1 = 3.3051708$
	$\implies$	$k_{1,1} = f(0.1, 3.3051708) = -2 * 0.1 + 3.3051708 = 3.1051708$
	$\implies$	$y_1 + \frac{h}{2}k_{1,1} = 3.3051708 + 0.05 * 3.1051708 = 3.4604293$
	$\implies$	$k_{1,2} = f(0.15, 3.4604293) = -2 * 0.15 + 3.4604293 = 3.1604293$
	$\implies$	$y_1 + \frac{h}{2}k_{1,2} = 3.3051708 + 0.05 * 3.1604293 = 3.4631923$
	$\implies$	$k_{1,3} = f(0.15, 3.4631923) = -2 * 0.15 + 3.4631923 = 3.1631923$
	$\implies$	$y_1 + hk_{1,3} = 3.3051708 + 0.1 * 3.4631923 = 3.62149$
	$\implies$	$k_{1,4} = f(0.2, 3.62149) = -2 * 0.2 + 3.62149 = 3.22149$
	$\implies$	$y_2 = 3.3051708 + \frac{0.1}{6} [3.1051708 + 2 * 3.1604293 +$ $+ 2 * 3.1631923 + 3.22149] = 3.6214025$
$t_2 = 0.2$		$y_2 = 3.6214025$

and here is a table giving the first five steps. The intermediate data is only given to three decimal places even though the computation has been done to many more.

$n$	$t_n$	$y_n$	$k_{n1}$	$y_{n1}$	$k_{n2}$	$y_{n2}$	$k_{n3}$	$y_{n3}$	$k_{n4}$	$y_{n+1}$
0	0.0	3.000	3.000	3.150	3.050	3.153	3.053	3.305	3.105	3.305170833
1	0.1	3.305	3.105	3.460	3.160	3.463	3.163	3.621	3.221	3.621402571
2	0.2	3.621	3.221	3.782	3.282	3.786	3.286	3.950	3.350	3.949858497
3	0.3	3.950	3.350	4.117	3.417	4.121	3.421	4.292	3.492	4.291824240
4	0.4	4.292	3.492	4.466	3.566	4.470	3.570	4.649	3.649	4.648720639
5	0.5	4.648								

These notes have, hopefully, motivated the Euler, improved Euler and Runge-Kutta algorithms. So far we not attempted to see how efficient and how accurate the algorithms are. A first look at those questions is provided in the notes “Simple ODE Solvers – Error Behaviour”.