Commun. Math. Phys. (2004)
Digital Object Identifier (DOI) 10.1007/s00220-004-1038-2

Communications in
Mathematical Physics

Particle–Hole Ladders

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Received: 21 September 2002 / Accepted: 12 August 2003
Published online: ■ –

Abstract: A self contained analysis demonstrates that the sum of all particle-hole ladder contributions for a two dimensional, weakly coupled fermion gas with a strictly convex Fermi curve at temperature zero is bounded. This is used in our construction of two dimensional Fermi liquids.

Contents

⋆ Research supported in part by the Natural Sciences and Engineering Research Council of Canada and the Forschungsinstitut für Mathematik, ETH Zürich.

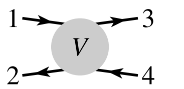
I. Introduction

This article is one of a series, starting with [FKTf1], that provides a construction of a class of two dimensional Fermi liquids. The concept of a Fermi liquid was introduced by L.D. Landau in [L1, L2, L3] and has become the generally accepted explanation for the success of the independent electron approximation. The phenomenological implications of Fermi liquid theory are derived from the structure of the single particle density n_{k} and Landau's quasiparticle interaction and forward scattering amplitude. The single particle density is constructed as a relatively straightforward limit of the one particle Green's function. The quasiparticle interaction and forward scattering amplitude, by contrast, are defined through two different limits of the transfer momentum flowing through the particle/hole channel of the two particle Green's function. This subtlety arises because the two particle Green's function is bounded but not continuous at transfer momentum zero.

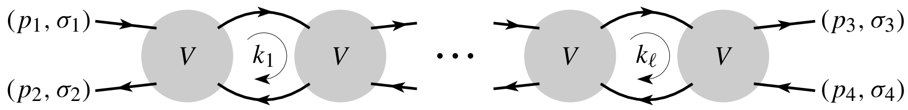
In [FKTr2] we showed that the leading contributions to the two particle Green's function are the, so-called, ladders. In this paper we extract sufficiently detailed information about particle/hole ladders to demonstrate the existence of the limits defining the quasiparticle interaction and forward scattering amplitude. In fact, in our construction of the full models, we are forced to this level of detail to formulate hypotheses on the sequence of effective interactions (see [FKTr2, Defs. IX.1 and IX.2]) that enable us to make an inductive construction. In other words, we would not be able to construct any of the Green's functions without the present fine analysis of the particle/hole channel. The control of the particle hole channel is in some ways the most subtle part of the argument in the construction of a Fermi liquid in that it results from a cancellation involving essentially all scales. Roughly speaking, it is like picking up the singularity of a Fourier series from its partial trigonometric sums.

Philip Anderson [A1, A2] suggested that, because of an instability arising from the particle/hole channel, two dimensional Fermi gases should exhibit behavior similar to a one dimensional Luttinger liquid, with the single particle density n_{k} having a vertical tangent at the Fermi surface rather than a jump discontinuity. In this series of papers, we show that this is not the case for the class of models considered here.

Formally, the amputated four–point Green's function, $G_4((p_1,\sigma_1),(p_2,\sigma_2),(p_3,\sigma_3),(p_4,\sigma_4))$ with incoming particles of momenta $p_1, p_4 \in \mathbb{R} \times \mathbb{R}^d$ and spins $\sigma_1, \sigma_4 \in \{\uparrow, \downarrow\}$ and outgoing particles of momenta p_2, p_3 and spins σ_2, σ_3, can be written as a sum of values of Feynman diagrams with four external legs. The propagator of these diagrams is $C(k) = \frac{1}{\imath k_0 - e(\mathbf{k})}$, where $k = (k_0, \mathbf{k}) \in \mathbb{R} \times \mathbb{R}^d$ and the dispersion relation $e(\mathbf{k})$ (into which the chemical potential has been absorbed) characterizes the independent fermion approximation. The interaction of the model determines the diagram vertices, $V((k_1,\sigma_1),(k_2,\sigma_2),(k_3,\sigma_3),(k_4,\sigma_4))$, $k_1 + k_4 = k_2 + k_3$. Here, the incoming momenta are k_1, k_4 and the outgoing momenta are k_2, k_3.



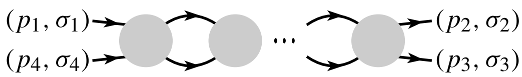
1. Ladders in Momentum Space. The most important contributions to this four–point function are ladders. The contribution of the particle–hole ladder with $\ell + 1$ rungs



is

$$\sum_{\substack{\tau_{i,1},\tau_{i,2}\in\{\uparrow,\downarrow\}\\ i=1,\cdots,\ell}} \int \frac{d^{d+1}k_1}{(2\pi)^{d+1}} \cdots \frac{d^{d+1}k_\ell}{(2\pi)^{d+1}}$$
$$\times\, V\big((p_1,\sigma_1),(p_2,\sigma_2),(p_1+k_1,\tau_{1,1}),(p_2+k_1,\tau_{1,2})\big)C(p_1+k_1)C(p_2+k_1)$$
$$\times\, V\big((p_1+k_1,\tau_{1,1}),(p_2+k_1,\tau_{1,2}),\cdots\big)\cdots V\big(\cdots,(p_1+k_\ell,\tau_{\ell,1}),(p_2+k_\ell,\tau_{\ell,2})\big)$$
$$\times\, C(p_1+k_\ell)C(p_2+k_\ell)V\big((p_1+k_\ell,\tau_{\ell,1}),(p_2+k_\ell,\tau_{\ell,2}),(p_3,\sigma_3),(p_4,\sigma_4)\big).$$

The contribution of the particle–particle ladder with $\ell + 1$ rungs



is

$$\sum_{\substack{\tau_{i,1},\tau_{i,2}\in\{\uparrow,\downarrow\}\\ i=1,\cdots,\ell}} \int \frac{d^{d+1}k_i}{(2\pi)^{d+1}} \cdots \frac{d^{d+1}k_\ell}{(2\pi)^{d+1}}$$
$$\times\, V\big((p_1,\sigma_1),(p_1+k_1,\tau_{1,1}),(p_4-k_1,\tau_{1,2}),(p_4,\sigma_4)\big)C(p_1+k_1)C(p_4-k_1)$$
$$\times\, V\big((p_1+k_1,\tau_{1,1}),\cdots,(p_4-k_1,\tau_{1,2})\big)\cdots V\big(\cdots,(p_1+k_\ell,\tau_{\ell,1}),(p_4-k_\ell,\tau_{\ell,2}),\cdots\big)$$
$$\times\, C(p_1+k_\ell)C(p_4-k_\ell)V\big((p_1+k_\ell,\tau_{\ell,1}),(p_2,\sigma_2),(p_3,\sigma_3),(p_4-k_\ell,\tau_{\ell,2})\big).$$

Ladders with two rungs are called bubbles. The values of the bubbles with dispersion relation $e(\mathbf{k}) = \frac{|\mathbf{k}|^2}{2m} - \mu$ and interaction $V\big((p_1,\sigma_1),(p_2,\sigma_2),(p_3,\sigma_3),(p_4,\sigma_4)\big) = \lambda\big(\delta_{\sigma_1,\sigma_2}\delta_{\sigma_3,\sigma_4} - \delta_{\sigma_1,\sigma_3}\delta_{\sigma_2,\sigma_4}\big)$ are well–known for $d = 2, 3$ [FHN]. The particle–particle bubble has a logarithmic singularity [FKST, Prop. II.1b] at transfer momentum $p_1 + p_4 = 0$ which is responsible for the formation of Cooper pairs and the onset of superconductivity. This singularity persists in models having dispersion relations that are symmetric about the origin, i.e. $e(\mathbf{k}) = e(-\mathbf{k})$. On the other hand, if $e(\mathbf{k})$ is strongly asymmetric in the sense of Definition I.10 of [FKTf1] then the particle–particle bubble remains continuous and, in particular, bounded [FKLT1, p. 297].

For the particle–hole bubble with $d = 2$ and $e(\mathbf{k}) = \frac{|\mathbf{k}|^2}{2m} - \mu$,

$$\int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3}\, C(k+p_1)\, C(k+p_2) = \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} \frac{1}{i(k_0+t_0/2)-e(\mathbf{k}+\mathbf{t}/2)} \frac{1}{i(k_0-t_0/2)-e(\mathbf{k}-\mathbf{t}/2)}$$

$$= \begin{cases} -\frac{m}{2\pi} + \frac{m}{2\pi|\mathbf{t}|^2} \operatorname{Re}\sqrt{|\mathbf{t}|^2(|\mathbf{t}|^2-4k_F^2)-4m^2t_0^2-4\imath m t_0|\mathbf{t}|^2} & \text{if } t_0,\ |\mathbf{t}| \neq 0 \text{ or } |\mathbf{t}| \geq 2k_F \\ -\frac{m}{2\pi} & \text{if } t_0 = 0 \text{ and } 0 < |\mathbf{t}| \leq 2k_F \\ 0 & \text{if } t_0 \neq 0 \text{ and } \mathbf{t} = 0 \end{cases},$$

where $t = p_1 - p_2$ is the transfer momentum, $k_F = \sqrt{2m\mu}$ is the radius of the Fermi surface and $\sqrt{\ }$ is the square root with nonnegative real part and cut along the negative real axis. See, for example, [FHN, (2.22) or FKST, Prop. II.1a]. This is C^∞ on $\{\, t \in \mathbb{R} \times \mathbb{R}^2 \mid t_0 \neq 0 \text{ or } |\mathbf{t}| > 2k_F \,\}$, is Hölder continuous of degree 1 in a neighbourhood of any t with $t_0 = 0,\ 0 < |\mathbf{t}| < 2k_F$ and is Hölder continuous of degree $\frac{1}{2}$ in a neighbourhood of any t with $t_0 = 0,\ |\mathbf{t}| = 2k_F$, but cannot be continuously extended to $t = 0$. However its restriction to $t_0 = 0$ does have a C^∞ extension at the point $\mathbf{t} = 0$. The discontinuity at $t = 0$ persists for general, even strongly asymmetric, $e(\mathbf{k})$. For this reason, bounds on particle–hole ladders in position space are not straightforward.

That the restriction of the particle–hole bubble to $t_0 = 0$ does have a C^∞ extension for a large class of smooth dispersion relations may be seen by the following argument, which was shown to us by Manfred Salmhofer [S]. A generalization of this argument is used in Proposition III.27.

Lemma I.1. *Choose a "scale parameter" $M > 1$ and a function $\nu \in C_0^\infty\big(\big[\frac{1}{M}, 2M\big]\big)$ that takes values in $[0,1]$, is identically 1 on $\big[\frac{2}{M}, M\big]$, is monotone on $\big[\frac{1}{M}, \frac{2}{M}\big]$ and $[M, 2M]$, and obeys*

$$\sum_{j=0}^{\infty} \nu\big(M^{2j}x\big) = 1 \tag{I.1}$$

for $0 < x < 1$. Set $\nu_0^{[0,j]}(k_0) = \sum_{\ell=0}^{j} \nu(M^{2\ell}k_0^2)$ and let $u(k,\mathbf{t})$ be a bounded C^∞ function with compact support in $\mathbf{k}$ and bounded derivatives. Let $e(\mathbf{k})$ be a C^∞ function that obeys $\lim_{|\mathbf{k}|\to\infty} e(\mathbf{k}) = +\infty$. Assume that the gradient of $e(\mathbf{k})$ does not vanish on the Fermi surface $F = \{\, \mathbf{k} \in \mathbb{R}^d \mid e(\mathbf{k}) = 0 \,\}$. Then

$$B(\mathbf{t}) = \lim_{j\to\infty} \int dk\, \frac{\nu_0^{[0,j]}(k_0)u(k,\mathbf{t})}{[ik_0 - e(\mathbf{k})][ik_0 - e(\mathbf{k}+\mathbf{t})]}$$

is C^∞ for $\mathbf{t}$ in a neighbourhood of 0.

Proof. Write

$$B_j(\mathbf{t}) = \int dk\, \frac{\nu_0^{[0,j]}(k_0)u(k,\mathbf{t})}{[ik_0-e(\mathbf{k})][ik_0-e(\mathbf{k}+\mathbf{t})]} = \int dk\, \frac{\nu_0^{[0,j]}(k_0)u(k,\mathbf{t})}{e(\mathbf{k})-e(\mathbf{k}+\mathbf{t})}\Big[\frac{1}{ik_0-e(\mathbf{k})} - \frac{1}{ik_0-e(\mathbf{k}+\mathbf{t})}\Big]$$

$$= \int dk\, \frac{\nu_0^{[0,j]}(k_0)u(k,\mathbf{t})}{e(\mathbf{k})-e(\mathbf{k}+\mathbf{t})} \int_0^1 ds\, \frac{d}{ds}\frac{1}{ik_0-E(\mathbf{k},\mathbf{t},s)} = \int dk \int_0^1 ds\, \frac{\nu_0^{[0,j]}(k_0)u(k,\mathbf{t})}{[ik_0-E(\mathbf{k},\mathbf{t},s)]^2},$$

where

$$E(\mathbf{k},\mathbf{t},s) = se(\mathbf{k}) + (1-s)e(\mathbf{k}+\mathbf{t}).$$

Make, for each fixed s and k_0, the change of variables from $\mathbf{k}$ to E and $d-1$ variables θ on F. Denote by $J(E,\mathbf{t},\theta,s)$ the Jacobian of this change of variables and set

$$f(k_0,E,\theta,\mathbf{t},s)=u\big((k_0,\mathbf{k}(E,\theta,\mathbf{t},s)),\mathbf{t}\big)J(E,\theta,\mathbf{t},s).$$

Because u has compact support in $\mathbf{k}$, f vanishes unless $|E|\le\mathcal{E}$, for some finite $\mathcal{E}$. Thus

$$B_j(\mathbf{t})=\int_0^1 ds\int d\theta\int dk_0\int_{-\mathcal{E}}^{\mathcal{E}} dE\ \tfrac{\nu_0^{[0,j]}(k_0)f(k_0,E,\theta,\mathbf{t},s)}{[ik_0-E]^2}.$$

Set

$$B_j'(\mathbf{t})=\int_0^1 ds\int d\theta\int dk_0\int_{-\mathcal{E}}^{\mathcal{E}} dE\ \tfrac{\nu_0^{[0,j]}(k_0)f(k_0,0,\theta,\mathbf{t},s)}{[ik_0-E]^2}.$$

Since

$$\Big|\partial_{\mathbf{t}}^{\alpha}\Big[\tfrac{\nu_0^{[0,j]}(k_0)f(k_0,E,\theta,\mathbf{t},s)}{[ik_0-E]^2}-\tfrac{\nu_0^{[0,j]}(k_0)f(k_0,0,\theta,\mathbf{t},s)}{[ik_0-E]^2}\Big]\Big|\le \mathrm{const}_\alpha\,\tfrac{|E|}{k_0^2+E^2}$$

is integrable on $\mathbb{R}\times[-\mathcal{E},\mathcal{E}]$, $\lim\limits_{j\to\infty} B_j(\mathbf{t})-B_j'(\mathbf{t})$ exists and is C^∞ by the Lebesgue dominated convergence theorem. So it suffices to consider

$$B_j'(\mathbf{t})=-2\mathcal{E}\int_0^1 ds\int d\theta\int dk_0\ \tfrac{\nu_0^{[0,j]}(k_0)f(k_0,0,\theta,\mathbf{t},s)}{k_0^2+\mathcal{E}^2}.$$

Since

$$\Big|\partial_{\mathbf{t}}^{\alpha}\tfrac{\nu_0^{[0,j]}(k_0)f(k_0,0,\theta,\mathbf{t},s)}{k_0^2+\mathcal{E}^2}\Big|\le \mathrm{const}_\alpha\,\tfrac{1}{k_0^2+\mathcal{E}^2}$$

is integrable on $\mathbb{R}$, $\lim\limits_{j\to\infty} B_j'(\mathbf{t})$ exists and is C^∞ by the Lebesgue dominated convergence theorem. □

2. Scales and Sectors. In this paper, we derive position space bounds for generalized particle–hole ladders in two space dimensions as they arise in a multiscale analysis. The main result is Theorem I.20, which is used in [FKTf2], under the name Theorem D.2, to help construct a Fermi liquid. We assume that the dispersion relation $e(\mathbf{k})$ is C^{r_e+3} for some $r_e\ge 6$, that its gradient does not vanish on the Fermi curve $F=\big\{\,\mathbf{k}\in\mathbb{R}^2\ \big|\ e(\mathbf{k})=0\,\big\}$ and that the Fermi curve is nonempty, connected, compact and strictly convex (meaning that its curvature does not vanish anywhere). We also fix the number $r_0\ge 6$ of derivatives in k_0 that we wish to control.

We introduce scales as in [FKTf1, Def. I.2] and [FKTo2, §VIII]:

Definition I.2. *i) For $j\ge 1$, the j^{th} scale function on $\mathbb{R}\times\mathbb{R}^2$ is defined as*

$$\nu^{(j)}(k)=\nu\Big(M^{2j}(k_0^2+e(\mathbf{k})^2)\Big),$$

where ν is the function of (I.1). It may be constructed by choosing a function $\varphi\in C_0^\infty\big((-2,2)\big)$ that is identically one on $[-1,1]$ and setting $\nu(x)=\varphi(x/M)-\varphi(Mx)$ for $x>0$ and zero otherwise. By construction, $\nu^{(j)}$ is identically one on

$$\Big\{\,k=(k_0,\mathbf{k})\in\mathbb{R}\times\mathbb{R}^2\ \Big|\ \sqrt{\tfrac{2}{M}}\,\tfrac{1}{M^j}\le|ik_0-e(\mathbf{k})|\le\sqrt{M}\,\tfrac{1}{M^j}\,\Big\}.$$

The support of $\nu^{(j)}$ *is called the* $j^{\rm th}$ *shell. By construction, it is contained in*

$$\Big\{ k \in \mathbb{R} \times \mathbb{R}^2 \ \Big|\ \tfrac{1}{\sqrt{M}} \tfrac{1}{M^j} \le |ik_0 - e(\mathbf{k})| \le \sqrt{2M} \tfrac{1}{M^j} \Big\}.$$

The momentum k *is said to be of scale* j *if* k *lies in the* $j^{\rm th}$ *shell.*

ii) For $j \ge 1$*, set*

$$\nu^{(\ge j)}(k) = \sum_{i \ge j} \nu^{(i)}(k)$$

for $|ik_0 - e(\mathbf{k})| > 0$ *and* $\nu^{(\ge j)}(k) = 1$ *for* $|ik_0 - e(\mathbf{k})| = 0$*. Equivalently,* $\nu^{(\ge j)}(k) = \varphi\big(M^{2j-1}(k_0^2 + e(\mathbf{k})^2)\big)$*. By construction,* $\nu^{(\ge j)}$ *is identically* 1 *on*

$$\Big\{ k \in \mathbb{R} \times \mathbb{R}^2 \ \Big|\ |ik_0 - e(\mathbf{k})| \le \sqrt{M} \tfrac{1}{M^j} \Big\}.$$

The support of $\nu^{(\ge j)}$ *is called the* $j^{\rm th}$ *neighbourhood of the Fermi surface. By construction, it is contained in*

$$\Big\{ k \in \mathbb{R} \times \mathbb{R}^2 \ \Big|\ |ik_0 - e(\mathbf{k})| \le \sqrt{2M} \tfrac{1}{M^j} \Big\}.$$

The support of $\varphi\big(M^{2j-2}(k_0^2 + e(\mathbf{k})^2)\big)$ *is called the* $j^{\rm th}$ *extended neighbourhood. It is contained in*

$$\Big\{ k \in \mathbb{R} \times \mathbb{R}^d \ \Big|\ |ik_0 - e(\mathbf{k})| \le \sqrt{2}M \tfrac{1}{M^j} \Big\}.$$

To estimate functions in position space and still make use of conservation of momentum, we use sectorization. See [FKTf1, Ex. A.1]. The following definition is also made in [FKTf2, §VI] and [FKTo3, §XII].

Definition I.3 (Sectors and sectorizations).

i) Let I *be an interval on the Fermi surface* F *and* $j \ge 1$*. Then*

$$s = \Big\{ k \text{ in the } j^{\rm th} \text{ neighbourhood} \ \Big|\ \pi_F(k) \in I \Big\}$$

is called a sector of length $|I|$ *at scale* j*. Here* $k \mapsto \pi_F(k)$ *is a projection on the Fermi surface. Two different sectors* s *and* s' *are called neighbours if* $s' \cap s \ne \emptyset$*.*

ii) A sectorization of length $\mathfrak{l}$ *at scale* j *is a set* Σ *of sectors of length* $\mathfrak{l}$ *at scale* j *that obeys*

- *the set* Σ *of sectors covers the Fermi surface*
- *each sector in* Σ *has precisely two neighbours in* Σ*, one to its left and one to its right*
- *if* $s,\ s' \in \Sigma$ *are neighbours then* $\frac{1}{16}\mathfrak{l} \le |s \cap s' \cap F| \le \frac{1}{8}\mathfrak{l}$*.*

Observe that there are at most $2\,\mathrm{length}(F)/\mathfrak{l}$ *sectors in* Σ*.*

In the renormalization group map of [FKTf1] and [FKTo3], we integrate over fields whose arguments (x, σ, s) lie in $\mathcal{B}^\updownarrow \times \Sigma$, where $\mathcal{B}^\updownarrow = (\mathbb{R} \times \mathbb{R}^2) \times \{\uparrow, \downarrow\}$ is the set of all "(positions, spins)". On the other hand, we are interested in the dependence of the two and four–point functions on external momenta. To distinguish between the set of all positions and the set of all momenta, we denote by $\mathbb{M} = \mathbb{R} \times \mathbb{R}^2$, the set of all possible momenta. The set of all possible positions shall still be denoted $\mathbb{R} \times \mathbb{R}^2$. Thus the external variables (k, σ) lie in $\check{\mathcal{B}}^\updownarrow = \mathbb{M} \times \{\uparrow, \downarrow\}$. In total, legs of four–legged kernels may lie in the disjoint union $\mathfrak{Y}^\updownarrow_\Sigma = \check{\mathcal{B}}^\updownarrow \cup (\mathcal{B}^\updownarrow \times \Sigma)$ for some sectorization Σ. The four–legged kernels over $\mathfrak{Y}^\updownarrow_\Sigma$ that we consider here arise in [FKTf2, §VII] as particle–hole reductions (as in Def. VII.4

of [FKTf2]) of four–legged kernels on $\mathfrak{X}_\Sigma = \check{\mathcal{B}} \cup (\mathcal{B} \times \Sigma)$ where $\check{\mathcal{B}} = \check{\mathcal{B}}^\updownarrow \times \{0,1\}$ and $\mathcal{B} = \mathcal{B}^\updownarrow \times \{0,1\}$ and $\{0,1\}$ is the set of creation/annihilation indices. Particle–hole reduction sets the creation/annihilation index to zero for legs number one and four and to one for legs number two and three. To simplify the notation in this paper, we shall eliminate the spin variables so that the legs lie in

$$\mathfrak{Y}_\Sigma = \mathbb{M} \cup \big((\mathbb{R} \times \mathbb{R}^2) \times \Sigma\big).$$

Sometimes a four–legged kernel will have different sectorizations Σ, Σ' on its two left hand legs and on its two right hand legs. Therefore, we introduce the space

$$\mathfrak{Y}^{(4)}_{\Sigma,\Sigma'} = \mathfrak{Y}^2_\Sigma \times \mathfrak{Y}^2_{\Sigma'}.$$

Since $\mathfrak{Y}_\Sigma$ is the disjoint union of $\mathbb{M}$ and $(\mathbb{R} \times \mathbb{R}^2) \times \Sigma$, the space $\mathfrak{Y}^{(4)}_{\Sigma,\Sigma'}$ is the disjoint union

$$\mathfrak{Y}^{(4)}_{\Sigma,\Sigma'} = \dot{\bigcup_{i_1,i_2,i_3,i_4 \in \{0,1\}}} \mathfrak{Y}_{i_1,\Sigma} \times \mathfrak{Y}_{i_2,\Sigma} \times \mathfrak{Y}_{i_3,\Sigma'} \times \mathfrak{Y}_{i_4,\Sigma'}, \tag{I.2}$$

where $\mathfrak{Y}_{0,\Sigma} = \mathbb{M}$ and $\mathfrak{Y}_{1,\Sigma} = (\mathbb{R} \times \mathbb{R}^2) \times \Sigma$. If f is a function on $\mathfrak{Y}^{(4)}_{\Sigma,\Sigma'}$, we denote by $f\big|_{(i_1,\cdots,i_4)}$ its restriction to $\mathfrak{Y}_{i_1,\Sigma} \times \mathfrak{Y}_{i_2,\Sigma} \times \mathfrak{Y}_{i_3,\Sigma'} \times \mathfrak{Y}_{i_4,\Sigma'}$ under the identification (I.2).

Definition I.4 (Translation invariance). *Let Σ and Σ' be sectorizations.*

i) Let $y \in \mathfrak{Y}_\Sigma$ and $t \in \mathbb{R} \times \mathbb{R}^2$. We set

$$T_t y = \begin{cases} k & \text{if } y = k \in \mathbb{M} \\ (x+t, s) & \text{if } y = (x,s) \in \big(\mathbb{R} \times \mathbb{R}^2\big) \times \Sigma \end{cases}.$$

ii) Let $i_1, \cdots, i_4 \in \{0,1\}$. A function f on $\mathfrak{Y}_{i_1,\Sigma} \times \mathfrak{Y}_{i_2,\Sigma} \times \mathfrak{Y}_{i_3,\Sigma'} \times \mathfrak{Y}_{i_4,\Sigma'}$ is called translation invariant, if for all $t \in \mathbb{R} \times \mathbb{R}^2$,

$$f(T_t y_1, \cdots, T_t y_4) = \Big(\prod_{\substack{1\le\mu\le4 \\ i_\mu=0}} e^{\imath(-1)^{b_\mu} < y_\mu, t>_-} \Big) f(y_1, \cdots, y_4),$$

where

$$b_\mu = \begin{cases} 0 & \text{if } \mu = 1,4 \\ 1 & \text{if } \mu = 2,3 \end{cases} \tag{I.3}$$

and $< k, x >_- = -k_0x_0 + \mathbf{k}_1\mathbf{x}_1 + \mathbf{k}_2\mathbf{x}_2$. This choice of b_μ reflects our image of f as a particle–hole kernel, with first and fourth, resp. second and third, arguments being creation, resp. annihilation, arguments.

iii) A function f on $\mathfrak{Y}^{(4)}_{\Sigma,\Sigma'}$ is translation invariant if $f\big|_{(i_1,\cdots,i_4)}$ is translation invariant for all $i_1, \cdots, i_4 \in \{0,1\}$. A function f on $\big(\mathfrak{Y}^\updownarrow_\Sigma\big)^4$ is translation invariant if $f\big((\cdot,\sigma_1),(\cdot,\sigma_2),(\cdot,\sigma_3),(\cdot,\sigma_4)\big)$ is translation invariant for all $\sigma_1, \cdots, \sigma_4 \in \{\uparrow, \downarrow\}$.

Definition I.5 (Fourier transform). *Let Σ, Σ' be sectorizations. Set $\mathfrak{Y}_{2,\Sigma} = \mathbb{M} \times \Sigma$.*

i) Let $i_1, \cdots, i_4 \in \{0, 1, 2\}$ and $1 \le \mu \le 4$ such that $i_\mu = 1$. The Fourier transform of a function f on $\mathfrak{Y}_{i_1,\Sigma} \times \mathfrak{Y}_{i_2,\Sigma} \times \mathfrak{Y}_{i_3,\Sigma'} \times \mathfrak{Y}_{i_4,\Sigma'}$ with respect to the $\mu^{\rm th}$ variable is the function on $\mathfrak{Y}_{i'_1,\Sigma} \times \mathfrak{Y}_{i'_2,\Sigma} \times \mathfrak{Y}_{i'_3,\Sigma'} \times \mathfrak{Y}_{i'_4,\Sigma'}$ with

$$i'_\nu = \begin{cases} i_\nu & \text{if } \nu \ne \mu \\ 2 & \text{if } \nu = \mu \end{cases}$$

defined by

$$(\Phi_\mu f)(y_1,\cdots,y_{\mu-1},(k,s),y_{\mu+1},\cdots,y_4) = \int e^{\imath(-1)^{b_\mu} <k,x>_-} f(y_1,\cdots,y_{\mu-1},(x,s),y_{\mu+1},\cdots,y_4)\, d^3x.$$

ii) Let $i_1, \cdots, i_4 \in \{0, 1\}$ with $i_\mu = 1$ for at least one $1 \le \mu \le 4$. The total Fourier transform $\check f$ of a translation invariant function f on $\mathfrak{Y}_{i_1,\Sigma} \times \mathfrak{Y}_{i_2,\Sigma} \times \mathfrak{Y}_{i_3,\Sigma'} \times \mathfrak{Y}_{i_4,\Sigma'}$ is defined by

$$\check f(y_1, y_2, y_3, y_4)\,(2\pi)^3 \delta(k_1 - k_2 - k_3 + k_4) = \Big(\prod_{\substack{1\le\mu\le4 \\ i_\mu=1}} \Phi_\mu\ f\Big)(y_1, y_2, y_3, y_4),$$

where $y_\mu = k_\mu$ when $i_\mu = 0$ and $y_\mu = (k_\mu, s_\mu)$ when $i_\mu = 1$. $\check f$ is defined on the set of all $(y_1, y_2, y_3, y_4) \in \mathfrak{Y}_{2i_1,\Sigma} \times \mathfrak{Y}_{2i_2,\Sigma} \times \mathfrak{Y}_{2i_3,\Sigma'} \times \mathfrak{Y}_{2i_4,\Sigma'}$ for which $k_1 - k_2 = k_3 - k_4$.

Definition I.6 (Sectorized functions). *Let Σ and Σ' be sectorizations.*

i) Let $i_1, \cdots, i_4 \in \{0, 1\}$. A translation invariant function f on $\mathfrak{Y}_{i_1,\Sigma} \times \mathfrak{Y}_{i_2,\Sigma} \times \mathfrak{Y}_{i_3,\Sigma'} \times \mathfrak{Y}_{i_4,\Sigma'}$ is sectorized if, for each $1 \le \mu \le 4$ with $i_\mu = 1$, the total Fourier transform $\check f(y_1,\cdots,y_{\mu-1},(k,s),y_{\mu+1},\cdots,y_4)$ vanishes unless k is in the $j^{\rm th}$ extended neighbourhood and $\pi_F(k) \in s$.
ii) A translation invariant function f on $\mathfrak{Y}^{(4)}_{\Sigma,\Sigma'}$ is sectorized if $f\big|_{(i_1,\cdots,i_4)}$ is sectorized for all $i_1, \cdots, i_4 \in \{0, 1\}$. A translation invariant function f on $\big(\mathfrak{Y}^{\updownarrow}_\Sigma\big)^4$ is sectorized if $f((\cdot,\sigma_1),(\cdot,\sigma_2),(\cdot,\sigma_3),(\cdot,\sigma_4))$ is sectorized for all $\sigma_1, \cdots, \sigma_4 \in \{\uparrow, \downarrow\}$.

Remark I.7. If f is a function in the space $\check{\mathcal{F}}_{4,\Sigma}$ of Def. XIV.6 of [FKTf2] (or Def. XVI.7.iii of [FKTo3]), then its particle–hole reduction is a sectorized function on $\big(\mathfrak{Y}^{\updownarrow}_\Sigma\big)^4$.

3. Particle–Hole Ladders.

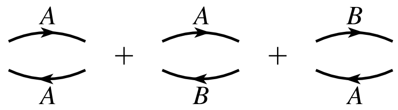
Definition I.8. *i) A (spin independent) propagator is a translation invariant function on $\big(\mathbb{R} \times \mathbb{R}^2\big)^2$. If $A(x, x')$ is a propagator, then its transpose is $A^t(x, x') = A(x', x)$.*
ii) A (spin independent) bubble propagator is a translation invariant function on $\big(\mathbb{R} \times \mathbb{R}^2\big)^4$. If A and B are propagators, we define the bubble propagator

$$A \otimes B(x_1, x_2, x_3, x_4) = A(x_1, x_3)B(x_2, x_4).$$

We set

$$\mathcal{C}(A, B) = (A + B) \otimes (A + B)^t - B \otimes B^t$$
$$= A \otimes A^t \quad + \quad A \otimes B^t \quad + \quad B \otimes A^t$$

=



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iii) Let Σ, Σ', Σ'' *be sectorizations,* P *be a bubble propagator and* F *be a function on* $\mathfrak{Y}_{i_1,\Sigma''} \times \mathfrak{Y}_{i_2,\Sigma''} \times (\mathbb{R} \times \mathbb{R}^2)^2$. *If* K *is a function on* $\mathfrak{Y}_\Sigma \times \mathfrak{Y}_\Sigma \times \mathfrak{Y}_{1,\Sigma'} \times \mathfrak{Y}_{1,\Sigma'}$, *we set*

$$(K \bullet P)(y_1,y_2;x_3,x_4) = \sum_{s_1',s_2' \in \Sigma'} \int dx_1' dx_2' \; K(y_1,y_2,(x_1',s_1'),(x_2',s_2')) \; P(x_1',x_2';x_3,x_4).$$

If K *is a function on* $\mathfrak{Y}_{1,\Sigma} \times \mathfrak{Y}_{1,\Sigma} \times \mathfrak{Y}_{i_3,\Sigma'} \times \mathfrak{Y}_{i_4,\Sigma'}$, *we set, when* i_1, i_2, i_3, i_4 *are not all 0,*

$$(F \bullet K)(y_1,y_2,y_3,y_4) = \sum_{s_1,s_2 \in \Sigma} \int dx_1 dx_2 \; F(y_1,y_2;x_1,x_2) K((x_1,s_1),(x_2,s_2),y_3,y_4),$$

and when $i_1, i_2, i_3, i_4 = 0$,

$$(F \bullet K)(k_1,k_2,k_3,k_4)(2\pi)^3 \delta(k_1-k_2-k_3+k_4)$$
$$= \sum_{s_1,s_2 \in \Sigma} \int dx_1 dx_2 \; F(k_1,k_2;x_1,x_2) K((x_1,s_1),(x_2,s_2),k_3,k_4).$$

Observe that $K \bullet P$ *is a function on* $\mathfrak{Y}_\Sigma^2 \times (\mathbb{R} \times \mathbb{R}^2)^2$ *and* $F \bullet K$ *is a function on* $\mathfrak{Y}^{(4)}_{\Sigma,\Sigma'}$. *If* K' *is a function on* $(\mathfrak{Y}^\updownarrow_\Sigma)^4$ *and* F' *is a function on* $(\mathfrak{Y}^\updownarrow_\Sigma)^2 \times (\mathcal{B}^\updownarrow)^2$ *we set*

$$(K' \bullet P)((\cdot,\sigma_1),(\cdot,\sigma_2),(\cdot,\sigma_3),(\cdot,\sigma_4)) = K'((\cdot,\sigma_1),(\cdot,\sigma_2),(\cdot,\sigma_3),(\cdot,\sigma_4)) \bullet P$$

and

$$(F' \bullet K')((\cdot,\sigma_1),(\cdot,\sigma_2),(\cdot,\sigma_3),(\cdot,\sigma_4)) = \sum_{\tau_1,\tau_2 \in \{\uparrow,\downarrow\}} F'((\cdot,\sigma_1),(\cdot,\sigma_2),(\cdot,\tau_1),(\cdot,\tau_2))$$
$$\bullet K'((\cdot,\tau_1),(\cdot,\tau_2),(\cdot,\sigma_3),(\cdot,\sigma_4)).$$

iv) Let $\ell \geq 1$. *Let, for* $1 \leq i \leq \ell+1$, $\Sigma^{(i)}$, $\Sigma'^{(i)}$ *be sectorizations and* K_i *a function on* $\mathfrak{Y}^{(4)}_{\Sigma^{(i)},\ \Sigma'^{(i)}}$. *Furthermore, let* $P_1, \cdots, P_\ell$ *be bubble propagators. The ladder with rungs* $K_1, \cdots, K_{\ell+1}$ *and bubble propagators* $P_1, \cdots, P_\ell$ *is defined to be*

$$K_1 \bullet P_1 \bullet K_2 \bullet P_2 \bullet \cdots \bullet K_\ell \bullet P_\ell \bullet K_{\ell+1}.$$

If Σ *is a sectorization and* $K_1', \cdots, K_{\ell+1}'$ *are functions on* $\left(\mathfrak{Y}^\updownarrow_\Sigma\right)^4$, *the ladder with rungs* $K_1', \cdots, K_{\ell+1}'$ *and bubble propagators* $P_1, \cdots, P_\ell$ *is defined to be*

$$K_1' \bullet P_1 \bullet K_2' \bullet P_2 \bullet \cdots \bullet K_\ell' \bullet P_\ell \bullet K_{\ell+1}'.$$

Remark I.9. We typically use $\mathcal{C}(A, B)$ with A being the part, $\nu^{(j)}(k)C(k)$, of the propagator, $C(k)$, having momentum in the j^{th} shell and B being the part, $\nu^{(\ge j+1)}(k)C(k)$, of the propagator having momentum in the $(j+1)^{\text{st}}$ neighbourhood. The bubble propagator $\mathcal{C}(A, B)$ always contains at least one "hard line" A and may or may not contain one "soft line" B. The latter are created by Wick ordering. See [FKTf1, §II, Subsect. 9].

Remark I.10. If $F_1,\ F_2$ are functions on $\big(\mathfrak{X}_\Sigma\big)^4$ and $A,\ B$ are propagators over $\mathcal{B}$ in the sense of Def. VII.1.i of [FKTf2], then the particle–hole reduction of $F_1 \bullet \mathcal{C}(A, B) \bullet F_2$ (with the $\mathcal{C}(A, B)$ of Def. VII.1.i of [FKTf2]) is equal to

$$-F_1^{\text{ph}} \bullet \mathcal{C}\big(A((\,\cdot\,1),(\,\cdot\,0)),\ B((\,\cdot\,1),(\,\cdot\,0))\big) \bullet F_2^{\text{ph}}$$

(with the $\mathcal{C}$ of Def. I.8) since $B((x,\sigma,0),(x',\sigma',1)) = -B((\,\cdot\,1),(\,\cdot\,0))^t((x,\sigma),(x',\sigma'))$.

4. Norms. In the momentum space variables, we take suprema of the function and its derivatives. In the position space variables, we will apply the L^1–L^∞ norm of Def. I.11, below, to the function and to the function multiplied by various coordinate differences.

Definition I.11. *Let f be a function on $\big(\mathbb{R} \times \mathbb{R}^2\big)^n$. Its L^1–L^∞ norm is*

$$|||f|||_{1,\infty} = \max_{1\le j_0\le n}\ \sup_{x_{j_0}\in\mathbb{R}\times\mathbb{R}^2} \int \prod_{\substack{j=1,\cdots,n\\ j\ne j_0}} dx_j\ |f(x_1,\cdots,x_n)|.$$

Multiple derivatives are labeled by a multiindex $\delta = (\delta_0, \delta_1, \delta_2) \in \mathbb{N}_0 \times \mathbb{N}_0^2$. For such a multiindex, we set $|\delta| = \delta_0+\delta_1+\delta_2$, $\delta! = \delta_0!\,\delta_1!\,\delta_2!$ and $x^\delta = x_0^{\delta_0}x_1^{\delta_1}x_2^{\delta_2}$ for $x \in \mathbb{R}\times\mathbb{R}^2$.

Definition I.12. *Let Σ be a sectorization and A a function on $\big((\mathbb{R} \times \mathbb{R}^2) \times \Sigma\big)^2$. For a multiindex $\delta \in \mathbb{N}_0 \times \mathbb{N}_0^2$, we define*

$$|A|_{1,\Sigma}^{\delta} = \max_{i=1,2}\ \max_{s_i\in\Sigma} \sum_{s_{3-i}\in\Sigma} \Big|\Big|\Big|(x-y)^\delta\, A\big((x,s_1),(y,s_2)\big)\Big|\Big|\Big|_{1,\infty}.$$

Variables for four–point functions may be momenta or position/sector pairs. Therefore we introduce differential–decay operators that differentiate momentum space variables and multiply position space variables by coordinate differences. We again use the identification

$$\mathfrak{Y}_{\Sigma,\Sigma'}^{(4)} = \dot{\bigcup_{i_1,i_2,i_3,i_4\in\{0,1\}}} \mathfrak{Y}_{i_1,\Sigma} \times \mathfrak{Y}_{i_2,\Sigma} \times \mathfrak{Y}_{i_3,\Sigma'} \times \mathfrak{Y}_{i_4,\Sigma'}$$

of (I.2).

Definition I.13 (Differential–decay operators). *Let Σ and Σ' be sectorizations, $\delta = (\delta_0, \delta_1, \delta_2) \in \mathbb{N}_0 \times \mathbb{N}_0^2$ a multiindex and $\mu, \mu' \in \{1, 2, 3, 4\}$ with $\mu \ne \mu'$.*

i) Let $i_1, \cdots, i_4 \in \{0, 1\}$ and f be a function on $\mathfrak{Y}_{i_1,\Sigma} \times \mathfrak{Y}_{i_2,\Sigma} \times \mathfrak{Y}_{i_3,\Sigma'} \times \mathfrak{Y}_{i_4,\Sigma'}$. If $i_\mu = 0$, multiplication by the δ^{th} power of the position variable dual to k_μ (see Def. I.5) is implemented by

$$\mathrm{D}_\mu^\delta f(\cdots, k_\mu, \cdots) = (-1)^{\delta_1+\delta_2}(-1)^{b_\mu|\delta|}\,\imath^{|\delta|}\ \frac{\partial^{\delta_0}}{\partial k_{\mu,0}^{\delta_0}}\ \frac{\partial^{\delta_1}}{\partial \mathbf{k}_{\mu,1}^{\delta_1}}\ \frac{\partial^{\delta_2}}{\partial \mathbf{k}_{\mu,2}^{\delta_2}} f(\cdots, k_\mu, \cdots).$$

In general, set

$$\mathrm{D}^{\delta}_{\mu;\mu'} f = \begin{cases} \big(\mathrm{D}^{\delta}_{\mu} - \mathrm{D}^{\delta}_{\mu'}\big) f & \text{if } i_\mu = i_{\mu'} = 0 \\ \big(\mathrm{D}^{\delta}_{\mu} - x^{\delta}_{\mu'}\big) f & \text{if } i_\mu = 0,\, i_{\mu'} = 1 \\ \big(x^{\delta}_{\mu} - \mathrm{D}^{\delta}_{\mu'}\big) f & \text{if } i_\mu = 1,\ i_{\mu'} = 0 \\ \big(x^{\delta}_{\mu} - x^{\delta}_{\mu'}\big) f & \text{if } i_\mu = i_{\mu'} = 1 \end{cases}.$$

Here, when $i_\mu = 1$, the μ^{th} argument of f is (x_μ, s_μ).

ii) If f is a function on $\mathfrak{Y}^{(4)}_{\Sigma,\Sigma'}$, then $\big(\mathrm{D}^{\delta}_{\mu;\mu'} f\big)\big|_{(i_1,\cdots,i_4)} = \mathrm{D}^{\delta}_{\mu;\mu'}\big(f\big|_{(i_1,\cdots,i_4)}\big)$ for all $i_1, \cdots, i_4 \in \{0, 1\}$.

Definition I.14. *Let Σ, Σ' be sectorizations.*

i) Let $i_1, \cdots, i_4 \in \{0, 1\}$ and f be a function on $\mathfrak{Y}_{i_1,\Sigma} \times \mathfrak{Y}_{i_2,\Sigma} \times \mathfrak{Y}_{i_3,\Sigma'} \times \mathfrak{Y}_{i_4,\Sigma'}$. For multiindices $\delta_{\mathrm{l}},\ \delta_{\mathrm{c}},\ \delta_{\mathrm{r}} \in \mathbb{N}_0 \times \mathbb{N}_0^2$, we define

$$|f|^{(\delta_{\mathrm{l}},\delta_{\mathrm{c}},\delta_{\mathrm{r}})}_{\Sigma,\Sigma'} = \max_{\substack{s_\nu \in \Sigma \\ \nu=1,2 \\ \text{with } i_\nu=1}} \ \max_{\substack{s_\nu \in \Sigma' \\ \nu=3,4 \\ \text{with } i_\nu=1}} \ \sup_{\substack{k_\nu \in \mathbb{M} \\ \nu=1,2,3,4 \\ \text{with } i_\nu=0}} \ \max_{\substack{\mu=1,2 \\ \mu'=3,4}} \big|\!\big|\!\big|\mathrm{D}^{\delta_{\mathrm{l}}}_{1;2}\mathrm{D}^{\delta_{\mathrm{c}}}_{\mu;\mu'}\mathrm{D}^{\delta_{\mathrm{r}}}_{3;4} f\big|\!\big|\!\big|_{1,\infty}.$$

Here, the ν^{th} argument of f is k_ν when $i_\nu = 0$ and (x_ν, s_ν) when $i_\nu = 1$. The $|\!|\!|\cdot|\!|\!|_{1,\infty}$ of Def. I.11 is applied to all spatial arguments of $\mathrm{D}^{\delta_{\mathrm{l}}}_{1;2}\mathrm{D}^{\delta_{\mathrm{c}}}_{\mu;\mu'}\mathrm{D}^{\delta_{\mathrm{r}}}_{3;4} f$.

ii) If f is a function on $\mathfrak{Y}^{(4)}_{\Sigma,\Sigma'}$, we define

$$|f|^{(\delta_{\mathrm{l}},\delta_{\mathrm{c}},\delta_{\mathrm{r}})}_{\Sigma,\Sigma'} = \sum_{i_1,i_2,i_3,i_4 \in \{0,1\}} \big|f\big|_{(i_1,\cdots,i_4)}\big|^{(\delta_{\mathrm{l}},\delta_{\mathrm{c}},\delta_{\mathrm{r}})}_{\Sigma,\Sigma'}.$$

In this definition, the system $(\delta_{\mathrm{l}}, \delta_{\mathrm{c}}, \delta_{\mathrm{r}})$ of multiindices indicates, roughly speaking, that one takes δ_{l} derivatives with respect to the momentum flowing between the two left legs, δ_{r} derivatives with respect to the momentum flowing between the two right legs and δ_{c} derivatives with respect to momenta flowing from the left hand side to the right-hand side.

In [FKTf1, FKTf2, FKTf3] and [FKTo1, FKTo2, FKTo3, FKTo4], we combine the norms of all derivatives of a function in a formal power series. We denote by $\mathfrak{N}_3$ the set of all formal power series $X = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^2} X_\delta t^\delta$ in the variables $t = (t_0, t_1, t_2)$ with coefficients $X_\delta \in \mathbb{R}_+ \cup \{\infty\}$. See Def. V.2 of [FKTf2] or Def. II.4 of [FKTo1].
A quantity in $\mathfrak{N}_3$ characteristic of the power counting for derivatives in scale j is

$$\mathfrak{c}_j = \sum_{\substack{\delta_1+\delta_2 \le r_e \\ |\delta_0| \le r_0}} M^{j|\delta|}\, t^\delta + \sum_{\substack{\delta_1+\delta_2 > r_e \\ \text{or } |\delta_0| > r_0}} \infty\, t^\delta. \tag{I.4}$$

Definition I.15. *Let Σ be a sectorization.*

i) For a function A on $\big((\mathbb{R} \times \mathbb{R}^2) \times \Sigma\big)^2$, we define

$$|A|_{1,\Sigma} = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^2} \tfrac{1}{\delta!} |A|^{\delta}_{1,\Sigma}\, t^\delta.$$

ii) For a function f *on* $\mathfrak{Y}_\Sigma^4 = \mathfrak{Y}_{\Sigma,\Sigma}^{(4)}$*, we define*

$$|f|_\Sigma = \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \tfrac{1}{\delta!}\Big(\max_{\delta_1+\delta_c+\delta_r=\delta} |f|_{\Sigma,\Sigma}^{(\delta_1,\delta_c,\delta_r)}\Big) t^\delta .$$

iii) For a function f *on* $\big(\mathfrak{Y}_\Sigma^\updownarrow\big)^4$*, we define*

$$|f|_\Sigma = \sum_{\sigma_1,\cdots,\sigma_4\in\{\uparrow,\downarrow\}} \big|f\big((\,\cdot\,,\sigma_1),(\,\cdot\,,\sigma_2),(\,\cdot\,,\sigma_3),(\,\cdot\,,\sigma_4)\big)\big|_\Sigma .$$

The following lemma, whose proof follows immediately from the various definitions and Lemma D.2.ii of [FKTo3], compares these norms with the norms of Def. VI.6 of [FKTf2].

Lemma I.16. *Let* Σ *be a sectorization.*

i) Let f *be a sectorized, translation invariant function on* $\big(\mathfrak{Y}_\Sigma^\updownarrow\big)^4$ *and* $V_{\rm ph}(f)$ *its particle–hole value as in Def. VII.4 of [FKTf2]. Let* $|\cdot|_{3,\Sigma}^{\sim}$ *be the norm of Def. XIII.12 of [FKTf3] (or Def. XVI.4 of [FKTo3]). Then there is a constant* $\it const$*, that depends only on* r_0 *and* r*, such that*

$$\big|V_{\rm ph}(f)\big|_{3,\Sigma}^{\sim} \le {\it const}\, |f|_\Sigma + \sum_{\substack{\delta_1+\delta_2>r \\ \text{or } \delta_0>r_0}} \infty\, t^\delta .$$

ii) Let g *be a function in the space* $\check{\mathcal{F}}_{4,\Sigma}$ *of Def. XIV.6 of [FKTf2] (or Def. XVI.7.iii of [FKTo3]) and* $g^{\rm ph}$ *its particle–hole reduction as in Def. VII.4 of [FKTf2]. Then there is a universal* $\it const$ *such that*

$$|g^{\rm ph}|_\Sigma \le {\it const}\, |g|_{3,\Sigma}^{\sim} .$$

5. The Propagators. The propagators we use in the multiscale analysis of [FKTf1, FKTf2, FKTf3] are of the form

$$C_v^{(j)}(k) = \tfrac{\nu^{(j)}(k)}{ik_0-e(\mathbf{k})-v(k)} \qquad C_v^{(\ge j)}(k) = \tfrac{\nu^{(\ge j)}(k)}{ik_0-e(\mathbf{k})-v(k)}$$

with functions $v(k)$ satisfying $|v(k)| \le \frac{1}{2}|\imath k_0 - e(\mathbf{k})|$. Their Fourier transforms are

$$C_v^{(j)}(x,y) = \int \tfrac{d^3k}{(2\pi)^3} e^{\imath<k,x-y>_-} C_v^{(j)}(k) \quad C_v^{(\ge j)}(x,y) = \int \tfrac{d^3k}{(2\pi)^3} e^{\imath<k,x-y>_-} C_v^{(\ge j)}(k),$$
$$C_v^{(j)}(y) = \int \tfrac{d^3k}{(2\pi)^3} e^{-\imath<k,y>_-} C_v^{(j)}(k) \quad C_v^{(\ge j)}(y) = \int \tfrac{d^3k}{(2\pi)^3} e^{-\imath<k,y>_-} C_v^{(\ge j)}(k).$$

The function $v(k)$ will be the sum of Fourier transforms of sectorized, translation invariant functions $p\big((x,s),(x,s')\big)$ on $\Big(\big(\mathbb{R}\times\mathbb{R}^2\big)\times\Sigma\Big)^2$ for various sectorizations Σ. The Fourier transform of such a function is defined as

$$\check{p}(k) = \sum_{s,s'\in\Sigma} \int d^3x\ e^{\imath<k,x>_-}\ p\big((0,s),(x,s')\big).$$

6. Resectorization. We now fix $\frac{1}{2} < \aleph < \frac{2}{3}$ and set $\mathfrak{l}_j = \frac{1}{M^{\aleph j}}$. Furthermore, we select, for each $j \geq 1$, a sectorization Σ_j of length $\mathfrak{l}_j$ at scale j and a partition of unity $\{ \chi_s \mid s \in \Sigma_j \}$ of the j^{th} neighbourhood which fulfills Lemma XII.3 of [FKTo3] with $\Sigma = \Sigma_j$. The Fourier transform of χ_s is

$$\hat{\chi}_s(x) = \int e^{-\imath <k,x>_-} \chi_s(k) \frac{d^3k}{(2\pi)^3}.$$

Definition I.17 (Resectorization). *Let $j, j', j_l, j_l', j_r, j_r' \geq 1$.*

i) Let p be a sectorized, translation invariant function on $\left(\left(\mathbb{R} \times \mathbb{R}^2\right) \times \Sigma_j\right)^2$. Then, for $j' \neq j$, the j'–resectorization of p is

$$p_{\Sigma_{j'}}((x_1,s_1),(x_2,s_2)) = \sum_{s_1',s_2' \in \Sigma_j} \int dx_1'\, dx_2'\ \hat{\chi}_{s_1}(x_1 - x_1')\ p((x_1',s_1'),(x_2',s_2'))\ \hat{\chi}_{s_2}(x_2' - x_2).$$

It is a sectorized, translation invariant function on $\left(\left(\mathbb{R} \times \mathbb{R}^2\right) \times \Sigma_{j'}\right)^2$. If $j = j'$, we set $p_{\Sigma_{j'}} = p$.

ii) Let $i_1, \cdots, i_4 \in \{0,1\}$ and f be a function on $\mathfrak{Y}_{i_1,\Sigma_{j_l}} \times \mathfrak{Y}_{i_2,\Sigma_{j_l}} \times \mathfrak{Y}_{i_3,\Sigma_{j_r}} \times \mathfrak{Y}_{i_4,\Sigma_{j_r}}$ that is sectorized and translation invariant. Then the (j_l', j_r')–resectorization of f is the sectorized, translation invariant function on $\mathfrak{Y}_{i_1,\Sigma_{j_l'}} \times \mathfrak{Y}_{i_2,\Sigma_{j_l'}} \times \mathfrak{Y}_{i_3,\Sigma_{j_r'}} \times \mathfrak{Y}_{i_4,\Sigma_{j_r'}}$ defined by

$$f_{\Sigma_{j_l'},\Sigma_{j_r'}}(y_1,y_2,y_3,y_4) = \sum_{\substack{s_\mu' \in \Sigma_{j_l} \\ \mu \in \{1,2\} \cap S}} \sum_{\substack{s_\mu' \in \Sigma_{j_r} \\ \mu \in \{3,4\} \cap S}} \int \prod_{\mu \in S} \left(dx_\mu'\ \hat{\chi}_{s_\mu}((-1)^{b_\mu}(x_\mu - x_\mu')) \right) f(y_1',y_2',y_3',y_4'),$$

where

$$S = \{ \mu \mid i_\mu = 1 \} \cap \begin{cases} \{1,2,3,4\} & \text{if } j_l' \neq j_l,\ j_r' \neq j_r \\ \{1,2\} & \text{if } j_l' \neq j_l,\ j_r' = j_r \\ \{3,4\} & \text{if } j_l' = j_l,\ j_r' \neq j_r \\ \emptyset & \text{if } j_l' = j_l,\ j_r' = j_r \end{cases}$$

and $y_\mu' = y_\mu$ for $\mu \notin S$ and, for $\mu \in S$,

$$y_\mu = (x_\mu, s_\mu) \qquad y_\mu' = (x_\mu', s_\mu').$$

iii) If f is a sectorized, translation invariant function on $\mathfrak{Y}^{(4)}_{\Sigma_{j_l},\Sigma_{j_r}}$, then $\left(f_{\Sigma_{j_l'},\Sigma_{j_r'}}\right)\big|_{(i_1,\cdots,i_4)} = \left(f\big|_{(i_1,\cdots,i_4)}\right)_{\Sigma_{j_l'},\Sigma_{j_r'}}$ for all $i_1, \cdots, i_4 \in \{0,1\}$. If $j_l' = j_r' = j'$, we set $f_{\Sigma_{j'}} = f_{\Sigma_{j'},\Sigma_{j'}}$.

iv) If f is a sectorized, translation invariant function on $\left(\mathfrak{Y}^{\updownarrow}_{\Sigma_j}\right)^4$, then

$$f_{\Sigma_{j'}}((\cdot,\sigma_1),(\cdot,\sigma_2),(\cdot,\sigma_3),(\cdot,\sigma_4)) = \Big(f((\cdot,\sigma_1),(\cdot,\sigma_2),(\cdot,\sigma_3),(\cdot,\sigma_4)) \Big)_{\Sigma_{j'}}$$

for all $\sigma_1, \cdots, \sigma_4 \in \{\uparrow, \downarrow\}$.

Remark I.18. Let K and H be sectorized translation invariant functions on $\mathfrak{Y}^{(4)}_{\Sigma_{i_1},\Sigma_{j_1}}$ and $\mathfrak{Y}^{(4)}_{\Sigma_{i_r},\Sigma_{j_r}}$ respectively. Let P be a bubble propagator. If the Fourier transform

$$\int \prod_{\mu=1}^{4} dx_\mu \ \prod_{\mu=1}^{4} e^{-\imath(-1)^{b_\mu}<k_\mu,x_\mu>_-} \ P(x_1,x_2,x_3,x_4)$$

of P is supported on the $\max\{j_1', i_r'\}^{\text{th}}$ neighbourhood, then

$$\Big[K \bullet P \bullet H\Big]_{\Sigma_{i_1'},\Sigma_{j_r'}} = K_{\Sigma_{i_1'},\Sigma_{j_1'}} \bullet P \bullet H_{\Sigma_{i_r'},\Sigma_{j_r'}}.$$

7. Compound Particle–Hole Ladders. Define, for any set $\mathcal{Z}$ and any function K on $\mathcal{Z}^4$, the flipped function

$$K^f(z_1,z_2,z_3,z_4) = -K(z_1,z_3,z_2,z_4). \tag{I.5}$$

Definition I.19. *Let $\vec{F} = \big(F^{(2)}, F^{(3)}, \cdots\big)$ be a sequence of sectorized, translation invariant functions $F^{(i)}$ on $\big(\mathfrak{Y}^{\updownarrow}_{\Sigma_i}\big)^4$ and $v(k)$ a function on $\mathbb{M}$ such that $|v(k)| \le \frac{1}{2}|\imath k_0 - e(\mathbf{k})|$. We define, recursively on $0 \le j < \infty$, the compound particle–hole (or wrong way) ladders up to scale j, denoted by $\mathcal{L}^{(j)} = \mathcal{L}^{(j)}_v(\vec{F})$, as*

$$\begin{aligned}\mathcal{L}^{(0)} &= 0,\\ \mathcal{L}^{(j+1)} &= \mathcal{L}^{(j)}_{\Sigma_j} + \sum_{\ell=1}^{\infty} \big(F + \mathcal{L}^{(j)}_{\Sigma_j} + \mathcal{L}^{(j)\,f}_{\Sigma_j}\big) \bullet \mathcal{C}^{(j)} \bullet \cdots \mathcal{C}^{(j)} \bullet \big(F + \mathcal{L}^{(j)}_{\Sigma_j} + \mathcal{L}^{(j)\,f}_{\Sigma_j}\big),\end{aligned}$$

where $F = \sum_{i=2}^{j} F^{(i)}_{\Sigma_j}$ and the ℓ^{th} term has ℓ bubble propagators $\mathcal{C}^{(j)} = \mathcal{C}\big(C^{(j)}_v, C^{(\ge j+1)}_v\big)$. Observe that $\mathcal{L}^{(1)} = \mathcal{L}^{(2)} = 0$.

Theorem I.20. *For every $\varepsilon > 0$ there are constants ρ_0,* const[1] *such that the following holds. Let $\vec{F} = \big(F^{(2)}, F^{(3)}, \cdots\big)$ be a sequence of sectorized, translation invariant spin independent*[2] *functions $F^{(i)}$ on $\big(\mathfrak{Y}^{\updownarrow}_{\Sigma_i}\big)^4$ and $\vec{p} = \big(p^{(2)}, p^{(3)}, \cdots\big)$ be a sequence of sectorized, translation invariant functions $p^{(i)}$ on $\Big(\big(\mathbb{R}\times\mathbb{R}^2\big)\times\Sigma_i\Big)^2$. Assume that there is $\rho \le \rho_0$ such that for $i \ge 2$,*

$$|F^{(i)}|_{\Sigma_i} \le \tfrac{\rho}{M^{\varepsilon i}}\mathfrak{c}_i, \qquad |p^{(i)}|_{1,\Sigma_i} \le \tfrac{\rho\,\mathfrak{l}_i}{M^i}\mathfrak{c}_i, \qquad \check{p}^{(i)}(0,\mathbf{k}) = 0.$$

Set $v(k) = \sum_{i=2}^{\infty} \check{p}^{(i)}(k)$. Then for all $j \ge 1$,

$$\big|\mathcal{L}^{(j+1)}_v(\vec{F})\big|_{\Sigma_j} \le \text{const}\,\rho^2\,\mathfrak{c}_j.$$

[1] Throughout this paper we use "const" to denote unimportant constants that depend only on the dispersion relation $e(\mathbf{k})$ and the scale parameter M. In particular, they do not depend on the scale j.

[2] "Spin independence" is formally defined in Def. II.6.

Remark I.21. Theorem I.20 and Theorem D.2 of [FKTf3] are equivalent. If one replaces the functions $F^{(i)}$ of Theorem D.2 of [FKTf3] by 24 times their particle–hole reductions, then, by Cor. D.7 of [FKTf3] and Remark I.10, the concepts of compound ladders of Def. I.19 and Def. D.1 of [FKTf3] coincide. Hence Theorem I.20 and Theorem D.2 of [FKTf3] are equivalent by Lemma I.16.

Theorem I.20 will be proven following Cor. II.24. The core of the proof consists of bounds on two types of ladder fragments, that look like

and are called particle–hole bubbles and double bubbles, and a combinatorial result, Cor. II.12, that enables one to express general ladders in terms of these fragments. The most subtle part of the bound, Theorem II.19, on particle–hole bubbles is a generalization of Lemma I.1. The bound, Theorem II.20, on double bubbles also exploits "volume improvement due to overlapping loops". A simple introduction to this phenomenon is provided at the beginning of §IV.

Ladders with external momenta have an infrared limit that behaves much like the model bubble of Lemma I.1.

Theorem I.22. *Under the hypotheses of Theorem I.20, the limit*

$$\mathfrak{L}(q, q', t, \sigma_1, \cdots \sigma_4) = \lim_{j\to\infty} \mathcal{L}_v^{(j)}(\vec{F})\big|_{i_1,i_2,i_3,i_4=0}\big((q+\tfrac{t}{2},\sigma_1),(q-\tfrac{t}{2},\sigma_2),(q'+\tfrac{t}{2},\sigma_3),(q'-\tfrac{t}{2},\sigma_4)\big)$$

exists for transfer momentum $t \neq 0$ *and is continuous in* (q, q', t) *for* $t \neq 0$*. The restrictions to* $\mathbf{t} = 0$ *and to* $t_0 = 0$*, namely,* $\mathfrak{L}(q, q', (t_0, \mathbf{0}), \sigma_1, \cdots \sigma_4)$ *and* $\mathfrak{L}(q, q', (0, \mathbf{t}), \sigma_1, \cdots \sigma_4)$*, have continuous extensions to* $t = 0$*.*

This Theorem is proven following Lemma II.29. Notation tables are provided at the end of the paper.

II. Reduction to Bubble Estimates

For the rest of the paper, we fix a sequence $\vec{F} = \big(F^{(2)}, F^{(3)}, \cdots\big)$ of sectorized, translation invariant, spin independent functions $F^{(i)}$ on $\big(\mathfrak{Y}^{\updownarrow}_{\Sigma_i}\big)^4$ and a sequence $\vec{p} = \big(p^{(2)}, p^{(3)}, \cdots\big)$ of sectorized, translation invariant functions $p^{(i)}$ on $\Big(\big(\mathbb{R}\times\mathbb{R}^2\big)\times\Sigma_i\Big)^2$ as in Theorem I.20 , and we set $v(k) = \sum_{i=2}^{\infty} \check{p}^{(i)}(k)$. Denote $\mathcal{L}^{(j+1)} = \mathcal{L}_v^{(j+1)}(\vec{F})$ and define the particle–hole bubble propagator of scale j by

$$\mathcal{C}^{(j)} = \mathcal{C}\big(C_v^{(j)}, C_v^{(\geq j+1)}\big) = \sum_{\substack{i_1,i_2\geq 1\\ \min(i_1,i_2)=j}} C_v^{(i_1)} \otimes C_v^{(i_2)\,t}.$$

In Def. I.19 of compound particle–hole ladders, every bubble propagator $\mathcal{C}^{(j)}$ has a hard line of a single specified scale, j. With this description of ladders, one cannot exploit the cancelation between scales illustrated in Lemma I.1. In Prop. II.3, we express the sum of all compound particle–hole ladders using particle–hole ladders whose bubble propagators

$$\mathcal{C}^{[j_1,j_2]} = \sum_{j=j_1}^{j_2} \mathcal{C}^{(j)} = C_v^{(\geq j_1)} \otimes C_v^{(\geq j_1)\,t} - C_v^{(\geq j_2+1)} \otimes C_v^{(\geq j_2+1)\,t}$$

have an interval of hard line scales blocked together. This then allows us to factor the ladder into individual bubbles and double bubbles that can be estimated separately. See Theorems II.19 and II.20. In the course of the reduction, we analyse the possible distribution of spins so that the bubble and double bubble estimates can be formulated in a spin independent way.

1. Combinatorial Structure of Compound Ladders. In this section, we use the following

Convention II.1. *Let K and K' be functions on $\big(\mathfrak{Y}_{\Sigma_j}\big)^4$ and $\big(\mathfrak{Y}_{\Sigma_{j'}}\big)^4$, respectively. Then the notation $K + K'$ denotes the function $K_{\Sigma_{\max\{j,j'\}}} + K'_{\Sigma_{\max\{j,j'\}}}$ on $\big(\mathfrak{Y}_{\Sigma_{\max\{j,j'\}}}\big)^4$. The same convention is used when K and K' are functions on $\big(\mathfrak{Y}^{\updownarrow}_{\Sigma_j}\big)^4$ and $\big(\mathfrak{Y}^{\updownarrow}_{\Sigma_{j'}}\big)^4$.*

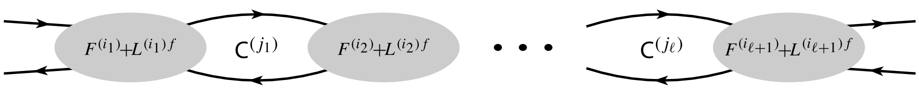
Definition II.2. *We define, recursively on $0 \leq j < \infty$, sectorized, translation invariant, spin independent functions $L^{(j)}$, on $\big(\mathfrak{Y}^{\updownarrow}_{\Sigma_{j-1}}\big)^4$ by*

$$\begin{aligned} &L^{(0)} = L^{(1)} = L^{(2)} = 0,\\ &L^{(j+1)} = \sum_{\ell=1}^{\infty} \sideset{}{'}\sum_{\substack{i_1,\cdots,i_{\ell+1}\geq 2\\ j_1,\cdots,j_\ell\geq 0}} \Big[\big(F^{(i_1)} + L^{(i_1)\,f}\big) \bullet \mathcal{C}^{(j_1)} \bullet \cdots \bullet \mathcal{C}^{(j_\ell)} \bullet \big(F^{(i_{\ell+1})} + L^{(i_{\ell+1})\,f}\big)\Big]_{\Sigma_j}, \end{aligned}$$

where the sum $\sum'$ imposes the constraints

$$\max\{j_1,\cdots,j_\ell\} = j \quad i_m \leq \min\{j_{m-1}, j_m\} \quad \text{for all } 1 \leq m \leq \ell+1.$$

When $m = 1$, $\min\{j_{m-1}, j_m\} = j_1$ and when $m = \ell+1$, $\min\{j_{m-1}, j_m\} = j_\ell$,



Observe that $L^{(j)}$ depends only on the components $F^{(2)}, \cdots, F^{(j-1)}$ of $\vec{F}$.

Proposition II.3.

i) $\mathcal{L}^{(j+1)} = \sum_{i=0}^{j+1} L^{(i)}_{\Sigma_j}$,

$$ii)\ \mathcal{L}^{(j+1)} = \sum_{\ell=1}^{\infty} \sum_{i_1,\cdots,i_{\ell+1}=2}^{j} \Big[\big(F^{(i_1)} + L^{(i_1)\,f}\big) \bullet \mathcal{C}^{[\max\{i_1,i_2\},j]} \bullet \big(F^{(i_2)} + L^{(i_2)\,f}\big) \cdots \bullet \mathcal{C}^{[\max\{i_\ell,i_{\ell+1}\},j]} \bullet \big(F^{(i_{\ell+1})} + L^{(i_{\ell+1})\,f}\big)\Big]_{\Sigma_j},$$

$$iii)\ L^{(j+1)} = \Big(\sum_{i=2}^{j} F^{(i)}_{\Sigma_j} + \mathcal{L}^{(j)\,f}_{\Sigma_j} + \mathcal{L}^{(j)}_{\Sigma_j}\Big) \bullet \mathcal{C}^{(j)} \bullet \Big(\sum_{i=2}^{j} F^{(i)}_{\Sigma_j} + \mathcal{L}^{(j)\,f}_{\Sigma_j} + \mathcal{L}^{(j+1)}\Big).$$

To prove Proposition II.3, we define

$$\tilde{\mathcal{L}}^{(j+1)} = \sum_{i=0}^{j+1} L^{(i)}_{\Sigma_j}$$

and verify, in Lemmas II.4 and II.5, parts (ii) and (iii) of the proposition, but with $\mathcal{L}^{(k)}$ replaced by $\tilde{\mathcal{L}}^{(k)}$. Then we prove that $\tilde{\mathcal{L}}^{(k)} = \mathcal{L}^{(k)}$.

Lemma II.4.

$$\tilde{\mathcal{L}}^{(j+1)} = \sum_{\ell=1}^{\infty} \sum_{i_1,\cdots,i_{\ell+1}=2}^{j} \Big[\big(F^{(i_1)} + L^{(i_1)\,f}\big) \bullet \mathcal{C}^{[\max\{i_1,i_2\},j]} \bullet \big(F^{(i_2)} + L^{(i_2)\,f}\big) \cdots \bullet \mathcal{C}^{[\max\{i_\ell,i_{\ell+1}\},j]} \bullet \big(F^{(i_{\ell+1})} + L^{(i_{\ell+1})\,f}\big)\Big]_{\Sigma_j}.$$

Proof.

$$\begin{aligned}
\tilde{\mathcal{L}}^{(j+1)} &= \sum_{\ell=1}^{\infty} \sum_{j_1,\cdots,j_\ell=0}^{j} \sum_{\substack{i_m=2\\ 1\le m\le \ell+1}}^{\min\{j_{m-1},j_m\}} \Big[\big(F^{(i_1)} + L^{(i_1)\,f}\big) \bullet \mathcal{C}^{(j_1)} \bullet \big(F^{(i_2)} + L^{(i_2)\,f}\big) \cdots \bullet \mathcal{C}^{(j_\ell)} \bullet \big(F^{(i_{\ell+1})} + L^{(i_{\ell+1})\,f}\big)\Big]_{\Sigma_j} \\
&= \sum_{\ell=1}^{\infty} \sum_{i_1,\cdots,i_{\ell+1}=2}^{j} \sum_{\substack{j_m=\max\{i_m,i_{m+1}\}\\ 1\le m\le \ell}}^{j} \Big[\big(F^{(i_1)} + L^{(i_1)\,f}\big) \bullet \mathcal{C}^{(j_1)} \bullet \big(F^{(i_2)} + L^{(i_2)\,f}\big) \cdots \bullet \mathcal{C}^{(j_\ell)} \bullet \big(F^{(i_{\ell+1})} + L^{(i_{\ell+1})\,f}\big)\Big]_{\Sigma_j}. \quad \square
\end{aligned}$$

Lemma II.5.

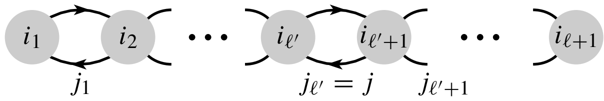
$$i)\ L^{(j+1)} = \Big(\sum_{i=2}^{j} F^{(i)}_{\Sigma_j} + \tilde{\mathcal{L}}^{(j)\,f}_{\Sigma_j} + \tilde{\mathcal{L}}^{(j)}_{\Sigma_j}\Big) \bullet \mathcal{C}^{(j)} \bullet \Big(\sum_{i=2}^{j} F^{(i)}_{\Sigma_j} + \tilde{\mathcal{L}}^{(j)\,f}_{\Sigma_j} + \tilde{\mathcal{L}}^{(j+1)}\Big),$$

$$ii)\ L^{(j+1)} = \sum_{\ell=1}^{\infty} \Big[\Big(\sum_{i=2}^{j} F^{(i)}_{\Sigma_j} + \tilde{\mathcal{L}}^{(j)\,f}_{\Sigma_j} + \tilde{\mathcal{L}}^{(j)}_{\Sigma_j}\Big) \bullet \mathcal{C}^{(j)}\Big]^{\ell} \bullet \Big(\sum_{i=2}^{j} F^{(i)}_{\Sigma_j} + \tilde{\mathcal{L}}^{(j)\,f}_{\Sigma_j} + \tilde{\mathcal{L}}^{(j)}_{\Sigma_j}\Big).$$

Proof. i)

$$L^{(j+1)} = \sum_{\ell=1}^{\infty} \sum_{\ell'=1}^{\ell} \sum_{\substack{j_1,\cdots,j_\ell=0 \\ j_1,\cdots,j_{\ell'-1}\le j-1 \\ j_{\ell'}=j}}^{j} \sum_{\substack{i_m=2 \\ 1\le m\le \ell+1}}^{\min\{j_{m-1},j_m\}} \Big[\big(F^{(i_1)} + L^{(i_1)\,f}\big) \bullet \mathcal{C}^{(j_1)}$$

$$\cdots \bullet \big(F^{(i_2)} + L^{(i_2)\,f}\big) \bullet \mathcal{C}^{(j_\ell)} \bullet \big(F^{(i_{\ell+1})} + L^{(i_{\ell+1})\,f}\big)\Big]_{\Sigma_j}$$

$=$  .

Splitting up the sum according to whether $\ell' = 1$, $1 < \ell' < \ell$ or $\ell' = \ell$, we have

$$L^{(j+1)} = \Big[\sum_{i=2}^{j} \big(F^{(i)}_{\Sigma_j} + L^{(i)\,f}_{\Sigma_j}\big)\Big] \bullet \mathcal{C}^{(j)} \bullet \Big[\sum_{i=2}^{j} \big(F^{(i)}_{\Sigma_j} + L^{(i)\,f}_{\Sigma_j}\big)\Big]$$

$$+ \sum_{\ell=2}^{\infty} \sum_{\substack{j_1,\cdots,j_\ell=0 \\ j_1=j}}^{j} \sum_{i=2}^{j} \sum_{\substack{i_m=2 \\ 2\le m\le \ell+1}}^{\min\{j_{m-1},j_m\}} \big(F^{(i)}_{\Sigma_j} + L^{(i)\,f}_{\Sigma_j}\big) \bullet \mathcal{C}^{(j)}$$

$$\bullet \Big[\big(F^{(i_2)} + L^{(i_2)\,f}\big) \bullet \mathcal{C}^{(j_2)} \bullet \cdots \bullet \mathcal{C}^{(j_\ell)} \bullet \big(F^{(i_{\ell+1})} + L^{(i_{\ell+1})\,f}\big)\Big]_{\Sigma_j}$$

$$+ \sum_{\ell=3}^{\infty} \sum_{\ell'=2}^{\ell-1} \Big[\sum_{j_1,\cdots,j_{\ell'-1}=0}^{j-1} \sum_{\substack{i_1,\cdots,i_{\ell'}\ge 2 \\ i_m\le \min\{j_{m-1},j_m\} \\ \text{for } m=1,\cdots,\ell'-1 \\ i_{\ell'}\le j_{\ell'-1}}} \big(F^{(i_1)} + L^{(i_1)\,f}\big)$$

$$\bullet \mathcal{C}^{(j_1)} \cdots \big(F^{(i_{\ell'})} + L^{(i_{\ell'})\,f}\big)\Big]_{\Sigma_j} \bullet \mathcal{C}^{(j)}$$

$$\times \Big[\sum_{j_{\ell'+1},\cdots,j_\ell=0}^{j} \sum_{\substack{i_{\ell'+1},\cdots,i_{\ell+1}\ge 2 \\ i_m\le \min\{j_{m-1},j_m\} \\ \text{for } m=\ell'+2,\cdots,\ell+1 \\ i_{\ell'+1}\le j_{\ell'+1}}} \big(F^{(i_{\ell'+1})} + L^{(i_{\ell'+1})\,f}\big)$$

$$\bullet \mathcal{C}^{(j_{\ell'+1})} \cdots \big(F^{(i_{\ell+1})} + L^{(i_{\ell+1})\,f}\big)\Big]_{\Sigma_j}$$

$$+ \sum_{\ell=2}^{\infty} \sum_{j_1,\cdots,j_{\ell-1}=0}^{j-1} \sum_{\substack{i_m=2 \\ 1\le m\le \ell}}^{\min\{j_{m-1},j_m\}} \sum_{i=2}^{j} \Big[\big(F^{(i_1)} + L^{(i_1)\,f}\big)$$

$$\bullet \mathcal{C}^{(j_1)} \cdots \mathcal{C}^{(j_{\ell-1})} \bullet \big(F^{(i_\ell)} + L^{(i_\ell)\,f}\big)\Big]_{\Sigma_j} \bullet \mathcal{C}^{(j)} \bullet \big(F^{(i)}_{\Sigma_j} + L^{(i)\,f}_{\Sigma_j}\big)$$

$$= \Big[\sum_{i=2}^{j} \big(F^{(i)}_{\Sigma_j} + L^{(i)\,f}_{\Sigma_j} \big) \Big] \bullet \mathcal{C}^{(j)} \bullet \Big[\sum_{i=2}^{j} \big(F^{(i)}_{\Sigma_j} + L^{(i)\,f}_{\Sigma_j} \big) \Big]$$

$$+ \Big[\sum_{i=2}^{j} \big(F^{(i)}_{\Sigma_j} + L^{(i)\,f}_{\Sigma_j} \big) \Big] \bullet \mathcal{C}^{(j)} \bullet \tilde{\mathcal{L}}^{(j+1)}$$

$$+ \tilde{\mathcal{L}}^{(j)}_{\Sigma_j} \bullet \mathcal{C}^{(j)} \bullet \tilde{\mathcal{L}}^{(j+1)} \ + \ \tilde{\mathcal{L}}^{(j)}_{\Sigma_j} \bullet \mathcal{C}^{(j)} \bullet \Big[\sum_{i=2}^{j} \big(F^{(i)}_{\Sigma_j} + L^{(i)\,f}_{\Sigma_j} \big) \Big]$$

$$= \Big[\sum_{i=2}^{j} F^{(i)}_{\Sigma_j} + \tilde{\mathcal{L}}^{(j)\,f}_{\Sigma_j} + \tilde{\mathcal{L}}^{(j)}_{\Sigma_j} \Big] \bullet \mathcal{C}^{(j)} \bullet \Big[\sum_{i=2}^{j} F^{(i)}_{\Sigma_j} + \tilde{\mathcal{L}}^{(j)\,f}_{\Sigma_j} + \tilde{\mathcal{L}}^{(j+1)} \Big].$$

ii) Substituting $\tilde{\mathcal{L}}^{(j+1)} = \tilde{\mathcal{L}}^{(j)}_{\Sigma_j} + L^{(j+1)}$ into part (i) gives

$$L^{(j+1)} = \Big(\sum_{i=2}^{j} F^{(i)}_{\Sigma_j} + \tilde{\mathcal{L}}^{(j)\,f}_{\Sigma_j} + \tilde{\mathcal{L}}^{(j)}_{\Sigma_j} \Big) \bullet \mathcal{C}^{(j)} \bullet \Big(\sum_{i=2}^{j} F^{(i)}_{\Sigma_j} + \tilde{\mathcal{L}}^{(j)\,f}_{\Sigma_j} + \tilde{\mathcal{L}}^{(j)}_{\Sigma_j} \Big)$$
$$+ \Big(\sum_{i=2}^{j} F^{(i)}_{\Sigma_j} + \tilde{\mathcal{L}}^{(j)\,f}_{\Sigma_j} + \tilde{\mathcal{L}}^{(j)}_{\Sigma_j} \Big) \bullet \mathcal{C}^{(j)} \bullet L^{(j+1)}.$$

Now just iterate. □

Proof of Proposition II.3. By Lemma II.5.ii,

$$\tilde{\mathcal{L}}^{(j+1)} = \tilde{\mathcal{L}}^{(j)}_{\Sigma_j} + L^{(j+1)}$$
$$= \tilde{\mathcal{L}}^{(j)}_{\Sigma_j} + \sum_{\ell=1}^{\infty} \big(F + \tilde{\mathcal{L}}^{(j)}_{\Sigma_j} + \tilde{\mathcal{L}}^{(j)\,f}_{\Sigma_j} \big) \bullet \mathcal{C}^{(j)} \bullet \cdots \mathcal{C}^{(j)} \bullet \big(F + \tilde{\mathcal{L}}^{(j)}_{\Sigma_j} + \tilde{\mathcal{L}}^{(j)\,f}_{\Sigma_j} \big),$$

where $F = \sum_{i=2}^{j} F^{(i)}_{\Sigma_j}$ and the $\ell^{\rm th}$ term has ℓ bubble propagators $\mathcal{C}^{(j)}$. Thus $\tilde{\mathcal{L}}^{(j)}$ obeys the same initial condition and recursion relation as that defining $\mathcal{L}^{(j)}$ in Def. I.19. Therefore, they are equal. Hence the proposition follows from Lemma II.4 and Lemma II.5.i. □

2. Spin Independence. The following discussion shows how spin independent functions on $\big(\mathfrak{Y}^{\updownarrow}_{\Sigma}\big)^4$ are related to functions on $\mathfrak{Y}^4_{\Sigma}$.

Definition II.6 (Spin independence). *Let $\mathfrak{Z}_{\rm l}$ and $\mathfrak{Z}_{\rm r}$ be sets and let f be a function on $\big(\mathfrak{Z}_{\rm l} \times \{\uparrow, \downarrow\}\big)^2 \times \big(\mathfrak{Z}_{\rm r} \times \{\uparrow, \downarrow\}\big)^2$. Set, for each $A \in SU(2)$,*

$$f^A\big((\cdot, \sigma_1), \cdots, (\cdot, \sigma_4)\big) = \sum_{\tau_1, \cdots \tau_4} f\big((\cdot, \tau_1), \cdots, (\cdot, \tau_4)\big)\, A_{\tau_1,\sigma_1} \bar{A}_{\tau_2,\sigma_2} \bar{A}_{\tau_3,\sigma_3} A_{\tau_4,\sigma_4}.$$

f is called (particle–hole) spin independent if $f = f^A$ for all $A \in SU(2)$.

Remark II.7. Let F be a four–legged kernel on $\mathfrak{X}_\Sigma$. If F is spin independent in the sense of Def. B.1.S of [FKTo2], then its particle–hole reduction is spin independent in the sense of Def. II.6.

Lemma II.8 (Charge spin representation). *Let* $\mathfrak{Z}_{\rm l}$ *and* $\mathfrak{Z}_{\rm r}$ *be sets and let* f *be a spin independent function on* $\big(\mathfrak{Z}_{\rm l}\times\{\uparrow,\downarrow\}\big)^2\times\big(\mathfrak{Z}_{\rm r}\times\{\uparrow,\downarrow\}\big)^2$. *Then, there are functions* f_C *and* f_S *on* $\mathfrak{Z}_{\rm l}^2\times\mathfrak{Z}_{\rm r}^2$ *such that*

$$
\begin{aligned}
&f\big((z_1,\sigma_1),(z_2,\sigma_2),(z_3,\sigma_3),(z_4,\sigma_4)\big)\\
&\quad=\tfrac12 f_C(z_1,z_2,z_3,z_4)\delta_{\sigma_1,\sigma_2}\delta_{\sigma_3,\sigma_4}\\
&\qquad+f_S(z_1,z_2,z_3,z_4)\big[\delta_{\sigma_1,\sigma_3}\delta_{\sigma_2,\sigma_4}-\tfrac12\delta_{\sigma_1,\sigma_2}\delta_{\sigma_3,\sigma_4}\big].
\end{aligned}
$$

Proof. The statement is essentially [N, (1–7)]. The proof is outlined in [N] between (3–40) and (3–41). For the readers' convenience, we include a detailed proof.

The z's play no role, so we suppress them. Then the function $f(\sigma_1,\sigma_2,\sigma_3,\sigma_4)$ can be viewed as an element of $\mathbb{C}^{16}=\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$ and $M_A:f\mapsto f^A$ is a linear map on $\mathbb{C}^{16}$. The map $A\mapsto M_A$ is a representation of $SU(2)$ on $\mathbb{C}^{16}$. Denote by S_n the standard $(2n+1)$ dimensional "spin n" irreducible representation of $SU(2)$. In particular, the identity representation $A\mapsto A$ is $S_{1/2}$. Since the representation $A\mapsto\bar A$ is unitarily equivalent to $S_{1/2}$, the representation $A\mapsto M_A$ is unitarily equivalent to $S_{1/2}\otimes S_{1/2}\otimes S_{1/2}\otimes S_{1/2}\cong(S_0\oplus S_1)\otimes(S_0\oplus S_1)\cong 2S_0\oplus 3S_1\oplus S_2$. Thus the dimension of the subspace $\big\{\,f\in\mathbb{C}^{16}\,\big|\,f=f^A\ \forall A\in SU(2)\,\big\}$ is exactly two. Since $f(\sigma_1,\sigma_2,\sigma_3,\sigma_4)=\delta_{\sigma_1,\sigma_2}\delta_{\sigma_3,\sigma_4}$ and $f(\sigma_1,\sigma_2,\sigma_3,\sigma_4)=\delta_{\sigma_1,\sigma_3}\delta_{\sigma_2,\sigma_4}-\frac12\delta_{\sigma_1,\sigma_2}\delta_{\sigma_3,\sigma_4}$ are two independent elements of that subspace, every $f\in\mathbb{C}^{16}$ obeying $f=f^A$ for all $A\in SU(2)$ is a linear combination of $\delta_{\sigma_1,\sigma_2}\delta_{\sigma_3,\sigma_4}$ and $\delta_{\sigma_1,\sigma_3}\delta_{\sigma_2,\sigma_4}-\frac12\delta_{\sigma_1,\sigma_2}\delta_{\sigma_3,\sigma_4}$. ◻

Remark II.9.

$$
\begin{aligned}
f_C&=f\big((\cdot,\uparrow),(\cdot,\uparrow),(\cdot,\uparrow),(\cdot,\uparrow)\big)+f\big((\cdot,\uparrow),(\cdot,\uparrow),(\cdot,\downarrow),(\cdot,\downarrow)\big)\\
&=f\big((\cdot,\downarrow),(\cdot,\downarrow),(\cdot,\downarrow),(\cdot,\downarrow)\big)+f\big((\cdot,\downarrow),(\cdot,\downarrow),(\cdot,\uparrow),(\cdot,\uparrow)\big),\\
f_S&=f\big((\cdot,\uparrow),(\cdot,\downarrow),(\cdot,\uparrow),(\cdot,\downarrow)\big)=f\big((\cdot,\downarrow),(\cdot,\uparrow),(\cdot,\downarrow),(\cdot,\uparrow)\big).
\end{aligned}
$$

Lemma II.10. *If* K *is a spin independent function on* $(\mathfrak{Z}\times\{\uparrow,\downarrow\})^4$, *then*

$$
\big(K^f\big)_C=\tfrac12\big(K_C+3K_S\big)^f,\qquad \big(K^f\big)_S=\tfrac12\big(K_C-K_S\big)^f,
$$

where K^f *is the flipped function of (I.5).*

Proof.

$$
\begin{aligned}
&K^f\big((z_1,\sigma_1),(z_2,\sigma_2),(z_3,\sigma_3),(z_4,\sigma_4)\big)\\
&\quad=-K\big((z_1,\sigma_1),(z_3,\sigma_3),(z_2,\sigma_2),(z_4,\sigma_4)\big)\\
&\quad=-\tfrac12 K_C(z_1,z_3,z_2,z_4)\delta_{\sigma_1,\sigma_3}\delta_{\sigma_2,\sigma_4}-K_S(z_1,z_3,z_2,z_4)\\
&\qquad\times\big[\delta_{\sigma_1,\sigma_2}\delta_{\sigma_3,\sigma_4}-\tfrac12\delta_{\sigma_1,\sigma_3}\delta_{\sigma_2,\sigma_4}\big]\\
&\quad=K_S^f(z_1,z_2,z_3,z_4)\delta_{\sigma_1,\sigma_2}\delta_{\sigma_3,\sigma_4}+\tfrac12\big(K_C^f-K_S^f\big)(z_1,z_2,z_3,z_4)\delta_{\sigma_1,\sigma_3}\delta_{\sigma_2,\sigma_4}\\
&\quad=\tfrac14\big(K_C^f+3K_S^f\big)(z_1,z_2,z_3,z_4)\delta_{\sigma_1,\sigma_2}\delta_{\sigma_3,\sigma_4}\\
&\quad+\tfrac12\big(K_C^f-K_S^f\big)(z_1,z_2,z_3,z_4)\big[\delta_{\sigma_1,\sigma_3}\delta_{\sigma_2,\sigma_4}-\tfrac12\delta_{\sigma_1,\sigma_2}\delta_{\sigma_3,\sigma_4}\big].\quad\square
\end{aligned}
$$

Lemma II.11. *If H' and K' are spin independent functions on $(\mathfrak{Y}^{\updownarrow}_{\Sigma})^4$ and P is a bubble propagator, then*

$$(H' \bullet P \bullet K')_C = H'_C \bullet P \bullet K'_C \quad (H' \bullet P \bullet K')_S = H'_S \bullet P \bullet K'_S.$$

Proof. This lemma follows directly from Remark II.9. □

Parts (ii) and (iii) of Prop. II.3, Lemma II.10 and Lemma II.11 give a coupled system of recursion relations for $\mathcal{L}_C^{(j)}$, $\mathcal{L}_S^{(j)}$, $L_C^{(j)}$ and $L_S^{(j)}$.

Corollary II.12.

$$i)\ \mathcal{L}_C^{(j+1)} = \sum_{\ell=1}^{\infty} \sum_{i_1,\cdots,i_{\ell+1}=2}^{j} \Big[\big(F_C^{(i_1)} + \tfrac{1}{2}L_C^{(i_1)\,f} + \tfrac{3}{2}L_S^{(i_1)\,f}\big) \bullet \mathcal{C}^{[\max\{i_1,i_2\},j]} \bullet \cdots \bullet \mathcal{C}^{[\max\{i_\ell,i_{\ell+1}\},j]} \bullet \big(F_C^{(i_{\ell+1})} + \tfrac{1}{2}L_C^{(i_{\ell+1})\,f} + \tfrac{3}{2}L_S^{(i_{\ell+1})\,f}\big)\Big]_{\Sigma_j},$$

$$\mathcal{L}_S^{(j+1)} = \sum_{\ell=1}^{\infty} \sum_{i_1,\cdots,i_{\ell+1}=2}^{j} \Big[\big(F_S^{(i_1)} + \tfrac{1}{2}L_C^{(i_1)\,f} - \tfrac{1}{2}L_S^{(i_1)\,f}\big) \bullet \mathcal{C}^{[\max\{i_1,i_2\},j]} \bullet \cdots \bullet \mathcal{C}^{[\max\{i_\ell,i_{\ell+1}\},j]} \bullet \big(F_S^{(i_{\ell+1})} + \tfrac{1}{2}L_C^{(i_{\ell+1})\,f} - \tfrac{1}{2}L_S^{(i_{\ell+1})\,f}\big)\Big]_{\Sigma_j}.$$

$$ii)\ L_C^{(j+1)} = \Big(\sum_{i=2}^{j} F^{(i)}_{C\,\Sigma_j} + \tfrac{1}{2}\mathcal{L}^{(j)\,f}_{C\,\Sigma_j} + \tfrac{3}{2}\mathcal{L}^{(j)\,f}_{S\,\Sigma_j} + \mathcal{L}^{(j)}_{C\,\Sigma_j}\Big) \bullet \mathcal{C}^{(j)} \bullet\Big(\sum_{i=2}^{j} F^{(i)}_{C\,\Sigma_j} + \tfrac{1}{2}\mathcal{L}^{(j)\,f}_{C\,\Sigma_j} + \tfrac{3}{2}\mathcal{L}^{(j)\,f}_{S\,\Sigma_j} + \mathcal{L}_C^{(j+1)}\Big),$$

$$L_S^{(j+1)} = \Big(\sum_{i=2}^{j} F^{(i)}_{S\,\Sigma_j} + \tfrac{1}{2}\mathcal{L}^{(j)\,f}_{C\,\Sigma_j} - \tfrac{1}{2}\mathcal{L}^{(j)\,f}_{S\,\Sigma_j} + \mathcal{L}^{(j)}_{S\,\Sigma_j}\Big) \bullet \mathcal{C}^{(j)} \bullet\Big(\sum_{i=2}^{j} F^{(i)}_{S\,\Sigma_j} + \tfrac{1}{2}\mathcal{L}^{(j)\,f}_{C\,\Sigma_j} - \tfrac{1}{2}\mathcal{L}^{(j)\,f}_{S\,\Sigma_j} + \mathcal{L}_S^{(j+1)}\Big).$$

Theorem I.20 will be proven by bounding each term on the right-hand side of Cor. II.12.i. Each such term is a particle–hole ladder of the form

$$\big(G^{(i_1)} + K^{(i_1)\,f}\big) \bullet \mathcal{C}^{[\max\{i_1,i_2\},j]} \bullet \cdots \bullet \mathcal{C}^{[\max\{i_\ell,i_{\ell+1}\},j]} \bullet \big(G^{(i_{\ell+1})} + K^{(i_{\ell+1})\,f}\big),$$

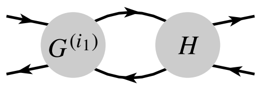
where $G^{(i)}$ is either $F_C^{(i)}$ or $F_S^{(i)}$ and $K^{(i)}$ is a linear combination of $L_C^{(i)}$ and $L_S^{(i)}$. This ladder has rungs $\big(G^{(i_\nu)} + K^{(i_\nu)\,f}\big)$ which are connected by particle–hole propagators $\mathcal{C}^{[i,j]}$. The induction step will consist in adding an additional rung to the left of the ladder. More precisely, we will prove a bound on

$$\big(G^{(i_1)} + K^{(i_1)\,f}\big) \bullet \mathcal{C}^{[i,j]} \bullet H$$

with $H = \big(G^{(i_2)} + K^{(i_2)\,f}\big) \bullet \mathcal{C}^{[\max\{i_2,i_3\},j]} \bullet \cdots \bullet \big(G^{(i_{\ell+1})} + K^{(i_{\ell+1})\,f}\big)$, assuming bounds on H. The expression

$$G^{(i_1)} \bullet \mathcal{C}^{[i,j]} \bullet H$$

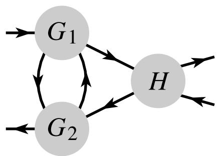
is a particle–hole bubble



We will derive the necessary bounds on general particle–hole bubbles in Theorem II.19. By Corollary II.12.ii,

$$K^{(i_1)\,f} \bullet \mathcal{C}^{[i,j]} \bullet H = \Big(G_1^{(i_1)} \bullet \mathcal{C}^{(i_1-1)} \bullet G_2^{(i_1)} \Big)^f \bullet \mathcal{C}^{[i,j]} \bullet H,$$

with $G_1^{(i)}$ and $G_2^{(i)}$ linear combinations of $\sum\limits_{k=2}^{i-1} F_C^{(k)}$, $\sum\limits_{k=2}^{i-1} F_S^{(k)}$, $\mathcal{L}_C^{(i-1)}$, $\mathcal{L}_S^{(i-1)}$, $\mathcal{L}_C^{(i-1)\,f}$, $\mathcal{L}_S^{(i-1)\,f}$, $\mathcal{L}_C^{(i)}$, $\mathcal{L}_S^{(i)}$. It is a double bubble



Bounds on double bubbles will be obtained in Theorem II.20.

3. Scaled Norms. In the induction procedure outlined above the various ladders naturally have different sectorization scales at their left and right hand ends. This was the motivation for Def. I.14.

Convention II.13. *Introduce, for scales ℓ, r, the short hand notation*

$$\mathfrak{Y}_{\ell,r} = \mathfrak{Y}^{(4)}_{\Sigma_\ell,\Sigma_r}.$$

Definition II.14. *For a function f on $\mathfrak{Y}_{\ell,r}$ and multiindices $\delta_{\rm l}, \delta_{\rm c}, \delta_{\rm r} \in \mathbb{N}_0 \times \mathbb{N}_0^2$, set*

$$\begin{aligned} \|f\|_{\ell,r}^{(\delta_{\rm l},\delta_{\rm c},\delta_{\rm r})} &= \tfrac{1}{M^{\ell|\delta_{\rm l}|+|\delta_{\rm c}|\max(\ell,r)+r|\delta_{\rm r}|}} |f|_{\Sigma_\ell,\Sigma_r}^{(\delta_{\rm l},\delta_{\rm c},\delta_{\rm r})}, \\ |f|_{\ell,r}^{[\delta_{\rm l},\delta_{\rm c},\delta_{\rm r}]} &= \max_{\substack{\delta'_{\rm l}\le\delta_{\rm l}\\ \delta'_{\rm c}\le\delta_{\rm c}\\ \delta'_{\rm r}\le\delta_{\rm r}}} \|f\|_{\ell,r}^{(\delta'_{\rm l},\delta'_{\rm c},\delta'_{\rm r})}. \end{aligned}$$

The norm $|\,\cdot\,|_{\Sigma_\ell,\Sigma_r}^{(\delta_{\rm l},\delta_{\rm c},\delta_{\rm r})}$ was defined in Def. I.14. If $\ell = r = j$, set

$$|f|_j^{[\![\delta]\!]} = \max_{\substack{\delta_{\rm l},\delta_{\rm c},\delta_{\rm r}\in\mathbb{N}_0\times\mathbb{N}_0^2\\ \delta_{\rm l}+\delta_{\rm c}+\delta_{\rm r}\le\delta}} \|f\|_{j,j}^{(\delta_{\rm l},\delta_{\rm c},\delta_{\rm r})}.$$

Set

$$\begin{aligned} \Delta &= \big\{\, \delta \in \mathbb{N}_0 \times \mathbb{N}_0^2 \;\big|\; \delta_0 \le r_0,\ \delta_1 + \delta_2 \le r_e \,\big\}, \\ \vec{\Delta} &= \big\{\, \vec{\delta} = (\delta_{\rm l}, \delta_{\rm c}, \delta_{\rm r}) \in \big(\mathbb{N}_0 \times \mathbb{N}_0^2\big)^3 \;\big|\; \delta_{\rm l} + \delta_{\rm c} + \delta_{\rm r} \in \Delta \,\big\}, \end{aligned} \tag{II.1}$$

where $r_e + 3$ is the degree of differentiability of the dispersion relation $e(\mathbf{k})$ and r_0 is the number of k_0 derivatives that we wish to control. The numbers r_e and r_0 also determine the number of finite coefficients in the formal power series $\mathfrak{c}_j$ of (I.4). The following remark relates the formal power series norms of Def. I.15.ii to the norms of Def. II.14.

Remark II.15. There is a constant const, depending only on r_e and r_0 such that the following holds. Let f be a sectorized, translation invariant function on $\mathfrak{Y}^4_{\Sigma_j}$.

i)

$$|\!|\!| f |\!|\!|_{\Sigma_j} \le \Big[\max_{\delta\in\Delta} |f|_j^{[\![\delta]\!]} \Big]\, \mathfrak{c}_j .$$

ii) If there is a number γ such that $|\!|\!| f |\!|\!|_{\Sigma_j} \le \gamma\, \mathfrak{c}_j$, then

$$|f|_j^{[\![\delta]\!]} \le \mathrm{const}\,\gamma \qquad \text{for all } \delta \in \Delta .$$

Thus to prove Theorem I.20, it suffices to prove that

$$\max_{\delta\in\Delta} |\mathcal{L}_C^{(j+1)}|_j^{[\![\delta]\!]} \le \mathrm{const}\,\rho^2 \qquad \max_{\delta\in\Delta} |\mathcal{L}_S^{(j+1)}|_j^{[\![\delta]\!]} \le \mathrm{const}\,\rho^2 .$$

Definition II.16 (Norms and resectorization). *Let $\ell, \ell', r, r' \ge 0$. For a sectorized, translation invariant, function f on $\mathfrak{Y}_{\ell',r'}$ and multiindices $\vec{\delta} \in \big(\mathbb{N}_0 \times \mathbb{N}_0^2\big)^3$, set*

$$|f|_{\ell,r}^{[\vec{\delta}]} = \big|f_{\Sigma_\ell,\Sigma_r}\big|_{\ell,r}^{[\vec{\delta}]} .$$

If $\ell = r = j$ and $\delta \in \mathbb{N}_0 \times \mathbb{N}_0^2$, set

$$|f|_j^{[\![\delta]\!]} = \max_{\substack{\delta_{\rm l},\delta_{\rm c},\delta_{\rm r} \in \mathbb{N}_0\times\mathbb{N}_0^2 \\ \delta_{\rm l}+\delta_{\rm c}+\delta_{\rm r}\le\delta}} |f|_{j,j}^{[\delta_{\rm l},\delta_{\rm c},\delta_{\rm r}]} .$$

As in Prop. XIX.4 of [FKTo4], one proves the following lemma, which gives the effect of changing the scale indices on the norm $|\cdot|_{\ell,r}^{[\vec{\delta}]}$.

Lemma II.17. *Let $\ell \ge \ell' \ge 1$ and $r \ge r' \ge 1$. Let f be a sectorized, translation invariant, function on $\mathfrak{Y}_{\ell',r'}$ and let $\vec{\delta} = (\delta_{\rm l}, \delta_{\rm c}, \delta_{\rm r}) \in \vec{\Delta}$. Then*

$$\begin{aligned} |f|_{\ell,r}^{[\vec{\delta}]} &\le \mathrm{const}\Big\{ \tfrac{1}{M^{\ell-\ell'}}\tfrac{1}{M^{r-r'}} |f|_{\ell',r'}^{[\vec{\delta}]} + \tfrac{1}{M^{\ell-\ell'}} |f|_{\ell',r'}^{[\delta_{\rm l},\delta_{\rm c},0]} + \tfrac{1}{M^{r-r'}} |f|_{\ell',r'}^{[0,\delta_{\rm c},\delta_{\rm r}]} + |f|_{\ell',r'}^{[0,\delta_{\rm c},0]} \Big\} \\ &\le \mathrm{const} |f|_{\ell',r'}^{[\vec{\delta}]} . \end{aligned}$$

The constant const depends only on Δ.

Proof. Let f be a function on $\mathfrak{Y}_{i_1,\Sigma_{\ell'}} \times \mathfrak{Y}_{i_2,\Sigma_{\ell'}} \times \mathfrak{Y}_{i_3,\Sigma r'} \times \mathfrak{Y}_{i_4,\Sigma_{r'}}$. We consider the case $i_1 = i_2 = i_3 = i_4 = 1$ and $\ell' < \ell$, $r' < r$. The other cases are similar, but easier. Recall from Def. I.17 that,

$$\begin{aligned} &f_{\Sigma_\ell,\Sigma_r}\big((x_1,s_1),(x_2,s_2),(x_3,s_3),(x_4,s_4)\big) \\ &\quad = \sum_{\substack{s'_\nu\in\Sigma_{\ell'}\\ \nu\in\{1,2\}}} \sum_{\substack{s'_\nu\in\Sigma_{r'}\\ \nu\in\{3,4\}}} \int \prod_{\nu=1}^4 \Big(dx'_\nu\ \hat{\chi}_{s_\nu}\big((-1)^{b_\nu}(x_\nu - x'_\nu)\big)\Big)\ f\big((x'_1,s'_1),(x'_2,s'_2),(x'_3,s'_3),(x'_4,s'_4)\big). \end{aligned}$$

First observe that, for any fixed $s_1, \cdots, s_4$, there are at most 3^4 choices of $(s'_1, \cdots, s'_4)$ for which the integral $\int \prod_{\nu=1}^4 \Big(dx'_\nu\ \hat{\chi}_{s_\nu}(\cdots)\Big)\ f(\cdots)$ fails to vanish identically, because

f is sectorized and $\ell' < \ell$, $r' < r$. So it suffices to consider any fixed $s'_1, \cdots, s'_4$. Hence by Leibniz's Rule (Lemma II.21), $\| f_{\Sigma_\ell,\Sigma_r}\|_{\ell,r}^{(\delta_{\rm l},\delta_{\rm c},\delta_{\rm r})}$ is bounded by a constant, which depends only on Δ, times the maximum of

$$\begin{aligned}
&\tfrac{1}{M^{\ell|\delta_{\rm l}|+|\delta_{\rm c}|\max(\ell,r)+r|\delta_{\rm r}|}} \int \prod_{\nu=2}^4 dx_\nu \int \prod_{\nu=1}^4 \Big(dx'_\nu \,\big|(x_\nu - x'_\nu)^{\beta_\nu}\,\hat\chi_{s_\nu}\big((-1)^{b_\nu}(x_\nu - x'_\nu)\big)\big|\Big) \\
&\qquad \times \big|{\rm D}_{1;2}^{\alpha_{\rm l}} {\rm D}_{\mu;\mu'}^{\alpha_{\rm c}} {\rm D}_{3;4}^{\alpha_{\rm r}} f\big((x'_1,s'_1),(x'_2,s'_2),(x'_3,s'_3),(x'_4,s'_4)\big)\big| \\
&\qquad \le \tfrac{1}{M^{\ell|\delta_{\rm l}|+|\delta_{\rm c}|\max(\ell,r)+r|\delta_{\rm r}|}} \Big(\prod_{\nu=1}^4 \big\|x_\nu^{\beta_\nu}\hat\chi_{s_\nu}(x_\nu)\big\|_{L^1}\Big) \big| f\big|_{\Sigma_{\ell'},\Sigma_{r'}}^{(\alpha_{\rm l},\alpha_{\rm c},\alpha_{\rm r})} \\
&\qquad = \tfrac{M^{\ell'|\alpha_{\rm l}|+|\alpha_{\rm c}|\max(\ell',r')+r'|\alpha_{\rm r}|}}{M^{\ell|\delta_{\rm l}|+|\delta_{\rm c}|\max(\ell,r)+r|\delta_{\rm r}|}} \Big(\prod_{\nu=1}^4 \big\|x_\nu^{\beta_\nu}\hat\chi_{s_\nu}(x_\nu)\big\|_{L^1}\Big) \big\| f\big\|_{\Sigma_{\ell'},\Sigma_{r'}}^{(\alpha_{\rm l},\alpha_{\rm c},\alpha_{\rm r})}
\end{aligned}$$

over $x_1, s_1, \cdots, s_4, s'_1, \cdots, s'_4$ and $\mu \in \{1, 2\}$, $\mu' \in \{3, 4\}$ and $\alpha_{\rm l}, \alpha_{\rm c}, \alpha_{\rm r}$ and

$$\beta_\nu = \beta_{\nu,{\rm l}} + \beta_{\nu,{\rm c}} + \beta_{\nu,{\rm r}}, \qquad \nu = 1, \cdots, 4$$

obeying

$$\begin{aligned}
&\beta_{1,{\rm l}} + \alpha_{\rm l} + \beta_{2,{\rm l}} = \delta_{\rm l}, \quad \beta_{\mu,{\rm c}} + \alpha_{\rm c} + \beta_{\mu',{\rm c}} = \delta_{\rm c}, \quad \beta_{3,{\rm r}} + \alpha_{\rm r} + \beta_{4,{\rm r}} = \delta_{\rm r}, \\
&\beta_{1,{\rm r}} = \beta_{2,{\rm r}} = \beta_{3,{\rm l}} = \beta_{4,{\rm l}} = \beta_{\nu,{\rm c}} = 0 \qquad \text{for } \nu \ne \mu, \mu'.
\end{aligned}$$

In particular

$$\ell|\delta_{\rm l}| + |\delta_{\rm c}|\max(\ell, r) + r|\delta_{\rm r}| \ge \ell|\alpha_{\rm l} + \beta_1 + \beta_2| + |\alpha_{\rm c}|\max(\ell, r) + r|\alpha_{\rm r} + \beta_3 + \beta_4|.$$

By Lemma XII.3 of [FKTo3],

$$\big\|x_\nu^{\beta_\nu}\hat\chi_{s_\nu}(x_\nu)\big\|_{L^1} \le const \begin{cases} M^{|\beta_\nu|\ell} & \text{if } \nu \in \{1,2\} \\ M^{|\beta_\nu| r} & \text{if } \nu \in \{3,4\} \end{cases} \tag{II.2}$$

so that

$$\begin{aligned}
&\tfrac{M^{\ell'|\alpha_{\rm l}|+|\alpha_{\rm c}|\max(\ell',r')+r'|\alpha_{\rm r}|}}{M^{\ell|\delta_{\rm l}|+|\delta_{\rm c}|\max(\ell,r)+r|\delta_{\rm r}|}} \prod_{\nu=1}^4 \big\|x_\nu^{\beta_\nu}\hat\chi_{s_\nu}(x_\nu)\big\|_{L^1} \\
&\qquad \le const\, \tfrac{M^{\ell'|\alpha_{\rm l}|+|\alpha_{\rm c}|\max(\ell',r')+r'|\alpha_{\rm r}|}}{M^{\ell|\delta_{\rm l}|+|\delta_{\rm c}|\max(\ell,r)+r|\delta_{\rm r}|}} M^{\ell|\beta_1+\beta_2|+r|\beta_3+\beta_4|} \\
&\qquad \le const\, \tfrac{1}{M^{(\ell-\ell')|\alpha_{\rm l}|+(r-r')|\alpha_{\rm r}|}}
\end{aligned}$$

and

$$\| f_{\Sigma_\ell,\Sigma_r}\|_{\ell,r}^{(\delta_{\rm l},\delta_{\rm c},\delta_{\rm r})} \le const \max_{\substack{\alpha_{\rm l}\le\delta_{\rm l}\\ \alpha_{\rm c}\le\delta_{\rm c}\\ \alpha_{\rm r}\le\delta_{\rm r}}} \tfrac{1}{M^{(\ell-\ell')|\alpha_{\rm l}|+(r-r')|\alpha_{\rm r}|}} \big\| f\big\|_{\Sigma_{\ell'},\Sigma_{r'}}^{(\alpha_{\rm l},\alpha_{\rm c},\alpha_{\rm r})},$$

and the lemma follows. □

4. Bubble and Double Bubble Bounds. We now formulate the bounds on bubbles and double bubbles that form the core of the proof of Theorem I.20.

Definition II.18. *Let $i \le j$. Then*

$$\mathcal{C}^{[i,j]} = \mathcal{C}^{[i,j]}_{\rm top} + \mathcal{C}^{[i,j]}_{\rm mid} + \mathcal{C}^{[i,j]}_{\rm bot},$$

where

$$\mathcal{C}^{[i,j]}_{\rm top} = \sum_{\substack{i \le i_t \le j \\ i_b > j}} C_v^{(i_t)} \otimes C_v^{(i_b)\,t},$$

$$\mathcal{C}^{[i,j]}_{\rm mid} = \sum_{\substack{i \le i_t \le j \\ i \le i_b \le j}} C_v^{(i_t)} \otimes C_v^{(i_b)\,t},$$

$$\mathcal{C}^{[i,j]}_{\rm bot} = \sum_{\substack{i_t > j \\ i \le i_b \le j}} C_v^{(i_t)} \otimes C_v^{(i_b)\,t}.$$

Theorem II.19 (Bubble bound). *Let $1 \le i, \ell \le j$ and $\delta_{\rm l}, \delta_{\rm r} \in \Delta$. Let g and h be sectorized, translation invariant functions on $\mathfrak{Y}_{\ell,i}$ and $\mathfrak{Y}_{i,j}$ respectively. Then*

a)

$$\big|g \bullet \mathcal{C}^{[i,j]} \bullet h\big|^{[\delta_{\rm l},0,\delta_{\rm r}]}_{\ell,j} \le \text{const } i \max_{\substack{\alpha_{\rm r},\alpha_{\rm l} \in \mathbb{N}_0 \times \mathbb{N}_0^2 \\ |\alpha_{\rm r}|+|\alpha_{\rm l}| \le 3}} \big|g\big|^{[\delta_{\rm l},0,\alpha_{\rm r}]}_{\ell,i} \big|h\big|^{[\alpha_{\rm l},0,\delta_{\rm r}]}_{i,j}.$$

b) For any $\beta \in \Delta$,

$$\tfrac{1}{M^{|\beta|j}} \big\| g \bullet {\rm D}^{\beta}_{1;3} \mathcal{C}^{[i,j]}_{\rm top} \bullet h \big\|^{(\delta_{\rm l},0,\delta_{\rm r})}_{\ell,j} \le \text{const } \big\|g\big\|^{(\delta_{\rm l},0,0)}_{\ell,i} \big\|h\big\|^{(0,0,\delta_{\rm r})}_{i,j},$$

$$\tfrac{1}{M^{|\beta|j}} \big\| g \bullet {\rm D}^{\beta}_{2;4} \mathcal{C}^{[i,j]}_{\rm bot} \bullet h \big\|^{(\delta_{\rm l},0,\delta_{\rm r})}_{\ell,j} \le \text{const } \big\|g\big\|^{(\delta_{\rm l},0,0)}_{\ell,i} \big\|h\big\|^{(0,0,\delta_{\rm r})}_{i,j}.$$

c)

$$\big\| g \bullet \mathcal{C}^{[i,j]}_{\rm mid} \bullet h \big\|^{(\delta_{\rm l},0,\delta_{\rm r})}_{\ell,j} \le \text{const}\,|j-i+1|\,\big\|g\big\|^{(\delta_{\rm l},0,0)}_{\ell,i} \big\|h\big\|^{(0,0,\delta_{\rm r})}_{i,j},$$

and for any $\beta \in \Delta$ with $|\beta| \ge 1$ and $(\mu, \mu') = (1,3), (2,4)$,

$$\tfrac{1}{M^{|\beta|j}} \big\| g \bullet {\rm D}^{\beta}_{\mu;\mu'} \mathcal{C}^{[i,j]}_{\rm mid} \bullet h \big\|^{(\delta_{\rm l},0,\delta_{\rm r})}_{\ell,j} \le \text{const } \big\|g\big\|^{(\delta_{\rm l},0,0)}_{\ell,i} \big\|h\big\|^{(0,0,\delta_{\rm r})}_{i,j}.$$

This theorem is proven in §III.

Theorem II.20 (Double bubble bound). *Let $1 \le \ell \le i \le j$, $\nu \in \mathbb{N}_0 \times \mathbb{N}_0^2$ and $\delta_{\rm l}, \delta_{\rm r} \in \Delta$. Let g_1, g_2 and h be sectorized, translation invariant functions on $\mathfrak{Y}_{\ell,\ell}$, $\mathfrak{Y}_{\ell,\ell}$ and $\mathfrak{Y}_{i,j}$ respectively. Let $\mathcal{D}$ be either*

$$\mathcal{D}^{(\ell)}_{\nu,{\rm up}}(x_1,x_2,x_3,x_4) = \tfrac{1}{M^{|\nu|\ell}} \sum_{m=\ell}^{\infty} {\rm D}^{\nu}_{1;3} C_v^{(\ell)}(x_1,x_3) C_v^{(m)}(x_4,x_2)$$

or

$$\mathcal{D}^{(\ell)}_{\nu,{\rm dn}}(x_1,x_2,x_3,x_4) = \tfrac{1}{M^{|\nu|\ell}} \sum_{m=\ell+1}^{\infty} C_v^{(m)}(x_1,x_3) {\rm D}^{\nu}_{2;4} C_v^{(\ell)}(x_4,x_2).$$

a) If $\nu + \delta_{\rm l} + \alpha \in \Delta$ for all $|\alpha| \le 3$, then

$$\big|(g_1 \bullet \mathcal{D} \bullet g_2)^f \bullet \mathcal{C}^{[i,j]} \bullet h\big|_{\ell,j}^{[\delta_{\rm l},0,\delta_{\rm r}]} \le \text{const } i\,\sqrt{\mathfrak{l}_\ell} \max_{\substack{\alpha_{\rm up},\alpha_{\rm dn},\alpha_{\rm l} \in \mathbb{N}_0\times\mathbb{N}_0^2 \\ |\alpha_{\rm up}|+|\alpha_{\rm dn}|+|\alpha_{\rm l}|\le 3}} \big|g_1\big|_\ell^{[\![\delta_{\rm l}+\alpha_{\rm up}]\!]} \times \big|g_2\big|_\ell^{[\![\delta_{\rm l}+\alpha_{\rm dn}]\!]} \big|h\big|_{i,j}^{[\alpha_{\rm l},0,\delta_{\rm r}]}.$$

b) If $\nu + \delta_{\rm l} \in \Delta$, then for any $\beta \in \Delta$,

$$\tfrac{1}{M^{|\beta|j}} \big\|(g_1 \bullet \mathcal{D} \bullet g_2)^f \bullet \mathrm{D}^\beta_{1;3}\mathcal{C}^{[i,j]}_{\rm top} \bullet h\big\|_{\ell,j}^{(\delta_{\rm l},0,\delta_{\rm r})} \le \text{const}\sqrt{\mathfrak{l}_\ell}\big|g_1\big|_{\ell,\ell}^{[0,\delta_{\rm l},0]}\big|g_2\big|_{\ell,\ell}^{[0,\delta_{\rm l},0]}\big\|h\big\|_{i,j}^{(0,0,\delta_{\rm r})},$$

$$\tfrac{1}{M^{|\beta|j}} \big\|(g_1 \bullet \mathcal{D} \bullet g_2)^f \bullet \mathrm{D}^\beta_{2;4}\mathcal{C}^{[i,j]}_{\rm bot} \bullet h\big\|_{\ell,j}^{(\delta_{\rm l},0,\delta_{\rm r})} \le \text{const}\sqrt{\mathfrak{l}_\ell}\big|g_1\big|_{\ell,\ell}^{[0,\delta_{\rm l},0]}\big|g_2\big|_{\ell,\ell}^{[0,\delta_{\rm l},0]}\big\|h\big\|_{i,j}^{(0,0,\delta_{\rm r})}.$$

c) If $\nu + \delta_{\rm l} \in \Delta$, then

$$\big\|(g_1 \bullet \mathcal{D} \bullet g_2)^f \bullet \mathcal{C}^{[i,j]}_{\rm mid} \bullet h\big\|_{\ell,j}^{(\delta_{\rm l},0,\delta_{\rm r})} \le \text{const}\,|j-i+1|\sqrt{\mathfrak{l}_\ell}\big|g_1\big|_{\ell,\ell}^{[0,\delta_{\rm l},0]}\big|g_2\big|_{\ell,\ell}^{[0,\delta_{\rm l},0]}\big\|h\big\|_{i,j}^{(0,0,\delta_{\rm r})},$$

and for any $\beta \in \Delta$ with $|\beta| \ge 1$ and $(\mu,\mu') = (1,3),\,(2,4)$,

$$\tfrac{1}{M^{|\beta|j}} \big\|(g_1 \bullet \mathcal{D} \bullet g_2)^f \bullet \mathrm{D}^\beta_{\mu;\mu'}\mathcal{C}^{[i,j]}_{\rm mid} \bullet h\big\|_{\ell,j}^{(\delta_{\rm l},0,\delta_{\rm r})} \le \text{const}\sqrt{\mathfrak{l}_\ell}\big|g_1\big|_{\ell,\ell}^{[0,\delta_{\rm l},0]}\big|g_2\big|_{\ell,\ell}^{[0,\delta_{\rm l},0]}\big\|h\big\|_{i,j}^{(0,0,\delta_{\rm r})}.$$

This theorem is proven in §IV.

Remark. Observe that

$$\mathcal{C}^{(\ell)} = \mathcal{D}^{(\ell)}_{0,\rm up} + \mathcal{D}^{(\ell)}_{0,\rm dn}.$$

We use Leibniz's rule to convert Theorems II.19 and II.20 into bounds on derivatives of $g \bullet \mathcal{C}^{[i,j]} \bullet h$ and $(g_1 \bullet \mathcal{C}^{(\ell)} \bullet g_2)^f \bullet \mathcal{C}^{[i,j]} \bullet h$ with respect to transfer momenta. These bounds are stated in Corollaries II.22, II.23 and II.24, below.

Lemma II.21 (Leibniz's Rule). *Let $\ell_1, r_1, \ell_2, r_2 \ge 1$, P be a bubble propagator and K_1, K_2 be sectorized, translation invariant functions on $\mathfrak{Y}_{\ell_1,r_1}$ and $\mathfrak{Y}_{\ell_2,r_2}$, respectively. Let $\mu, \nu \in \{1,2\}$, $\mu', \nu' \in \{3,4\}$ and $\delta \in \mathbb{N}_0 \times \mathbb{N}_0^2$. Then,*

$$\mathrm{D}^\delta_{\nu;\nu'}(K_1 \bullet P \bullet K_2) = \sum_{\substack{\beta_1,\beta_2,\beta_3 \in \mathbb{N}_0\times\mathbb{N}_0^2 \\ \beta_1+\beta_2+\beta_3=\delta}} \tbinom{\delta}{\beta_1,\beta_2,\beta_3} (\mathrm{D}^{\beta_1}_{\nu;\mu+2}K_1) \bullet (\mathrm{D}^{\beta_2}_{\mu;\mu'}P) \bullet (\mathrm{D}^{\beta_3}_{\mu'-2;\nu'}K_2).$$

Here $\binom{\delta}{\beta_1,\beta_2,\beta_3} = \frac{\delta!}{\beta_1!\beta_2!\beta_3!}$.

Proof. The proof is trivial. □

Corollary II.22. *Let $1 \le \ell \le i \le j$ and $\delta_{\rm l}, \delta_{\rm c}, \delta_{\rm r} \in \Delta$. Let g and h be sectorized, translation invariant functions on $\mathfrak{Y}_{\ell,\ell}$ and $\mathfrak{Y}_{i,j}$ respectively.*

a)

$$\big\| g \bullet \mathcal{C}^{[i,j]}_{\rm top} \bullet h \big\|^{(\delta_{\rm l},\delta_{\rm c},\delta_{\rm r})}_{\ell,j} \le {\rm const}\, \big| g \big|^{[\delta_{\rm l},\delta_{\rm c},0]}_{\ell,\ell} \big| h \big|^{[0,\delta_{\rm c},\delta_{\rm r}]}_{i,j},$$
$$\big\| g \bullet \mathcal{C}^{[i,j]}_{\rm bot} \bullet h \big\|^{(\delta_{\rm l},\delta_{\rm c},\delta_{\rm r})}_{\ell,j} \le {\rm const}\, \big| g \big|^{[\delta_{\rm l},\delta_{\rm c},0]}_{\ell,\ell} \big| h \big|^{[0,\delta_{\rm c},\delta_{\rm r}]}_{i,j}.$$

b) For $\mu \in \{1,2\}$ and $\mu' \in \{3,4\}$,

$$\mathrm{D}^{\delta_{\rm c}}_{\mu;\mu'}\big(g \bullet \mathcal{C}^{[i,j]}_{\rm mid} \bullet h\big) = \sum_{\substack{\beta_1,\beta_2,\beta_3 \in \mathbb{N}_0\times\mathbb{N}_0^2 \\ \beta_1+\beta_2+\beta_3=\delta_c}} \tbinom{\delta_c}{\beta_1,\beta_2,\beta_3} \mathrm{D}^{\beta_1}_{\mu;3} g \bullet \mathrm{D}^{\beta_2}_{1;3}\mathcal{C}^{[i,j]}_{\rm mid} \bullet \mathrm{D}^{\beta_3}_{1;\mu'} h,$$

and, for all $\beta_1+\beta_2+\beta_3=\delta_{\rm c}$,

$$\begin{aligned} &\tfrac{1}{M^{|\delta_{\rm c}|j}} \big\| \mathrm{D}^{\beta_1}_{\mu;3} g \bullet \mathrm{D}^{\beta_2}_{1;3}\mathcal{C}^{[i,j]}_{\rm mid} \bullet \mathrm{D}^{\beta_3}_{1;\mu'} h \big\|^{(\delta_{\rm l},0,\delta_{\rm r})}_{\ell,j} \\ &\le \tfrac{\rm const}{M^{|\beta_1|(j-\ell)}} \begin{cases} (j-i+1) \big\| g \big\|^{(\delta_{\rm l},\beta_1,0)}_{\ell,\ell} \big\| h \big\|^{(0,\beta_3,\delta_{\rm r})}_{i,j} & \text{if } \beta_2 = 0 \\ i \max\limits_{|\alpha_{\rm r}+\alpha_{\rm l}|\le 3} \big| g \big|^{[\delta_{\rm l},\beta_1,\alpha_{\rm r}]}_{\ell,\ell} \big| h \big|^{[\alpha_{\rm l},\beta_3,\delta_{\rm r}]}_{i,j} & \text{if } \beta_2 = 0 \\ \big\| g \big\|^{(\delta_{\rm l},\beta_1,0)}_{\ell,\ell} \big\| h \big\|^{(0,\beta_3,\delta_{\rm r})}_{i,j} & \text{if } \beta_2 \ne 0 \end{cases}. \end{aligned}$$

Proof. a) We consider the case of top. By Leibniz,

$$\mathrm{D}^{\delta_c}_{\mu;\mu'}\big(g \bullet \mathcal{C}^{[i,j]}_{\rm top} \bullet h\big) = \sum_{\substack{\beta_1,\beta_2,\beta_3 \in \mathbb{N}_0\times\mathbb{N}_0^2 \\ \beta_1+\beta_2+\beta_3=\delta_c}} \tbinom{\delta_c}{\beta_1,\beta_2,\beta_3} \mathrm{D}^{\beta_1}_{\mu;3} g \bullet \mathrm{D}^{\beta_2}_{1;3}\mathcal{C}^{[i,j]}_{\rm top} \bullet \mathrm{D}^{\beta_3}_{1;\mu'} h.$$

The desired inequality follows by the triangle inequality, Theorem II.19b and Lemma II.17, with $\mathrm{D}^{\beta_1}_{\mu;3} g$ in place of g and $\mathrm{D}^{\beta_3}_{1;\mu'} h$ in place of h.

b) The first statement is again Leibniz's Rule. By the first statement of Theorem II.19.c, with $\mathrm{D}^{\beta_1}_{\mu;3} g$ in place of g and $\mathrm{D}^{\beta_3}_{1;\mu'} h$ in place of h,

$$\begin{aligned} &\tfrac{1}{M^{|\delta_{\rm c}|j}} \big\| \mathrm{D}^{\beta_1}_{\mu;3} g \bullet \mathcal{C}^{[i,j]}_{\rm mid} \bullet \mathrm{D}^{\beta_3}_{1;\mu'} h \big\|^{(\delta_{\rm l},0,\delta_{\rm r})}_{\ell,j} \\ &\le {\rm const}\, \tfrac{j-i+1}{M^{|\delta_{\rm c}|j}} \big\| \mathrm{D}^{\beta_1}_{\mu;3} g \big\|^{(\delta_{\rm l},0,0)}_{\ell,i} \big\| \mathrm{D}^{\beta_3}_{1;\mu'} h \big\|^{(0,0,\delta_{\rm r})}_{i,j} \\ &\le {\rm const}\, \tfrac{j-i+1}{M^{|\delta_{\rm c}|j}} \big\| \mathrm{D}^{\beta_1}_{\mu;3} g \big\|^{(\delta_{\rm l},0,0)}_{\ell,\ell} \big\| \mathrm{D}^{\beta_3}_{1;\mu'} h \big\|^{(0,0,\delta_{\rm r})}_{i,j} \\ &\le {\rm const}\, \tfrac{j-i+1}{M^{|\delta_{\rm c}|j}} M^{|\beta_1|\ell} \big\| g \big\|^{(\delta_{\rm l},\beta_1,0)}_{\ell,\ell} M^{|\beta_3|j} \big\| h \big\|^{(0,\beta_3,\delta_{\rm r})}_{i,j} \\ &\le {\rm const}\, \tfrac{j-i+1}{M^{|\beta_1|(j-\ell)}} \big\| g \big\|^{(\delta_{\rm l},\beta_1,0)}_{\ell,\ell} \big\| h \big\|^{(0,\beta_3,\delta_{\rm r})}_{i,j}. \end{aligned}$$

For the second inequality, we used the variant

$$\begin{aligned} \big\| \mathrm{D}^{\beta_1}_{\mu;3} g \big\|^{(\delta_{\rm l},0,0)}_{\ell,i} &= \tfrac{1}{M^{|\delta_{\rm l}|\ell}} \big| \mathrm{D}^{\delta_{\rm l}}_{1;2} \mathrm{D}^{\beta_1}_{\mu;3} g \big|^{[0,0,0]}_{\ell,i} \\ &\le {\it const}\, \tfrac{1}{M^{|\delta_{\rm l}|\ell}} \big| \mathrm{D}^{\delta_{\rm l}}_{1;2} \mathrm{D}^{\beta_1}_{\mu;3} g \big|^{[0,0,0]}_{\ell,\ell} = {\it const}\, \big\| \mathrm{D}^{\beta_1}_{\mu;3} g \big\|^{(\delta_{\rm l},0,0)}_{\ell,\ell} \end{aligned}$$

of Lemma II.17. The proof of the second case is similar, but with

$$\begin{aligned}\big\|g\bullet\mathcal{C}^{[i,j]}_{\rm mid}\bullet h\big\|^{(\delta_{\rm l},0,\delta_{\rm r})}_{\ell,j}&\le\big|g\bullet\mathcal{C}^{[i,j]}\bullet h\big|^{[\delta_{\rm l},0,\delta_{\rm r}]}_{\ell,j}+\big|g\bullet\mathcal{C}^{[i,j]}_{\rm top}\bullet h\big|^{[\delta_{\rm l},0,\delta_{\rm r}]}_{\ell,j}\\&\quad+\big|g\bullet\mathcal{C}^{[i,j]}_{\rm bot}\bullet h\big|^{[\delta_{\rm l},0,\delta_{\rm r}]}_{\ell,j}\\&\le\text{const }i\max_{\substack{\alpha_{\rm r},\alpha_{\rm l}\in\mathbb{N}_0\times\mathbb{N}_0^2\\|\alpha_{\rm r}|+|\alpha_{\rm l}|\le3}}\big|g\big|^{[\delta_{\rm l},0,\alpha_{\rm r}]}_{\ell,i}\big|h\big|^{[\alpha_{\rm l},0,\delta_{\rm r}]}_{i,j}\end{aligned}$$

(by Theorem II.19.a,b) used in place of the first statement of Theorem II.19.c. The proof of the third case is again similar, but with the second statement of Theorem II.19.c used in place of the first statement of Theorem II.19.c. □

Corollary II.23. *Let $1\le\ell\le i\le j$ and $\delta_{\rm l},\delta_{\rm c},\delta_{\rm r}\in\Delta$. Let g_1, g_2 and h be sectorized, translation invariant functions on $\mathfrak{Y}_{\ell,\ell}$, $\mathfrak{Y}_{\ell,\ell}$ and $\mathfrak{Y}_{i,j}$ respectively. Let $\mu\in\{1,2\}$, $\mu'\in\{3,4\}$ and*

$$g=(g_1\bullet\mathcal{C}^{(\ell)}\bullet g_2)^f.$$

a) If $\delta_{\rm l}+\delta_{\rm c}\in\Delta$, then

$$\begin{aligned}\big\|g\bullet\mathcal{C}^{[i,j]}_{\rm top}\bullet h\big\|^{(\delta_{\rm l},\delta_{\rm c},\delta_{\rm r})}_{\ell,j}&\le\text{const}\sqrt{\mathfrak{l}_\ell}\;\big|g_1\big|^{[\![\delta_{\rm l}+\delta_{\rm c}]\!]}_{\ell}\big|g_2\big|^{[\![\delta_{\rm l}+\delta_{\rm c}]\!]}_{\ell}\big|h\big|^{[0,\delta_{\rm c},\delta_{\rm r}]}_{i,j},\\\big\|g\bullet\mathcal{C}^{[i,j]}_{\rm bot}\bullet h\big\|^{(\delta_{\rm l},\delta_{\rm c},\delta_{\rm r})}_{\ell,j}&\le\text{const}\sqrt{\mathfrak{l}_\ell}\;\big|g_1\big|^{[\![\delta_{\rm l}+\delta_{\rm c}]\!]}_{\ell,\ell}\big|g_2\big|^{[\![\delta_{\rm l}+\delta_{\rm c}]\!]}_{\ell}\big|h\big|^{[0,\delta_{\rm c},\delta_{\rm r}]}_{i,j}.\end{aligned}$$

b) Let $\beta_1+\beta_2+\beta_3=\delta_{\rm c}$. If $\delta_{\rm l}+\beta_1\in\Delta$, then

$$\begin{aligned}&\tfrac{1}{M^{|\delta_{\rm c}|j}}\big\|{\rm D}^{\beta_1}_{\mu;3}g\bullet{\rm D}^{\beta_2}_{1;3}\mathcal{C}^{[i,j]}_{\rm mid}\bullet{\rm D}^{\beta_3}_{1;\mu'}h\big\|^{(\delta_{\rm l},0,\delta_{\rm r})}_{\ell,j}\\&\le\tfrac{\rm const}{M^{|\beta_1|(j-\ell)}}\sqrt{\mathfrak{l}_\ell}\begin{cases}(j-i+1)\big|g_1\big|^{[\![\delta_{\rm l}+\beta_1]\!]}_{\ell}\big|g_2\big|^{[\![\delta_{\rm l}+\beta_1]\!]}_{\ell}\big\|h\big\|^{(0,\beta_3,\delta_{\rm r})}_{i,j}&\text{if }\beta_2=0\\\big|g_1\big|^{[\![\delta_{\rm l}+\beta_1]\!]}_{\ell}\big|g_2\big|^{[\![\delta_{\rm l}+\beta_1]\!]}_{\ell}\big\|h\big\|^{(0,\beta_3,\delta_{\rm r})}_{i,j}&\text{if }\beta_2\ne0\end{cases}.\end{aligned}$$

If $\beta_2=0$ and $\delta_{\rm l}+\beta_1+\alpha\in\Delta$ for all $|\alpha|\le3$, then

$$\begin{aligned}&\tfrac{1}{M^{|\delta_{\rm c}|j}}\big\|{\rm D}^{\beta_1}_{\mu;3}g\bullet{\rm D}^{\beta_2}_{1;3}\mathcal{C}^{[i,j]}_{\rm mid}\bullet{\rm D}^{\beta_3}_{1;\mu'}h\big\|^{(\delta_{\rm l},0,\delta_{\rm r})}_{\ell,j}\\&\le\tfrac{\rm const}{M^{|\beta_1|(j-\ell)}}\,i\sqrt{\mathfrak{l}_\ell}\max_{\substack{\alpha_{\rm up},\alpha_{\rm dn},\alpha_{\rm l}\in\mathbb{N}_0\times\mathbb{N}_0^2\\|\alpha_{\rm up}|+|\alpha_{\rm dn}|+|\alpha_{\rm l}|\le3}}\big|g_1\big|^{[\![\delta_{\rm l}+\beta_1+\alpha_{\rm up}]\!]}_{\ell}\big|g_2\big|^{[\![\delta_{\rm l}+\beta_1+\alpha_{\rm dn}]\!]}_{\ell}\big|h\big|^{[\alpha_{\rm l},\beta_3,\delta_{\rm r}]}_{i,j}.\end{aligned}$$

Proof. a) We consider the case of top. By Leibniz,

$$\begin{aligned}&{\rm D}^{\delta_{\rm c}}_{1;\mu'}\big(g\bullet\mathcal{C}^{[i,j]}_{\rm top}\bullet h\big)\\&\quad=\sum_{\substack{\beta_1,\beta_2,\beta_3\in\mathbb{N}_0\times\mathbb{N}_0^2\\\beta_1+\beta_2+\beta_3=\delta_{\rm c}}}\tbinom{\delta_{\rm c}}{\beta_1,\beta_2,\beta_3}{\rm D}^{\beta_1}_{1;3}(g_1\bullet\mathcal{C}^{(\ell)}\bullet g_2)^f\bullet{\rm D}^{\beta_2}_{1;3}\mathcal{C}^{[i,j]}_{\rm top}\bullet{\rm D}^{\beta_3}_{1;\mu'}h,\\&{\rm D}^{\delta_{\rm c}}_{2;\mu'}\big(g\bullet\mathcal{C}^{[i,j]}_{\rm top}\bullet h\big)\\&\quad=\sum_{\substack{\beta_1,\beta_2,\beta_3\in\mathbb{N}_0\times\mathbb{N}_0^2\\\beta_1+\beta_2+\beta_3=\delta_{\rm c}}}\tbinom{\delta_{\rm c}}{\beta_1,\beta_2,\beta_3}{\rm D}^{\beta_1}_{2;3}(g_1\bullet\mathcal{C}^{(\ell)}\bullet g_2)^f\bullet{\rm D}^{\beta_2}_{1;3}\mathcal{C}^{[i,j]}_{\rm top}\bullet{\rm D}^{\beta_3}_{1;\mu'}h.\end{aligned}$$

Substitute $\mathcal{C}^{(\ell)} = \mathcal{D}^{(\ell)}_{0,\mathrm{up}} + \mathcal{D}^{(\ell)}_{0,\mathrm{dn}}$. We consider the case of up. Then

$$\begin{aligned}
&\mathrm{D}^{\beta_1}_{1;3}(g_1 \bullet \mathcal{D}^{(\ell)}_{0,\mathrm{up}} \bullet g_2)^f \bullet \mathrm{D}^{\beta_2}_{1;3}\mathcal{C}^{[i,j]}_{\mathrm{top}} \bullet \mathrm{D}^{\beta_3}_{1;\mu'}h \\
&\quad = (\mathrm{D}^{\beta_1}_{1;2}g_1 \bullet \mathcal{D}^{(\ell)}_{0,\mathrm{up}} \bullet g_2)^f \bullet \mathrm{D}^{\beta_2}_{1;3}\mathcal{C}^{[i,j]}_{\mathrm{top}} \bullet \mathrm{D}^{\beta_3}_{1;\mu'}h, \\
&\mathrm{D}^{\beta_1}_{2;3}(g_1 \bullet \mathcal{D}^{(\ell)}_{0,\mathrm{up}} \bullet g_2)^f \bullet \mathrm{D}^{\beta_2}_{1;3}\mathcal{C}^{[i,j]}_{\mathrm{top}} \bullet \mathrm{D}^{\beta_3}_{1;\mu'}h \\
&\quad = \big(\mathrm{D}^{\beta_1}_{3;2}(g_1 \bullet \mathcal{D}^{(\ell)}_{0,\mathrm{up}} \bullet g_2)\big)^f \bullet \mathrm{D}^{\beta_2}_{1;3}\mathcal{C}^{[i,j]}_{\mathrm{top}} \bullet \mathrm{D}^{\beta_3}_{1;\mu'}h \\
&\quad = (-1)^{|\beta_1|}\sum_{\substack{\gamma_1,\gamma_2,\gamma_3\in\mathbb{N}_0\times\mathbb{N}_0^2 \\ \gamma_1+\gamma_2+\gamma_3=\beta_1}} M^{|\gamma_2|\ell}\tbinom{\beta_1}{\gamma_1,\gamma_2,\gamma_3}\big(\mathrm{D}^{\gamma_1}_{2;3}g_1 \bullet \mathcal{D}^{(\ell)}_{\gamma_2,\mathrm{up}} \\
&\qquad \bullet \mathrm{D}^{\gamma_3}_{1;3}g_2\big)^f \bullet \mathrm{D}^{\beta_2}_{1;3}\mathcal{C}^{[i,j]}_{\mathrm{top}} \bullet \mathrm{D}^{\beta_3}_{1;\mu'}h.
\end{aligned}$$

The $\frac{1}{M^{|\delta_c|j}}\|\cdot\|^{(\delta_l,0,\delta_r)}_{\ell,j}$ norm of each term is bounded by Theorem II.20.b.

b) As above, we must estimate the $\frac{1}{M^{|\delta_c|j}}\|\cdot\|^{(\delta_l,0,\delta_r)}_{\ell,j}$ norm of terms like

$$\big(\mathrm{D}^{\beta_1}_{1;2}g_1 \bullet \mathcal{D}^{(\ell)}_{0,\mathrm{up}} \bullet g_2\big)^f \bullet \mathrm{D}^{\beta_2}_{1;3}\mathcal{C}^{[i,j]}_{\mathrm{mid}} \bullet \mathrm{D}^{\beta_3}_{1;\mu'}h$$

and

$$M^{|\gamma_2|\ell}\big(\mathrm{D}^{\gamma_1}_{2;3}g_1 \bullet \mathcal{D}^{(\ell)}_{\gamma_2,\mathrm{up}} \bullet \mathrm{D}^{\gamma_3}_{1;3}g_2\big)^f \bullet \mathrm{D}^{\beta_2}_{1;3}\mathcal{C}^{[i,j]}_{\mathrm{mid}} \bullet \mathrm{D}^{\beta_3}_{1;\mu'}h$$

with $\gamma_1+\gamma_2+\gamma_3=\beta_1$. This is done using Theorem II.20. (In the last case, we write $\mathcal{C}^{[i,j]}_{\mathrm{mid}} = \mathcal{C}^{[i,j]} - \mathcal{C}^{[i,j]}_{\mathrm{top}} - \mathcal{C}^{[i,j]}_{\mathrm{bot}}$.) □

Corollary II.24. *Let $1 \le \ell \le i \le j$, $1 \le r \le j$ and $\delta_l, \delta_c, \delta_r \in \Delta$. Let $\mu \in \{1,2\}$ and $\mu' \in \{3,4\}$. Let h be a sectorized, translation invariant function on $\mathfrak{Y}_{i,r}$ and let $h' = h_{\Sigma_i,\Sigma_j}$ be its resectorization as in Def. I.17.i.*

a) Let g be a sectorized, translation invariant function on $\mathfrak{Y}_{\ell,i}$. Then

$$\frac{1}{M^{j|\delta_c|}}\big\|g \bullet \mathcal{C}^{[i,j]}_{\mathrm{mid}} \bullet \mathrm{D}^{\delta_c}_{\mu;\mu'}h'\big\|^{(\delta_l,0,\delta_r)}_{\ell,j} \le \mathrm{const}\max_{\substack{\alpha_r,\alpha_l\in\mathbb{N}_0\times\mathbb{N}_0^2 \\ |\alpha_r|+|\alpha_l|\le 3}} \times |g|^{[\delta_l,0,\alpha_r]}_{\ell,i}\Big(\tfrac{j-i+1}{M^{j-i}}|h|^{[0,\delta_c,\delta_r]}_{i,r} + i\,|h|^{[\alpha_l,0,0]}_{i,r}\Big).$$

b) Let g_1 and g_2 be sectorized, translation invariant functions on $\mathfrak{Y}_{\ell,\ell}$. If $\delta_l+\alpha \in \Delta$ for all $|\alpha| \le 3$, then

$$\begin{aligned}
&\frac{1}{M^{j|\delta_c|}}\big\|(g_1 \bullet \mathcal{C}^{(\ell)} \bullet g_2)^f \bullet \mathcal{C}^{[i,j]}_{\mathrm{mid}} \bullet \mathrm{D}^{\delta_c}_{\mu;\mu'}h'\big\|^{(\delta_l,0,\delta_r)}_{\ell,j} \\
&\quad \le \mathrm{const}\sqrt{\mathfrak{l}_\ell}\max_{\substack{\alpha_{\mathrm{up}},\alpha_{\mathrm{dn}},\alpha_l\in\mathbb{N}_0\times\mathbb{N}_0^2 \\ |\alpha_{\mathrm{up}}|+|\alpha_{\mathrm{dn}}|+|\alpha_l|\le 3}} \\
&\qquad \times |g_1|^{[\![\delta_l+\alpha_{\mathrm{up}}]\!]}_{\ell}|g_2|^{[\![\delta_l+\alpha_{\mathrm{dn}}]\!]}_{\ell}\Big(\tfrac{j-i+1}{M^{j-i}}|h|^{[0,\delta_c,\delta_r]}_{i,r} + i\,|h|^{[\alpha_l,0,0]}_{i,r}\Big).
\end{aligned}$$

Proof. We prove part a. First suppose that h is a function on $\mathfrak{Y}^2_{\Sigma_i} \times \big((\mathbb{R}\times\mathbb{R}^2)\times\Sigma_r\big)^2$. Then, for $s_3, s_4 \in \Sigma_j$,

$$h'\big(\,\cdot\,,\,\cdot\,,(\,\cdot\,,s_3),(\,\cdot\,,s_4)\big) = h \bullet \Big(\hat\chi_{s_3}\otimes\hat\chi^t_{s_4}\Big).$$

We have

$$\tfrac{1}{M^{j|\delta_c|}}\big\|g\bullet\mathcal{C}^{[i,j]}_{\rm mid}\bullet{\rm D}^{\delta_c}_{\mu;\mu'}h'\big\|^{(\delta_l,0,\delta_r)}_{\ell,j} = \tfrac{1}{M^{j(|\delta_c|+|\delta_r|)}}\big\|g\bullet\mathcal{C}^{[i,j]}_{\rm mid}\bullet{\rm D}^{\delta_c}_{\mu;\mu'}{\rm D}^{\delta_r}_{3;4}h'\big\|^{(\delta_l,0,0)}_{\ell,j}.$$

Apply Leibniz to ${\rm D}^{\delta_c}_{\mu;\mu'}{\rm D}^{\delta_r}_{3;4}\big(h\bullet(\hat\chi_{s_3}\otimes\hat\chi^t_{s_4})\big)$, yielding a sum of terms of the form

$${\rm D}^{\beta_1}_{\mu;\mu'}{\rm D}^{\gamma_1}_{3;4}h\bullet{\rm D}^{\beta_2+\gamma_2}_{1;3}{\rm D}^{\beta_3+\gamma_3}_{2;4}(\hat\chi_{s_3}\otimes\hat\chi^t_{s_4})$$

with $\beta_1+\beta_2+\beta_3=\delta_c$ and $\gamma_1+\gamma_2+\gamma_3=\delta_r$. If $\beta_1+\gamma_1=0$ we apply Theorem II.19a and otherwise we apply Theorem II.19c. The lemma follows from

$$\begin{aligned}
&\big|h\bullet{\rm D}^{\beta_2+\gamma_2}_{1;3}{\rm D}^{\beta_3+\gamma_3}_{2;4}(\hat\chi_{s_3}\otimes\hat\chi^t_{s_4})\big)\big|^{[\alpha_1,0,0]}_{i,j}\\
&\quad\le M^{j(|\delta_c+\delta_r|)}\big|h\big|^{[\alpha_1,0,0]}_{i,r},\\
&\big\|{\rm D}^{\beta_1}_{\mu;\mu'}{\rm D}^{\gamma_1}_{3;4}h\bullet{\rm D}^{\beta_2+\gamma_2}_{1;3}{\rm D}^{\beta_3+\gamma_3}_{2;4}(\hat\chi_{s_3}\otimes\hat\chi^t_{s_4})\big)\big\|^{(0,0,0)}_{i,j}\\
&\quad\le M^{j(|\beta_2+\gamma_2+\beta_3+\gamma_3|)}M^{|\beta_1|i+|\gamma_1|r}\big|h\big|^{[0,\delta_c,\delta_r]}_{i,r}\\
&\quad\le M^{j(|\delta_c+\delta_r|-1)}M^i\big|h\big|^{[0,\delta_c,\delta_r]}_{i,r}\quad \text{if } |\beta_1+\gamma_1|\ge 1,
\end{aligned}$$

which is proven in the same way as Lemma II.17 and, in particular, uses (II.2) with $r=j$. If one of the third or fourth arguments of h lie in momentum space, $\mathbb{M}$, the argument is similar, except that the corresponding $\hat\chi_{s_3}$ or $\hat\chi_{s_4}$ is omitted. The proof of part b is similar with Theorem II.20 used in place of Theorem II.19. □

Proof of Theorem I.20 (assuming Theorems II.19 and II.20). Let $\delta\in\Delta$. By the hypothesis of the theorem and Remark II.15.ii, there is a constant c_F such that

$$\max_{\delta\in\Delta}|F^{(i)}_C|^{[\![\delta]\!]}_i \le \tfrac{c_F}{M^{\varepsilon i}}\rho \qquad \max_{\delta\in\Delta}|F^{(i)}_S|^{[\![\delta]\!]}_i \le \tfrac{c_F}{M^{\varepsilon i}}\rho. \tag{II.3}$$

We prove by induction on j that

$$\max_{\delta\in\Delta}|\mathcal{L}^{(i)}_C|^{[\![\delta]\!]}_{i-1}\le c_{\mathcal{L}}\,\rho^2,\qquad \max_{\delta\in\Delta}|\mathcal{L}^{(i)}_S|^{[\![\delta]\!]}_{i-1}\le c_{\mathcal{L}}\,\rho^2\qquad\text{for all } i\le j, \tag{II.4}$$

with a constant $c_{\mathcal{L}}$, independent of j. See Remark II.15. By construction $\mathcal{L}^{(0)}=\mathcal{L}^{(1)}=\mathcal{L}^{(2)}=0$. Now assume that (II.4) holds for some $j\ge 2$. We prove that

$$\max_{\delta\in\Delta}|\mathcal{L}^{(j+1)}_S|^{[\![\delta]\!]}_j\le c_{\mathcal{L}}\,\rho^2. \tag{II.5}$$

The bound on $\mathcal{L}^{(j+1)}_C$ is similar.

For $i\le j$ we have, by Corollary II.12.ii,

$$L^{(i)}_C = G^{(i-1)}_{C,1}\bullet\mathcal{C}^{(i-1)}\bullet G^{(i-1)}_{C,2}\qquad L^{(i)}_S = G^{(i-1)}_{S,1}\bullet\mathcal{C}^{(i-1)}\bullet G^{(i-1)}_{S,2}$$

with

$$G_{C,1}^{(i-1)} = \sum_{i'=2}^{i-1} F_{C\,\Sigma_{i-1}}^{(i')} + \tfrac{1}{2}\mathcal{L}_{C\,\Sigma_{i-1}}^{(i-1)\,f} + \tfrac{3}{2}\mathcal{L}_{S\,\Sigma_{i-1}}^{(i-1)\,f} + \mathcal{L}_{C\,\Sigma_{i-1}}^{(i-1)},$$

$$G_{C,2}^{(i-1)} = \sum_{i'=2}^{i-1} F_{C\,\Sigma_{i-1}}^{(i')} + \tfrac{1}{2}\mathcal{L}_{C\,\Sigma_{i-1}}^{(i-1)\,f} + \tfrac{3}{2}\mathcal{L}_{S\,\Sigma_{i-1}}^{(i-1)\,f} + \mathcal{L}_{C}^{(i)},$$

$$G_{S,1}^{(i-1)} = \sum_{i'=2}^{i-1} F_{S\,\Sigma_{i-1}}^{(i')} + \tfrac{1}{2}\mathcal{L}_{C\,\Sigma_{i-1}}^{(i-1)\,f} - \tfrac{1}{2}\mathcal{L}_{S\,\Sigma_{i-1}}^{(i-1)\,f} + \mathcal{L}_{S\,\Sigma_{i-1}}^{(i-1)},$$

$$G_{S,2}^{(i-1)} = \sum_{i'=2}^{i-1} F_{S\,\Sigma_{i-1}}^{(i')} + \tfrac{1}{2}\mathcal{L}_{C\,\Sigma_{i-1}}^{(i-1)\,f} - \tfrac{1}{2}\mathcal{L}_{S\,\Sigma_{i-1}}^{(i-1)\,f} + \mathcal{L}_{S}^{(i)}.$$

The hypotheses (II.3) on $\vec{F}$ and the induction hypotheses (II.4) imply, via Lemma II.17, that, when ρ is small enough and M^{ε} is large enough,

$$\max_{\delta\in\Delta} \big|G_{C,\nu}^{(i-1)}\big|_{i-1}^{[\![\delta]\!]} \le c_F\,\rho \qquad \max_{\delta\in\Delta} \big|G_{S,\nu}^{(i-1)}\big|_{i-1}^{[\![\delta]\!]} \le c_F\,\rho \tag{II.6}$$

for $i \le j,\ \nu = 1, 2$.

Remark II.25. For $i \le j$,

$$\max_{\delta\in\Delta} \big|L_C^{(i)}\big|_{i-1}^{[\![\delta]\!]} \le \text{const}\, c_F^2\rho^2 \qquad \max_{\delta\in\Delta} \big|L_S^{(i)}\big|_{i-1}^{[\![\delta]\!]} \le \text{const}\, c_F^2\rho^2,$$

where const is $(2 + 3^{r_0+2r_e})$ times the constant of Cor. II.22.

Proof. We prove the remark for $L_C^{(i)}$. Fix $(\delta_{\rm l}, \delta_{\rm c}, \delta_{\rm r}) \in \vec{\Delta}$. Decomposing

$$\mathcal{C}^{[i-1,i-1]} = \big(\mathcal{C}_{\rm top}^{[i-1,i-1]} + \mathcal{C}_{\rm bot}^{[i-1,i-1]}\big) + \mathcal{C}_{\rm mid}^{[i-1,i-1]}$$

and applying Cor. II.22, parts a and b respectively, with ℓ, i, j all replaced by $i-1$, we have

$$\begin{aligned}
&\big|G_{C,1}^{(i-1)} \bullet \mathcal{C}^{[i-1,i-1]} \bullet G_{C,2}^{(i-1)}\big|_{i-1,i-1}^{[\delta_{\rm l},\delta_{\rm c},\delta_{\rm r}]} \\
&\quad\le \text{const}\,\big((1+1) + 3^{|\delta_{\rm c}|}\big)\big|G_{C,1}^{(i-1)}\big|_{i-1,i-1}^{[\delta_{\rm l},\delta_{\rm c},0]}\ \big|G_{C,2}^{(i-1)}\big|_{i-1,i-1}^{[0,\delta_{\rm c},\delta_{\rm r}]} \\
&\quad\le \text{const}\,(2 + 3^{r_0+2r_e})c_F^2\rho^2. \qquad \square
\end{aligned}$$

Set, for each $i \ge 1$,

$$\mathfrak{v}_i = i^2 \max\big\{\sqrt{\mathfrak{l}_i},\ \tfrac{1}{M^{\varepsilon i}}\big\},$$

and

$$K^{(i)} = F_S^{(i)} + \tfrac{1}{2}L_{C\,\Sigma_i}^{(i)\,f} - \tfrac{1}{2}L_{S\,\Sigma_i}^{(i)\,f}.$$

Then, by Cor. II.12.i,

$$\mathcal{L}_S^{(j+1)} = \sum_{\ell=1}^{\infty} \sum_{i_1,\cdots,i_{\ell+1}=2}^{j} \Big[K^{(i_1)} \bullet \mathcal{C}^{[\max\{i_1,i_2\},j]} \bullet K^{(i_2)} \bullet \cdots \bullet \mathcal{C}^{[\max\{i_\ell,i_{\ell+1}\},j]} \bullet K^{(i_{\ell+1})}\Big]_{\Sigma_j}.$$

We put the main estimates required to complete the proof of Theorem I.20 in □

Lemma II.26. *Let $\ell \ge 1$ and $i_1, \cdots, i_{\ell+1} \le j$.*

a) For $|\alpha| \le 3$ and $\delta \in \Delta$,

$$\Big| K^{(i_1)} \bullet \mathcal{C}^{[\max\{i_1,i_2\},j]} \bullet K^{(i_2)} \bullet \cdots \bullet K^{(i_{\ell+1})} \Big|_{i_1,j}^{[\alpha,0,\delta]} \le \mathrm{const}^\ell \left(c_F\rho\right)^{\ell+1} \tfrac{i_{\ell+1}}{i_1}\, \mathfrak{v}_{i_1} \cdots \mathfrak{v}_{i_\ell}.$$

b) For $(\delta_l, 0, \delta_r) \in \vec{\Delta}$,

$$\begin{aligned} &\Big| K^{(i_1)} \bullet \mathcal{C}^{[\max\{i_1,i_2\},j]} \bullet K^{(i_2)} \bullet \cdots \bullet K^{(i_{\ell+1})} \Big|_{j,j}^{[\delta_l,0,\delta_r]} \\ &\quad \le \mathrm{const}^\ell \left(c_F\rho\right)^{\ell+1} \mathfrak{v}_{i_2} \cdots \mathfrak{v}_{i_\ell} \min\left\{\mathfrak{v}_{i_1}, \mathfrak{v}_{i_{\ell+1}}\right\}. \end{aligned}$$

c) For $0 \ne \delta \in \Delta$, $\mu \in \{1,2\}$ and $\mu' \in \{3,4\}$, there are sectorized, translation invariant functions k', k'' on $\mathfrak{Y}_{i_1,j}$ such that

$$\tfrac{1}{M^{j|\delta|}} \mathrm{D}^\delta_{\mu;\mu'} \Big[\big(K^{(i_1)} \bullet \mathcal{C}^{[\max\{i_1,i_2\},j]} \bullet \cdots \bullet K^{(i_{\ell+1})} \big)_{\Sigma_{i_1},\Sigma_j} \Big] = k' + k''$$

and, for all $|\alpha| \le 3$ and all γ with $\gamma + \delta \in \Delta$,

$$\begin{aligned} \big|k'\big|_{i_1,j}^{[\alpha,0,\gamma]} &\le \mathrm{const}^\ell \left(c_F\rho\right)^{\ell+1} \tfrac{i_{\ell+1}}{i_1}\, \mathfrak{v}_{i_1} \cdots \mathfrak{v}_{i_\ell}, \\ \big|k''\big|_{i_1,j}^{[0,0,\gamma]} &\le \tfrac{j-i_1+1}{M^{j-i_1}} \mathrm{const}^\ell \left(c_F\rho\right)^{\ell+1} \tfrac{i_{\ell+1}}{i_1}\, \mathfrak{v}_{i_1} \cdots \mathfrak{v}_{i_\ell}. \end{aligned}$$

d) For $(\delta_l, \delta_c, \delta_r) \in \vec{\Delta}$ with $|\delta_c| \ge 1$,

$$\begin{aligned} &\Big| K^{(i_1)} \bullet \mathcal{C}^{[\max\{i_1,i_2\},j]} \bullet \cdots \bullet K^{(i_{\ell+1})} \Big|_{j,j}^{[\delta_l,\delta_c,\delta_r]} \\ &\quad \le \mathrm{const}^\ell \left(c_F\rho\right)^{\ell+1} \mathfrak{v}_{i_2} \cdots \mathfrak{v}_{i_\ell} \min\left\{\mathfrak{v}_{i_1}, \mathfrak{v}_{i_{\ell+1}}\right\}. \end{aligned}$$

Set $i = \max\{i_1, i_2\}$ and write, for $(\alpha_l, \alpha_c, \alpha_r) \in \vec{\Delta}$ and $\alpha_{\rm up}, \alpha_{\rm dn} \in \Delta$,

$$\begin{aligned} q(\alpha_l, \alpha_c, \alpha_r; \alpha_{\rm up}, \alpha_{\rm dn}) = i \, \big|F_S^{(i_1)}\big|_{i_1,i_1}^{[\alpha_l,\alpha_c,\alpha_r]} + i\sqrt{\mathfrak{l}_{i_1-1}} \Big[\big|G_{C,1}^{(i_1-1)}\big|_{i_1-1}^{[\![\alpha_{\rm up}]\!]} \big|G_{C,2}^{(i_1-1)}\big|_{i_1-1}^{[\![\alpha_{\rm dn}]\!]} \\ + \big|G_{S,1}^{(i_1-1)}\big|_{i_1-1}^{[\![\alpha_{\rm up}]\!]} \big|G_{S,2}^{(i_1-1)}\big|_{i_1-1}^{[\![\alpha_{\rm dn}]\!]} \Big]. \end{aligned}$$

By (II.3) and (II.6),

$$q(\alpha_l, \alpha_c, \alpha_r; \alpha_{\rm up}, \alpha_{\rm dn}) \le i \tfrac{c_F\rho}{M^{\varepsilon i_1}} + i\, 2c_F^2\rho^2 \sqrt{\mathfrak{l}_{i_1-1}} \le 2c_F\rho\, \mathfrak{v}_{i_1} \tfrac{i_2}{i_1} \tag{II.7}$$

for ρ sufficiently small. The proof of Lemma II.26 follows

Lemma II.27. *Let $1 \le i_1, i_2 \le i \le j$, $1 \le r \le j$ and $\delta_l, \delta_c, \delta_r \in \Delta$. Let $\mu \in \{1,2\}$ and $\mu' \in \{3,4\}$ and* loc $\in$ {top, bot, mid}*. If* loc $\in$ {top, mid}*, set $(\nu, \nu') = (1,3)$. If* loc = bot*, set $(\nu, \nu') = (2,4)$. Let H be a sectorized, translation invariant function on $\mathfrak{Y}_{i_2,r}$.*

a) Let $\beta_2 + \beta_3 = \delta$ with $\beta_2 \ne 0$ and either $|\delta_l| \le 3$, $\delta + \delta_r \in \Delta$ or $\delta_l + \delta_r + \delta \in \Delta$. Then

$$\begin{aligned} &\tfrac{1}{M^{j|\delta|}} \big| K^{(i_1)} \bullet \mathrm{D}^{\beta_2}_{\nu;\nu'} \mathcal{C}^{[i,j]}_{\rm loc} \bullet \mathrm{D}^{\beta_3}_{1;\mu'} H_{\Sigma_{i_2},\Sigma_j} \big|_{i_1,j}^{[\delta_l,0,\delta_r]} \\ &\quad \le \mathrm{const}\ q(\delta_l,0,0;\delta_l,\delta_l)\ \big|H\big|_{i_2,j}^{[0,\beta_3,\delta_r]}. \end{aligned}$$

b) Let $|\delta_{\rm l}| \le 3$ *and* $\delta + \delta_{\rm r} \in \Delta$. *Then*

$$\begin{aligned} &\tfrac{1}{M^{j|\delta|}} \big| K^{(i_1)} \bullet \mathcal{C}_{\rm loc}^{[i,j]} \bullet {\rm D}^{\delta}_{1;\mu'} H_{\Sigma_{i_2},\Sigma_j} \big|^{[\delta_{\rm l},0,\delta_{\rm r}]}_{i_1,j} \\ &\quad \le {\rm const} \max_{\substack{\alpha_{\rm up},\alpha_{\rm dn},\alpha_{\rm l}\in\mathbb{N}_0\times\mathbb{N}_0^2 \\ |\alpha_{\rm up}|+|\alpha_{\rm dn}|+|\alpha_{\rm l}|\le 3}} q\,(\delta_{\rm l},0,\alpha_{\rm up}+\alpha_{\rm dn};\delta_{\rm l}+\alpha_{\rm up},\delta_{\rm l}+\alpha_{\rm dn}) \,\Big(\big|H\big|^{[0,\delta,\delta_{\rm r}]}_{i_2,j} + \big|H\big|^{[\alpha_{\rm l},0,0]}_{i_2,j} \Big). \end{aligned}$$

c) Let $\beta_1 + \beta_2 + \beta_3 = \delta$ *and* $\beta_1 + \delta_{\rm l}, \delta, \delta_{\rm r} \in \Delta$. *Then*

$$\begin{aligned} &\tfrac{1}{M^{j|\delta|}} \big| {\rm D}^{\beta_1}_{\mu;3} K^{(i_1)} \bullet {\rm D}^{\beta_2}_{\nu;\nu'} \mathcal{C}_{\rm loc}^{[i,j]} \bullet {\rm D}^{\beta_3}_{1;\mu'} H_{\Sigma_{i_2},\Sigma_j} \big|^{[\delta_{\rm l},0,\delta_{\rm r}]}_{i_1,j} \\ &\quad \le {\rm const}\ \tfrac{j-i+1}{M^{|\beta_1|(j-i_1)}}\, q\,(\delta_{\rm l},\beta_1,0;\delta_{\rm l}+\beta_1,\delta_{\rm l}+\beta_1)\ \big|H\big|^{[0,\beta_3,\delta_{\rm r}]}_{i_2,j}. \end{aligned}$$

Proof. Subbing in the definition of $K^{(i_1)}$ and applying Lemma II.17,

$$\begin{aligned} &\big| {\rm D}^{\beta_1}_{\mu;3} K^{(i_1)} \bullet {\rm D}^{\beta_2}_{\nu;\nu'} \mathcal{C}_{\rm loc}^{[i,j]} \bullet {\rm D}^{\beta_3}_{1;\mu'} H_{\Sigma_{i_2},\Sigma_j} \big|^{[\delta_{\rm l},0,\delta_{\rm r}]}_{i_1,j} \\ &\quad \le \big| {\rm D}^{\beta_1}_{\mu;3} F_S^{(i_1)} \bullet {\rm D}^{\beta_2}_{\nu;\nu'} \mathcal{C}_{\rm loc}^{[i,j]} \bullet {\rm D}^{\beta_3}_{1;\mu'} H_{\Sigma_i,\Sigma_j} \big|^{[\delta_{\rm l},0,\delta_{\rm r}]}_{i_1,j} \\ &\qquad + {\rm const} \big| {\rm D}^{\beta_1}_{\mu;3} \big(G^{(i_1-1)}_{C,1} \bullet \mathcal{C}^{(i_1-1)} \bullet G^{(i_1-1)}_{C,2} \big)^f \bullet {\rm D}^{\beta_2}_{\nu;\nu'} \mathcal{C}_{\rm loc}^{[i,j]} \bullet {\rm D}^{\beta_3}_{1;\mu'} H_{\Sigma_i,\Sigma_j} \big|^{[\delta_{\rm l},0,\delta_{\rm r}]}_{i_1-1,j} \\ &\qquad + {\rm const} \big| {\rm D}^{\beta_1}_{\mu;3} \big(G^{(i_1-1)}_{S,1} \bullet \mathcal{C}^{(i_1-1)} \bullet G^{(i_1-1)}_{S,2} \big)^f \bullet {\rm D}^{\beta_2}_{\nu;\nu'} \mathcal{C}_{\rm loc}^{[i,j]} \bullet {\rm D}^{\beta_3}_{1;\mu'} H_{\Sigma_i,\Sigma_j} \big|^{[\delta_{\rm l},0,\delta_{\rm r}]}_{i_1-1,j}. \end{aligned}$$

We were able to replace $H_{\Sigma_{i_2},\Sigma_j}$ by H_{Σ_i,Σ_j} without changing the $\bullet$ products because $i \ge i_2$ and $\mathcal{C}_{\rm loc}^{[i,j]}$ is supported in the $i^{\rm th}$ neighbourhood.

a) By Cor. II.22.b with $\beta_1 = 0$ and $\delta_{\rm c} = \delta$ (and, when loc $=$ top, bot, Theorem II.19.b with $h = \frac{1}{M^{j|\beta_3|}} {\rm D}^{\beta_3}_{1;\mu'} H_{\Sigma_i,\Sigma_j}$)

$$\tfrac{1}{M^{j|\delta|}} \big| F_S^{(i_1)} \bullet {\rm D}^{\beta_2}_{\nu;\nu'} \mathcal{C}_{\rm loc}^{[i,j]} \bullet {\rm D}^{\beta_3}_{1;\mu'} H_{\Sigma_i,\Sigma_j} \big|^{[\delta_{\rm l},0,\delta_{\rm r}]}_{i_1,j} \le {\rm const} \big| F_S^{(i_1)} \big|^{[\delta_{\rm l},0,0]}_{i_1} \big|H\big|^{[0,\beta_3,\delta_{\rm r}]}_{i_2,j},$$

provided $\delta_{\rm l}, \delta_{\rm r}, \delta \in \Delta$. For $T = C, S$,

$$\begin{aligned} &\tfrac{1}{M^{j|\delta|}} \big| \big(G^{(i_1-1)}_{T,1} \bullet \mathcal{C}^{(i_1-1)} \bullet G^{(i_1-1)}_{T,2} \big)^f \bullet {\rm D}^{\beta_2}_{\nu;\nu'} \mathcal{C}_{\rm loc}^{[i,j]} \bullet {\rm D}^{\beta_3}_{1;\mu'} H_{\Sigma_i,\Sigma_j} \big|^{[\delta_{\rm l},0,\delta_{\rm r}]}_{i_1-1,j} \\ &\quad \le {\rm const} \sqrt{\mathfrak{l}_{i_1-1}}\ \big| G^{(i_1-1)}_{T,1} \big|^{[\![\delta_{\rm l}]\!]}_{i_1-1} \big| G^{(i_1-1)}_{T,2} \big|^{[\![\delta_{\rm l}]\!]}_{i_1-1} \big|H\big|^{[0,\beta_3,\delta_{\rm r}]}_{i_2,j} \end{aligned}$$

by Cor. II.23.b with $\ell = i_1 - 1$, $\beta_1 = 0$ and $\delta_{\rm c} = \delta$ (and Theorem II.20.b when loc $=$ top, bot), provided $\delta_{\rm l}, \delta_{\rm r}, \delta \in \Delta$.

b) By Cor. II.24.a with $\delta_{\rm c} = \delta$, $\ell = i_1$ and $r = j$ (and Cor. II.22.a when loc $=$ top, bot)

$$\begin{aligned} &\tfrac{1}{M^{j|\delta|}} \big| F_S^{(i_1)} \bullet \mathcal{C}_{\rm loc}^{[i,j]} \bullet {\rm D}^{\delta}_{1;\mu'} H_{\Sigma_i,\Sigma_j} \big|^{[\delta_{\rm l},0,\delta_{\rm r}]}_{i_1,j} \\ &\quad \le {\rm const} \max_{\substack{\alpha_{\rm l},\alpha_{\rm r}\in\mathbb{N}_0\times\mathbb{N}_0^2 \\ |\alpha_{\rm l}|+|\alpha_{\rm r}|\le 3}} \big| F_S^{(i_1)} \big|^{[\delta_{\rm l},0,\alpha_{\rm r}]}_{i_1} \Big(\big|H\big|^{[0,\delta,\delta_{\rm r}]}_{i_2,j} + i \big|H\big|^{[\alpha_{\rm l},0,0]}_{i_2,j} \Big), \end{aligned}$$

provided $\delta_l, \delta_r, \delta \in \Delta$. For $T = C, S$,

$$\begin{aligned}
&\tfrac{1}{M^{j|\delta|}}\big|\big(G_{T,1}^{(i_1-1)} \bullet \mathcal{C}^{(i_1-1)} \bullet G_{T,2}^{(i_1-1)}\big)^f \bullet \mathcal{C}_{\rm loc}^{[i,j]} \bullet \mathrm{D}^{\delta}_{1;\mu'} H_{\Sigma_i,\Sigma_j}\big|_{i_1-1,j}^{[\delta_l,0,\delta_r]} \\
&\quad \le \text{const} \sqrt{\mathfrak{l}_{i_1-1}} \max_{\substack{\alpha_{\rm up},\alpha_{\rm dn},\alpha_l\in\mathbb{N}_0\times\mathbb{N}_0^2 \\ |\alpha_{\rm up}|+|\alpha_{\rm dn}|+|\alpha_l|\le 3}} \big|G_{T,1}^{(i_1-1)}\big|_{i_1-1}^{[\![\delta_l+\alpha_{\rm up}]\!]} \\
&\qquad \times \big|G_{T,2}^{(i_1-1)}\big|_{i_1-1}^{[\![\delta_l+\alpha_{\rm dn}]\!]}\Big(\big|H\big|_{i_2,j}^{[0,\delta,\delta_r]} + i\big|H\big|_{i_2,j}^{[\alpha_l,0,0]}\Big)
\end{aligned}$$

by Cor. II.24.b with $\delta_c = \delta$ and $\ell = i_1 - 1$ (and Cor. II.23.a when loc = top, bot) provided $\delta_l + \alpha \in \Delta$ for all $|\alpha| \le 3$ (which is certainly the case when $|\delta_l| \le 3$) and $\delta_r, \delta \in \Delta$.

c) By Cor. II.22.b, with $\delta_c = \delta$ (and Theorem II.19.b with $\beta = \beta_2$, $g = \frac{1}{M^{i_1|\beta_1|}} \mathrm{D}^{\beta_1}_{\mu;3} F_S^{(i_1)}$ and $h = \frac{1}{M^{j|\beta_3|}} \mathrm{D}^{\beta_3}_{1;\mu'} H_{\Sigma_i,\Sigma_j}$, when loc = top, bot)

$$\begin{aligned}
&\tfrac{1}{M^{j|\delta|}}\big|\mathrm{D}^{\beta_1}_{\mu;3} F_S^{(i_1)} \bullet \mathrm{D}^{\beta_2}_{\nu;\nu'}\mathcal{C}_{\rm loc}^{[i,j]} \bullet \mathrm{D}^{\beta_3}_{1;\mu'} H_{\Sigma_i,\Sigma_j}\big|_{i_1,j}^{[\delta_l,0,\delta_r]} \\
&\quad \le \text{const}\, \tfrac{j-i+1}{M^{|\beta_1|(j-i_1)}} \big|F_S^{(i_1)}\big|_{i_1}^{[\delta_l,\beta_1,0]} \big|H\big|_{i_2,j}^{[0,\beta_3,\delta_r]},
\end{aligned}$$

provided $\delta_l, \delta_r, \delta \in \Delta$. For $T = C, S$,

$$\begin{aligned}
&\tfrac{1}{M^{j|\delta|}}\big|\mathrm{D}^{\beta_1}_{\mu;3}\big(G_{T,1}^{(i_1-1)} \bullet \mathcal{C}^{(i_1-1)} \bullet G_{T,2}^{(i_1-1)}\big)^f \bullet \mathrm{D}^{\beta_2}_{\nu;\nu'}\mathcal{C}_{\rm loc}^{[i,j]} \bullet \mathrm{D}^{\beta_3}_{1;\mu'} H_{\Sigma_i,\Sigma_j}\big|_{i_1-1,j}^{[\delta_l,0,\delta_r]} \\
&\quad \le \text{const}\, \tfrac{j-i+1}{M^{|\beta_1|(j-i_1)}} \sqrt{\mathfrak{l}_{i_1-1}} \big|G_{T,1}^{(i_1-1)}\big|_{i_1-1}^{[\![\delta_l+\beta_1]\!]} \big|G_{T,2}^{(i_1-1)}\big|_{i_1-1}^{[\![\delta_l+\beta_1]\!]} \big|H\big|_{i_2,j}^{[0,\beta_3,\delta_r]}
\end{aligned}$$

by Cor. II.23.b with $\delta_c = \delta$ and $\ell = i_1 - 1$ (and Theorem II.20.b when loc = top, bot – the $\mathrm{D}^{\beta_1}_{\mu;3}$ is treated as in the proof of Cor. II.23.a) provided $\delta_l + \beta_1, \delta_r, \delta \in \Delta$. □

Proof of Lemma II.26. The proof is by induction on ℓ. We begin the induction at $\ell = 1$. Observe that, by (II.3), Remark II.25 and Lemma II.17,

$$\max_{\delta\in\Delta} |K^{(i_2)}|_{i_2}^{[\![\delta]\!]} \le c_F\,\rho \tag{II.8}$$

for all $\delta \in \Delta$, if ρ is sufficiently small.

a) By Lemma II.27.b, with $\delta = 0$, $\delta_l = \alpha$ and $\delta_r = \delta$, (II.7), (II.8) and Lemma II.17,

$$\begin{aligned}
&\Big|K^{(i_1)} \bullet \mathcal{C}_{\rm loc}^{[i,j]} \bullet K^{(i_2)}\Big|_{i_1,j}^{[\alpha,0,\delta]} \\
&\quad \le \text{const} \max_{\substack{\alpha_{\rm up},\alpha_{\rm dn},\alpha_l\in\mathbb{N}_0\times\mathbb{N}_0^2 \\ |\alpha_{\rm up}|+|\alpha_{\rm dn}|+|\alpha_l|\le 3}} q(\alpha,0,\alpha_{\rm up}+\alpha_{\rm dn};\alpha+\alpha_{\rm up},\alpha+\alpha_{\rm dn})\Big[\big|K^{(i_2)}\big|_{i_2}^{[0,0,\delta]} + \big|K^{(i_2)}\big|_{i_2}^{[\alpha_l,0,0]}\Big] \\
&\quad \le \text{const}\, c_F^2\rho^2\, \mathfrak{v}_{i_1}\, \tfrac{i_2}{i_1}
\end{aligned}$$

for all of loc = mid, top, bot. Observe that $\alpha + \alpha_{\rm up} + \alpha_{\rm dn} + \alpha_l \in \Delta$, since, by hypothesis, $r_e, r_0 \ge 6$.

b) By symmetry, we may assume, without loss of generality that $i_1 \ge i_2$. Then $\mathfrak{v}_{i_2} \cdots \mathfrak{v}_{i_\ell}$ $\min\{\mathfrak{v}_{i_1}\ ,\ \mathfrak{v}_{i_{\ell+1}}\}$ reduces to $\mathfrak{v}_{i_1}$. By Lemma II.17, Lemma II.27.c, with $\beta_1 = \beta_2 = \beta_3 = 0$, (II.7), (II.8) and part a of this lemma with $\ell = 1$,

$$\begin{aligned}
&\Big| K^{(i_1)} \bullet \mathcal{C}^{[i,j]} \bullet K^{(i_2)} \Big|_{j,j}^{[\delta_\mathrm{l},0,\delta_\mathrm{r}]} \\
&\le \tfrac{\mathrm{const}}{M^{j-i}} \Big| K^{(i_1)} \bullet \mathcal{C}^{[i,j]} \bullet K^{(i_2)} \Big|_{i,j}^{[\delta_\mathrm{l},0,\delta_\mathrm{r}]} + \mathrm{const} \Big| K^{(i_1)} \bullet \mathcal{C}^{[i,j]} \bullet K^{(i_2)} \Big|_{i,j}^{[0,0,\delta_\mathrm{r}]} \\
&\le \tfrac{\mathrm{const}\,(j-i+1)}{M^{j-i}} q(\delta_\mathrm{l},0,0;\delta_\mathrm{l},\delta_\mathrm{l}) \big| K^{(i_2)} \big|_{i_2}^{[0,0,\delta_\mathrm{r}]} + \mathrm{const} \Big| K^{(i_1)} \bullet \mathcal{C}^{[i,j]} \bullet K^{(i_2)} \Big|_{i,j}^{[0,0,\delta_\mathrm{r}]} \\
&\le \mathrm{const}\, c_F^2 \rho^2\, \mathfrak{v}_{i_1}\, \tfrac{i_2}{i_1} \le \mathrm{const}\, c_F^2 \rho^2\, \mathfrak{v}_{i_1}.
\end{aligned}$$

c) Substitute $\mathcal{C}^{[i,j]} = \mathcal{C}^{[i,j]}_{\mathrm{top}} + \mathcal{C}^{[i,j]}_{\mathrm{mid}} + \mathcal{C}^{[i,j]}_{\mathrm{bot}}$ into

$$\mathrm{D}^{\delta}_{\mu;\mu'}\Big[\big(K^{(i_1)} \bullet \mathcal{C}^{[i,j]} \bullet K^{(i_2)}\big)_{\Sigma_{i_1},\Sigma_j}\Big] = \mathrm{D}^{\delta}_{\mu;\mu'}\Big[K^{(i_1)} \bullet \mathcal{C}^{[i,j]} \bullet K^{(i_2)}_{\Sigma_{i_2},\Sigma_j}\Big]$$

and apply Leibniz's rule (Lemma II.21) using the routing which gives $\mathrm{D}^{\beta_2}_{1;3}\mathcal{C}^{[i,j]}_{\mathrm{top}}$, $\mathrm{D}^{\beta_2}_{1;3}\mathcal{C}^{[i,j]}_{\mathrm{mid}}$ and $\mathrm{D}^{\beta_2}_{2;4}\mathcal{C}^{[i,j]}_{\mathrm{bot}}$. We define k' to be $\frac{1}{M^{j|\delta|}}$ times the sum of all resulting terms having no derivatives acting on $K^{(i_1)}$ and k'' to be $\frac{1}{M^{j|\delta|}}$ times the sum of all terms having at least one derivative acting on $K^{(i_1)}$. Fix any α, α' and γ, γ' obeying, $|\alpha| \le 3$, $\gamma + \delta \in \Delta$ and $\alpha' + \gamma' + \delta \in \Delta$. We show

$$\begin{aligned}
\big|k'\big|_{i_1,j}^{[\alpha,0,\gamma]} &\le \mathrm{const}\, c_F^2 \rho^2\, \mathfrak{v}_{i_1}\, \tfrac{i_2}{i_1}, \\
\big|k''\big|_{i_1,j}^{[0,0,\gamma]} &\le \mathrm{const}\, c_F^2\, \tfrac{j-i_1+1}{M^{j-i_1}}\, \rho^2\, \mathfrak{v}_{i_1}\, \tfrac{i_2}{i_1},
\end{aligned} \tag{II.9}$$

and

$$\big|k'\big|_{j,j}^{[\alpha',0,\gamma']} + \big|k''\big|_{j,j}^{[\alpha',0,\gamma']} \le \mathrm{const}\, c_F^2 \rho^2\, \mathfrak{v}_{i_1}\, \tfrac{i_2}{i_1}. \tag{II.10}$$

Let loc $\in$ {mid, top, bot}. If loc $\in$ {top, mid}, set $(\nu, \nu') = (1, 3)$. If loc = bot, set $(\nu, \nu') = (2, 4)$. The contributions to k' and k'' coming from $\mathcal{C}^{[i,j]}_{\mathrm{loc}}$ are

$$\begin{aligned}
k'_{\mathrm{loc}} &= \frac{1}{M^{j|\delta|}} \sum_{\substack{\beta_2,\beta_3 \in \mathbb{N}_0 \times \mathbb{N}_0^2 \\ \beta_2+\beta_3=\delta}} \binom{\delta}{\beta_2,\beta_3} K^{(i_1)} \bullet \mathrm{D}^{\beta_2}_{\nu;\nu'}\mathcal{C}^{[i,j]}_{\mathrm{loc}} \bullet \mathrm{D}^{\beta_3}_{1;\mu'} K^{(i_2)}_{\Sigma_{i_2},\Sigma_j}, \\
k''_{\mathrm{loc}} &= \frac{1}{M^{j|\delta|}} \sum_{\substack{\beta_1,\beta_2,\beta_3 \in \mathbb{N}_0 \times \mathbb{N}_0^2 \\ \beta_1+\beta_2+\beta_3=\delta \\ |\beta_1|>0}} \binom{\delta}{\beta_1,\beta_2,\beta_3} \mathrm{D}^{\beta_1}_{\mu;3} K^{(i_1)} \bullet \mathrm{D}^{\beta_2}_{\nu;\nu'}\mathcal{C}^{[i,j]}_{\mathrm{loc}} \bullet \mathrm{D}^{\beta_3}_{1;\mu'} K^{(i_2)}_{\Sigma_{i_2},\Sigma_j}.
\end{aligned}$$

We first bound k'_{loc}. Fix $\beta_2 + \beta_3 = \delta$. First consider $\beta_2 \ne 0$. Let $(\delta_\mathrm{l}, \delta_\mathrm{r}) = (\alpha, \gamma)$ or (α', γ'). By Lemma II.27.a,

$$\begin{aligned}
\tfrac{1}{M^{j|\delta|}} \Big| K^{(i_1)} \bullet \mathrm{D}^{\beta_2}_{\nu;\nu'}\mathcal{C}^{[i,j]}_{\mathrm{loc}} \bullet \mathrm{D}^{\beta_3}_{1;\mu'} K^{(i_2)}_{\Sigma_{i_2},\Sigma_j} \Big|_{i_1,j}^{[\delta_\mathrm{l},0,\delta_\mathrm{r}]} &\le \mathrm{const}\, q(\delta_\mathrm{l},0,0;\delta_\mathrm{l},\delta_\mathrm{l}) \big| K^{(i_2)} \big|_{i_2}^{[0,\beta_3,\delta_\mathrm{r}]} \\
&\le \mathrm{const}\, c_F^2 \rho^2 \mathfrak{v}_{i_1}\, \tfrac{i_2}{i_1}.
\end{aligned}$$

Next consider $\beta_2 = 0$. By Lemma II.27.b, with $\delta_l = \alpha$ and $\delta_r = \gamma$,

$$\begin{aligned}
&\tfrac{1}{M^{j|\delta|}}\Big| K^{(i_1)} \bullet \mathrm{D}^{\beta_2}_{\nu;\nu'}\mathcal{C}^{[i,j]}_{\rm loc} \bullet \mathrm{D}^{\beta_3}_{1;\mu'} K^{(i_2)}_{\Sigma_{i_2},\Sigma_j}\Big|^{[\alpha,0,\gamma]}_{i_1,j} \\
&\quad = \tfrac{1}{M^{j|\delta|}}\Big| K^{(i_1)} \bullet \mathcal{C}^{[i,j]}_{\rm loc} \bullet \mathrm{D}^{\delta}_{1;\mu'} K^{(i_2)}_{\Sigma_{i_2},\Sigma_j}\Big|^{[\alpha,0,\gamma]}_{i_1,j} \\
&\quad \le \mathrm{const} \max_{\substack{\alpha_{\rm up},\alpha_{\rm dn},\alpha_l\in\mathbb{N}_0\times\mathbb{N}_0^2 \\ |\alpha_{\rm up}|+|\alpha_{\rm dn}|+|\alpha_l|\le 3}} q(\alpha,0,\alpha_{\rm up}+\alpha_{\rm dn};\alpha+\alpha_{\rm up},\alpha+\alpha_{\rm dn})\Big[\big|K^{(i_2)}\big|^{[0,\delta,\gamma]}_{i_2} + \big|K^{(i_2)}\big|^{[\alpha_l,0,0]}_{i_2}\Big] \\
&\quad \le \mathrm{const}\, c_F^2 \rho^2 \mathfrak{v}_{i_1} \tfrac{i_2}{i_1}
\end{aligned}$$

and

$$\begin{aligned}
&\tfrac{1}{M^{j|\delta|}}\Big| K^{(i_1)} \bullet \mathrm{D}^{\beta_2}_{\nu;\nu'}\mathcal{C}^{[i,j]}_{\rm loc} \bullet \mathrm{D}^{\beta_3}_{1;\mu'} K^{(i_2)}_{\Sigma_{i_2},\Sigma_j}\Big|^{[\alpha',0,\gamma']}_{j,j} \\
&\quad \le \mathrm{const}\, \tfrac{1}{M^{j|\delta|}}\Big| K^{(i_1)} \bullet \mathcal{C}^{[i,j]}_{\rm loc} \bullet \mathrm{D}^{\delta}_{1;\mu'} K^{(i_2)}_{\Sigma_{i_2},\Sigma_j}\Big|^{[0,0,\gamma']}_{i_1,j} \\
&\qquad +\mathrm{const}\, \tfrac{1}{M^{j-i_1}}\tfrac{1}{M^{j|\delta|}}\Big| K^{(i_1)} \bullet \mathcal{C}^{[i,j]}_{\rm loc} \bullet \mathrm{D}^{\delta}_{1;\mu'} K^{(i_2)}_{\Sigma_{i_2},\Sigma_j}\Big|^{[\alpha',0,\gamma']}_{i_1,j} \\
&\quad \le \mathrm{const}\, \tfrac{1}{M^{j|\delta|}}\Big| K^{(i_1)} \bullet \mathcal{C}^{[i,j]}_{\rm loc} \bullet \mathrm{D}^{\delta}_{1;\mu'} K^{(i_2)}_{\Sigma_{i_2},\Sigma_j}\Big|^{[0,0,\gamma']}_{i_1,j} \\
&\qquad +\mathrm{const}\, \tfrac{j-i+1}{M^{j-i_1}} q(\alpha',0,0;\alpha',\alpha')\big|K^{(i_2)}\big|^{[0,\delta,\gamma']}_{i_2,j} \\
&\quad \le \mathrm{const}\, c_F^2\rho^2\mathfrak{v}_{i_1}\tfrac{i_2}{i_1}.
\end{aligned}$$

In the first step we applied Lemma II.17. In the second, we applied Lemma II.27.c with $\beta_1 = \beta_2 = 0$, $\beta_3 = \delta$, $\delta_l = \alpha'$ and $\delta_r = \gamma'$. In the third step we applied the conclusion of the last estimate and Lemma II.17.

To bound $\big|k''_{\rm loc}\big|^{[\delta_l,0,\delta_r]}_{i_1,j}$ with $\delta_l + \delta_r + \delta \in \Delta$ observe that, by Lemma II.27.c, for all $\beta_1 + \beta_2 + \beta_3 = \delta$ with $\beta_1 \ne 0$,

$$\begin{aligned}
&\tfrac{1}{M^{j|\delta|}}\Big| \mathrm{D}^{\beta_1}_{\mu;3}K^{(i_1)} \bullet \mathrm{D}^{\beta_2}_{\nu;\nu'}\mathcal{C}^{[i,j]}_{\rm loc} \bullet \mathrm{D}^{\beta_3}_{1;\mu'} K^{(i_2)}_{\Sigma_{i_2},\Sigma_j}\Big|^{[\delta_l,0,\delta_r]}_{i_1,j} \\
&\quad \le \mathrm{const}\, \tfrac{j-i+1}{M^{|\beta_1|(j-i_1)}} q(\delta_l,\beta_1,0;\delta_l+\beta_1,\delta_l+\beta_1)\big|K^{(i_2)}\big|^{[0,\beta_3,\delta_r]}_{i_2,j} \\
&\quad \le \mathrm{const}\, \tfrac{j-i_1+1}{M^{j-i_1}} c_F^2\rho^2\mathfrak{v}_{i_1}\tfrac{i_2}{i_1}.
\end{aligned}$$

Setting $(\delta_l, \delta_r) = (0, \gamma)$, we get the k'' estimate of (II.9). Setting $(\delta_l, \delta_r) = (\alpha', \gamma')$ and using Lemma II.17, we get the k'' estimate of (II.10).

When $\ell = 1$, part c follows from (II.9).

d) Again, we may assume, without loss of generality that $i_1 \ge i_2$. By part c and (II.10),

$$\begin{aligned}
\tfrac{1}{M^{j|\delta_c|}}\Big| \mathrm{D}^{\delta_c}_{\mu;\mu'}\Big[K^{(i_1)} \bullet \mathcal{C}^{[i,j]} \bullet K^{(i_2)}_{\Sigma_{i_2},\Sigma_j}\Big]\Big|^{[\delta_l,0,\delta_r]}_{j,j} &\le \big|k'\big|^{[\delta_l,0,\delta_r]}_{j,j} + \big|k''\big|^{[\delta_l,0,\delta_r]}_{j,j} \\
&\le \mathrm{const}\, c_F^2\rho^2\, \mathfrak{v}_{i_1}.
\end{aligned}$$

This finishes the case $\ell = 1$.

Induction step. We assume that the lemma holds for $\ell - 1$. Write

$$K^{(i_1)} \bullet \mathcal{C}^{[i,j]} \bullet K^{(i_2)} \bullet \cdots \bullet K^{(i_{\ell+1})} = K^{(i_1)} \bullet \mathcal{C}^{[i,j]} \bullet H$$

with

$$H = K^{(i_2)} \bullet \mathcal{C}^{[\max\{i_2,i_3\},j]} \bullet \cdots \bullet K^{(i_{\ell+1})}.$$

Set

$$\mathfrak{V} = \mathrm{const}^{\ell-1}\, (c_F\rho)^{\ell}\ \mathfrak{v}_{i_2} \cdots \mathfrak{v}_{i_\ell}.$$

The induction hypothesis applies to H. So, for all $|\alpha| \le 3$ and $\delta \in \Delta$,

$$\big|H\big|_{i_2,j}^{[\alpha,0,\delta]} \ \le\ \mathfrak{V}\, \tfrac{i_{\ell+1}}{i_2}.$$

Furthermore, for each $0 \ne \delta \in \Delta$, $\mu \in \{1,2\}$ and $\mu' \in \{3,4\}$, there is a decomposition

$$\tfrac{1}{M^{j|\delta|}} \mathrm{D}^{\delta}_{\mu;\mu'} H_{\Sigma_{i_2},\Sigma_j} = h'_{\delta,\mu,\mu'} + h''_{\delta,\mu,\mu'}$$

with, for all $|\alpha| \le 3$ and all γ with $\gamma + \delta \in \Delta$,

$$\begin{aligned} \big|h'_{\delta,\mu,\mu'}\big|_{i_2,j}^{[\alpha,0,\gamma]} &\le \mathfrak{V}\, \tfrac{i_{\ell+1}}{i_2},\\ \big|h''_{\delta,\mu,\mu'}\big|_{i_2,j}^{[0,0,\gamma]} &\le \tfrac{j-i_2+1}{M^{j-i_2}}\, \mathfrak{V}\, \tfrac{i_{\ell+1}}{i_2}. \end{aligned}$$

In particular,

$$\big|H\big|_{i_2,j}^{[0,\delta_{\rm c},\delta_{\rm r}]} \ \le\ 2\mathfrak{V}\, \tfrac{i_{\ell+1}}{i_2}$$

for $\delta_{\rm c} + \delta_{\rm r} \in \Delta$.

a) By Lemma II.27.b, with $\delta = 0$, $\delta_{\rm l} = \alpha$ and $\delta_{\rm r} = \delta$, (II.7) and Lemma II.17,

$$\begin{aligned} \Big|K^{(i_1)} \bullet \mathcal{C}^{[i,j]} \bullet H\Big|_{i_1,j}^{[\alpha,0,\delta]} &\le \mathrm{const} \max_{\substack{\alpha_{\rm up},\alpha_{\rm dn},\alpha_{\rm l}\in\mathbb{N}_0\times\mathbb{N}_0^2\\ |\alpha_{\rm up}|+|\alpha_{\rm dn}|+|\alpha_{\rm l}|\le 3}} q(\alpha,0,\alpha_{\rm up}+\alpha_{\rm dn};\alpha+\alpha_{\rm up},\alpha+\alpha_{\rm dn})\,|H|_{i_2,j}^{[\alpha_{\rm l},0,\delta]}\\ &\le \mathrm{const}\, c_F\rho\ \mathfrak{v}_{i_1} \tfrac{i_2}{i_1}\, \mathfrak{V}\, \tfrac{i_{\ell+1}}{i_2} \le \mathrm{const}\, c_F\rho\ \mathfrak{v}_{i_1}\, \mathfrak{V}\, \tfrac{i_{\ell+1}}{i_1}. \end{aligned}$$

b) Again, we may assume, without loss of generality that $i_1 \ge i_{\ell+1}$. Then the factor $\mathrm{const}^{\ell}\, (c_F\rho)^{\ell+1} \mathfrak{v}_{i_2} \cdots \mathfrak{v}_{i_\ell} \min\big\{\mathfrak{v}_{i_1}, \mathfrak{v}_{i_{\ell+1}}\big\}$ in the right-hand side of the statement reduces to $\mathrm{const}\, c_F\rho\, \mathfrak{v}_{i_1}\mathfrak{V}$. The remainder of the proof is virtually identical to that for $\ell = 1$.

c) Substitute $\mathcal{C}^{[i,j]} = \mathcal{C}^{[i,j]}_{\rm top} + \mathcal{C}^{[i,j]}_{\rm mid} + \mathcal{C}^{[i,j]}_{\rm bot}$ into

$$\mathrm{D}^{\delta}_{\mu;\mu'}\Big[\big(K^{(i_1)} \bullet \mathcal{C}^{[i,j]} \bullet H\big)_{\Sigma_{i_1},\Sigma_j}\Big] = \mathrm{D}^{\delta}_{\mu;\mu'}\Big[K^{(i_1)} \bullet \mathcal{C}^{[i,j]} \bullet H_{\Sigma_{i_2},\Sigma_j}\Big]$$

and apply Leibniz's rule (Lemma II.21) using the routing which gives $\mathrm{D}^{\beta_2}_{1;3}\mathcal{C}^{[i,j]}_{\rm top}$, $\mathrm{D}^{\beta_2}_{1;3}\mathcal{C}^{[i,j]}_{\rm mid}$ and $\mathrm{D}^{\beta_2}_{2;4}\mathcal{C}^{[i,j]}_{\rm bot}$. We define k' to be $\frac{1}{M^{j|\delta|}}$ times the sum of all resulting terms having no derivatives acting on $K^{(i_1)}$ and k'' to be $\frac{1}{M^{j|\delta|}}$ times the sum of all terms having at least one derivative acting on $K^{(i_1)}$. Fix any α, α' and γ, γ' obeying, $|\alpha| \le 3$, $\gamma + \delta \in \Delta$ and $\alpha' + \gamma' + \delta \in \Delta$. We show

$$\big|k'\big|_{i_1,j}^{[\alpha,0,\gamma]} \le \mathrm{const}\, c_F\rho\, \mathfrak{v}_{i_1}\, \mathfrak{V}\, \tfrac{i_{\ell+1}}{i_1},$$

$$\left|k''\right|_{i_1,j}^{[0,0,\gamma]} \le \text{const}\, c_F\, \tfrac{j-i_1+1}{M^{j-i_1}}\, \rho\, \mathfrak{v}_{i_1}\, \mathfrak{V}\, \tfrac{i_{\ell+1}}{i_1} \tag{II.11}$$

and

$$\left|k'\right|_{j,j}^{[\alpha',0,\gamma']} + \left|k''\right|_{j,j}^{[\alpha',0,\gamma']} \le \text{const}\, c_F\, \rho\, \mathfrak{v}_{i_1}\, \mathfrak{V}\, \tfrac{i_{\ell+1}}{i_1}. \tag{II.12}$$

Let loc $\in$ {mid, top, bot}. If loc $\in$ {top, mid}, set $(\nu, \nu') = (1, 3)$. If loc = bot, set $(\nu, \nu') = (2, 4)$. The contributions to k' and k'' coming from $\mathcal{C}_{\text{loc}}^{[i,j]}$ are

$$k'_{\text{loc}} = \tfrac{1}{M^{j|\delta|}} \sum_{\substack{\beta_2,\beta_3\in\mathbb{N}_0\times\mathbb{N}_0^2\\ \beta_2+\beta_3=\delta}} \binom{\delta}{\beta_2,\beta_3} K^{(i_1)} \bullet \mathrm{D}^{\beta_2}_{\nu;\nu'} \mathcal{C}_{\text{loc}}^{[i,j]} \bullet \mathrm{D}^{\beta_3}_{1;\mu'} H_{\Sigma_{i_2},\Sigma_j},$$

$$k''_{\text{loc}} = \tfrac{1}{M^{j|\delta|}} \sum_{\substack{\beta_1,\beta_2,\beta_3\in\mathbb{N}_0\times\mathbb{N}_0^2\\ \beta_1+\beta_2+\beta_3=\delta\\ |\beta_1|>0}} \binom{\delta}{\beta_1,\beta_2,\beta_3} \mathrm{D}^{\beta_1}_{\mu;3} K^{(i_1)} \bullet \mathrm{D}^{\beta_2}_{\nu;\nu'} \mathcal{C}_{\text{loc}}^{[i,j]} \bullet \mathrm{D}^{\beta_3}_{1;\mu'} H_{\Sigma_{i_2},\Sigma_j}.$$

We first bound k'_{loc}. Fix $\beta_2 + \beta_3 = \delta$. First consider $\beta_2 \ne 0$. Let $(\delta_{\text{l}}, \delta_{\text{r}}) = (\alpha, \gamma)$ or (α', γ'). By Lemma II.27.a,

$$\begin{aligned}\tfrac{1}{M^{j|\delta|}} \left| K^{(i_1)} \bullet \mathrm{D}^{\beta_2}_{\nu;\nu'} \mathcal{C}_{\text{loc}}^{[i,j]} \bullet \mathrm{D}^{\beta_3}_{1;\mu'} H_{\Sigma_{i_2},\Sigma_j} \right|_{i_1,j}^{[\delta_{\text{l}},0,\delta_{\text{r}}]} &\le \text{const}\, q(\delta_{\text{l}},0,0;\delta_{\text{l}},\delta_{\text{l}})\, |H|_{i_2,j}^{[0,\beta_3,\delta_{\text{r}}]} \\ &\le \text{const}\, c_F\, \rho\, \mathfrak{v}_{i_1}\, \tfrac{i_2}{i_1}\, \mathfrak{V}\, \tfrac{i_{\ell+1}}{i_2} \\ &\le \text{const}\, c_F\, \rho\, \mathfrak{v}_{i_1}\, \mathfrak{V}\, \tfrac{i_{\ell+1}}{i_1}.\end{aligned}$$

Next consider $\beta_2 = 0$. By (II.7),

$$\begin{aligned}&\tfrac{1}{M^{j|\delta|}} \left| K^{(i_1)} \bullet \mathrm{D}^{\beta_2}_{\nu;\nu'} \mathcal{C}_{\text{loc}}^{[i,j]} \bullet \mathrm{D}^{\beta_3}_{1;\mu'} H_{\Sigma_{i_2},\Sigma_j} \right|_{i_1,j}^{[\alpha,0,\gamma]} \\ &= \tfrac{1}{M^{j|\delta|}} \left| K^{(i_1)} \bullet \mathcal{C}_{\text{loc}}^{[i,j]} \bullet \mathrm{D}^{\delta}_{1;\mu'} H_{\Sigma_{i_2},\Sigma_j} \right|_{i_1,j}^{[\alpha,0,\gamma]} \\ &\le \left| K^{(i_1)} \bullet \mathcal{C}_{\text{loc}}^{[i,j]} \bullet h'_{\delta,1,\mu'} \right|_{i_1,j}^{[\alpha,0,\gamma]} + \left| K^{(i_1)} \bullet \mathcal{C}_{\text{loc}}^{[i,j]} \bullet h''_{\delta,1,\mu'} \right|_{i_1,j}^{[\alpha,0,\gamma]} \\ &\le \text{const} \max_{\substack{\alpha_{\text{up}},\alpha_{\text{dn}},\alpha_1\in\mathbb{N}_0\times\mathbb{N}_0^2\\ |\alpha_{\text{up}}|+|\alpha_{\text{dn}}|+|\alpha_1|\le 3}} q(\alpha,0,\alpha_{\text{up}}+\alpha_{\text{dn}};\alpha+\alpha_{\text{up}},\alpha+\alpha_{\text{dn}}) \left|h'_{\delta,1,\mu'}\right|_{i_2,j}^{[\alpha_1,0,\gamma]} \\ &\quad + \text{const}\, q(\alpha,0,0;\alpha,\alpha)(j-i+1) \left|h''_{\delta,1,\mu'}\right|_{i_2,j}^{[0,0,\gamma]} \\ &\le \text{const}\, c_F\, \rho\, \mathfrak{v}_{i_1}\, \mathfrak{V}\, \tfrac{i_{\ell+1}}{i_1},\end{aligned}$$

since $\frac{(j-i+1)(j-i_2-1)}{M^{j-i_2}} \le \text{const}$. The term with $h'_{\delta,1,\mu'}$ was bounded using Lemma II.27.b with $\delta = 0$. The term with $h''_{\delta,1,\mu'}$ was bounded using Lemma II.27.c with $\beta_1 = \beta_2 = \beta_3 = 0$. Again, with $\beta_2 = 0$,

$$\tfrac{1}{M^{j|\delta|}} \left| K^{(i_1)} \bullet \mathrm{D}^{\beta_2}_{\nu;\nu'} \mathcal{C}_{\text{loc}}^{[i,j]} \bullet \mathrm{D}^{\beta_3}_{1;\mu'} H_{\Sigma_{i_2},\Sigma_j} \right|_{j,j}^{[\alpha',0,\gamma']} \le \text{const}\, c_F\, \rho\, \mathfrak{v}_{i_1}\, \mathfrak{V}\, \tfrac{i_{\ell+1}}{i_1},$$

as in the proof of part (c) when $\ell = 1$.

We bound $\big|k''_{\rm loc}\big|^{[\delta_{\rm l},0,\delta_{\rm r}]}_{i_1,j}$ with $\delta_{\rm l}+\delta_{\rm r}+\delta\in\Delta$ as for $\ell=1$. Observe that, by Lemma II.27.c, for all $\beta_1+\beta_2+\beta_3=\delta$ with $\beta_1\ne 0$,

$$\begin{aligned}\tfrac{1}{M^{j|\delta|}}&\Big|{\rm D}^{\beta_1}_{\mu;3}K^{(i_1)}\bullet {\rm D}^{\beta_2}_{1,3}\mathcal{C}^{[i,j]}_{\rm loc}\bullet {\rm D}^{\beta_3}_{1;\mu'}H_{\Sigma_{i_2},\Sigma_j}\Big|^{[\delta_{\rm l},0,\delta_{\rm r}]}_{i_1,j}\\ &\le {\rm const}\ \tfrac{j-i+1}{M^{|\beta_1|(j-i_1)}}\,q\big(\delta_{\rm l},\beta_1,0;\delta_{\rm l}+\beta_1,\delta_{\rm l}+\beta_1\big)|H|^{[0,\beta_3,\delta_{\rm r}]}_{i_2,j}\\ &\le {\rm const}\ \tfrac{j-i_1+1}{M^{j-i_1}}c_F\rho\,\mathfrak{v}_{i_1}\ \mathfrak{V}\ \tfrac{i_{\ell+1}}{i_1}.\end{aligned}$$

Setting $(\delta_{\rm l},\delta_{\rm r})=(0,\gamma)$, we get the k'' estimate of (II.11). Setting $(\delta_{\rm l},\delta_{\rm r})=(\alpha',\gamma')$ and using Lemma II.17, we get the k'' estimate of (II.12).
d) Part (d) follows from part (c) and (II.12) as in the case $\ell=1$. □

Completion of the proof of Theorem I.20. We prove (II.5). Let $\vec\delta=(\delta_{\rm l},\delta_{\rm c},\delta_{\rm r})\in\vec\Delta$. By parts b) and d) of the lemma above, for $\ell\ge 1$,

$$\begin{aligned}&\Big|K^{(i_1)}\bullet\mathcal{C}^{[\max\{i_1,i_2\},j]}\bullet K^{(i_2)}\bullet\cdots\bullet K^{(i_{\ell+1})}\Big|^{[\vec\delta]}_{j,j}\\ &\quad\le {\rm const}^\ell\ (c_F\rho)^{\ell+1}\ \mathfrak{v}_{i_2}\cdots\mathfrak{v}_{i_\ell}\ \min\big\{\mathfrak{v}_{i_1},\mathfrak{v}_{i_{\ell+1}}\big\}.\end{aligned}$$

Therefore, by Cor. II.12.i,

$$\begin{aligned}\big|\mathcal{L}^{(j+1)}_S\big|^{[\vec\delta]}_{j,j}&\le\sum_{\ell=1}^\infty\ \sum_{i_1,\cdots,i_{\ell+1}=2}^{j}{\rm const}^\ell\ (c_F\rho)^{\ell+1}\ \mathfrak{v}_{i_2}\cdots\mathfrak{v}_{i_\ell}\ \min\big\{\mathfrak{v}_{i_1},\mathfrak{v}_{i_{\ell+1}}\big\}\\ &\le {\rm const}\,c_F^2\rho^2\ \sum_{\ell=1}^\infty\ \Big(({\rm const}\,c_F\rho)^{\ell-1}\sum_{i_2,\cdots,i_\ell=2}^\infty\mathfrak{v}_{i_2}\cdots\mathfrak{v}_{i_\ell}\Big)\\ &\quad\times\Big(\sum_{i_1,i_{\ell+1}=2}^\infty\min\big\{\mathfrak{v}_{i_1},\mathfrak{v}_{i_{\ell+1}}\big\}\Big)\\ &\le {\rm const}\,c_F^2\rho^2\ \sum_{\ell=1}^\infty\ ({\rm const}\,c_F\rho)^{\ell-1}\Big(\sum_{i_1\ge i_{\ell+1}}\mathfrak{v}_{i_1}+\sum_{i_1<i_{\ell+1}}\mathfrak{v}_{i_{\ell+1}}\Big)\\ &\le {\rm const}\,c_F^2\rho^2\ \sum_{i=2}^\infty(i-1)\mathfrak{v}_i\ \le\ {\rm const}\,c_F^2\rho^2\ =\ c_{\mathcal{L}}\rho^2,\end{aligned}$$

when ρ is small enough. This concludes the induction step in the proof of Theorem I.20.
□

5. The Infrared Limit. We now turn to the proof of Theorem I.22, which asserts that compound particle hole ladders have infrared limits. Define, for each $j\ge 2$, $\ell\ge 1$ and $\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}\ge 2$, the function

$$\mathfrak{L}^{(j)}_{\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}:\mathbb{M}^3\times\{\uparrow,\downarrow\}^4\to\mathbb{C}$$

by

$$\begin{aligned}&\mathfrak{L}^{(j)}_{\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}(q,q',t,\sigma_1,\cdots\sigma_4)\\ &\quad=\Big[\big(F^{(\mathrm{i}_1)}+L^{(\mathrm{i}_1)}f\big)\bullet\mathcal{C}^{[\max\{\mathrm{i}_1,\mathrm{i}_2\},j]}\bullet\big(F^{(\mathrm{i}_2)}+L^{(\mathrm{i}_2)}f\big)\bullet\mathcal{C}^{[\max\{\mathrm{i}_\ell,\mathrm{i}_{\ell+1}\},j]}\\ &\qquad\cdots\bullet\big(F^{(\mathrm{i}_{\ell+1})}+L^{(\mathrm{i}_{\ell+1})}f\big)\Big]_{i_1,i_2,i_3,i_4=0}\big((q+\tfrac{t}{2},\sigma_1),(q-\tfrac{t}{2},\sigma_2),(q'+\tfrac{t}{2},\sigma_3),(q'-\tfrac{t}{2},\sigma_4)\big).\end{aligned}$$

By Prop. II.3.ii,

$$\mathcal{L}^{(j+1)}\Big|_{i_1,i_2,i_3,i_4=0}\big((q+\tfrac{t}{2},\sigma_1),(q-\tfrac{t}{2},\sigma_2),(q'+\tfrac{t}{2},\sigma_3),(q'-\tfrac{t}{2},\sigma_4)\big)$$
$$=\sum_{\ell=1}^{\infty}\sum_{\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}=2}^{j}\mathfrak{L}^{(j)}_{\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}(q,q',t,\sigma_1,\cdots\sigma_4).$$

Lemma II.28.

$$\sum_{\ell=1}^{\infty}\sum_{\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}=2}^{\infty}\sup_{j\ge\max\{\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}\}}\sup_{\substack{q,q',t\in\mathbb{M}\\ \sigma_i\in\{\uparrow,\downarrow\}}}\left|\mathfrak{L}^{(j)}_{\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}(q,q',t,\sigma_1,\cdots\sigma_4)\right|<\infty.$$

Proof. By Lemma II.26.ii, with $\delta_{\mathrm{l}}=\delta_{\mathrm{r}}=0$, and the analogous bound for $\mathcal{L}_C^{(j+1)}$,

$$\sup_{\substack{q,q',t\in\mathbb{M}\\ \sigma_i\in\{\uparrow,\downarrow\}}}\left|\mathfrak{L}^{(j)}_{\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}(q,q',t,\sigma_1,\cdots\sigma_4)\right|\le\mathrm{const}^{\ell}\left(c_F\rho\right)^{\ell+1}\mathfrak{v}_{\mathrm{i}_2}\cdots\mathfrak{v}_{\mathrm{i}_\ell}\min\left\{\mathfrak{v}_{\mathrm{i}_1},\mathfrak{v}_{\mathrm{i}_{\ell+1}}\right\}$$

uniformly in j. Hence, as in the final part of the proof of Theorem I.20,

$$\begin{aligned}&\sum_{\ell=1}^{\infty}\sum_{\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}=2}^{\infty}\sup_{j\ge\max\{\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}\}}\sup_{\substack{q,q',t\in\mathbb{M}\\ \sigma_i\in\{\uparrow,\downarrow\}}}\left|\mathfrak{L}^{(j)}_{\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}(q,q',t,\sigma_1,\cdots\sigma_4)\right|\\
&\quad\le\mathrm{const}\,c_F^2\rho^2\sum_{\ell=1}^{\infty}\Big((\mathrm{const}\,c_F\rho)^{\ell-1}\sum_{\mathrm{i}_2,\cdots,\mathrm{i}_\ell=2}^{\infty}\mathfrak{v}_{\mathrm{i}_2}\cdots\mathfrak{v}_{\mathrm{i}_\ell}\Big)\Big(\sum_{\mathrm{i}_1,\mathrm{i}_{\ell+1}=2}^{\infty}\min\left\{\mathfrak{v}_{\mathrm{i}_1},\mathfrak{v}_{\mathrm{i}_{\ell+1}}\right\}\Big)\\
&\quad\le\mathrm{const}\,c_F^2\rho^2\sum_{\ell=1}^{\infty}(\mathrm{const}\,c_F\rho)^{\ell-1}\Big(\sum_{\mathrm{i}_1\ge\mathrm{i}_{\ell+1}}\mathfrak{v}_{\mathrm{i}_1}+\sum_{\mathrm{i}_1<\mathrm{i}_{\ell+1}}\mathfrak{v}_{\mathrm{i}_{\ell+1}}\Big)\\
&\quad\le\mathrm{const}\,c_F^2\rho^2\sum_{\mathrm{i}=2}^{\infty}(\mathrm{i}-1)\mathfrak{v}_{\mathrm{i}}\;<\;\infty,\end{aligned}$$

when ρ is small enough. □

Lemma II.29. *For $t\ne0$, the limit*

$$\mathfrak{L}_{\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}(q,q',t,\sigma_1,\cdots\sigma_4)=\lim_{j\to\infty}\mathfrak{L}^{(j)}_{\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}(q,q',t,\sigma_1,\cdots\sigma_4)$$

exists. The limit is continuous in (q,q',t) for $t\ne0$. The restrictions to $\mathbf{t}=0$ and to $t_0=0$, namely, $\mathfrak{L}_{\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}(q,q',(t_0,\mathbf{0}),\sigma_1,\cdots\sigma_4)$ and $\mathfrak{L}_{\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}(q,q',(0,\mathbf{t}),\sigma_1,\cdots\sigma_4)$, have continuous extensions to $t=0$.

Proof. It suffices to consider separately the spin and charge parts, in the sense of Lemma II.8, of $\mathfrak{L}^{(j)}_{\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}(q,q',t,\sigma_1,\cdots\sigma_4)$. We denote them $\mathfrak{L}^{(j)}_{X,\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}(q,q',t)$ with $X=S,C$. The existence and continuity of the limits when $t\ne0$ shall be proven in Lemma III.29.

Recall that the bubble propagator $\mathcal{C}^{[i,j]}$ has momentum space kernel

$$\mathcal{C}^{[i,j]}(p,k)=\frac{\nu^{(\ge i)}(p)\nu^{(\ge i)}(k)-\nu^{(\ge j+1)}(p)\nu^{(\ge j+1)}(k)}{[ip_0-e'(p)][ik_0-e'(k)]},$$

where $e'(k) = e(\mathbf{k}) - v(k)$. Define the model particle–hole bubble propagators

$$\begin{aligned} \mathcal{A}_{i,j}(p,k) &= \tfrac{\nu^{(\ge i)}(p)\nu^{(\ge i)}(k)[1-\nu_j(e(\mathbf{p}))\nu_j(e(\mathbf{k}))]}{[ip_0-e'(p)][ik_0-e'(k)]}, \\ \mathcal{B}_{i,j}(p,k) &= \tfrac{\nu^{(\ge i)}(p)\nu^{(\ge i)}(k)[1-\nu_j(p_0)\nu_j(k_0)]}{[ip_0-e'(p)][ik_0-e'(k)]}, \end{aligned} \tag{II.13}$$

where

$$\nu_j(\omega) = \sum_{m=j}^{\infty} \nu\big(M^{2m}\omega^2\big)$$

with ν being the single scale cutoff introduced in Def. I.2. Let

$$\begin{aligned} A^{(j)}_{X,\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}(q,q',t_0) = \big[K_X^{(\mathrm{i}_{\ell+1})} \bullet \mathcal{A}_{\max\{\mathrm{i}_{\ell+1},\mathrm{i}_\ell\},j} \bullet K_X^{(\mathrm{i}_\ell)} \bullet \cdots \bullet \mathcal{A}_{\max\{\mathrm{i}_2,\mathrm{i}_1\},j} \\ \bullet K_X^{(\mathrm{i}_1)}\big]_{i_1,i_2,i_3,i_4=0}(q+\tfrac{t}{2},q-\tfrac{t}{2},q'+\tfrac{t}{2},q'-\tfrac{t}{2})\big|_{\mathbf{t}=0} \end{aligned}$$

and

$$\begin{aligned} B^{(j)}_{X\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}(q,q',\mathbf{t}) = \big[K_X^{(\mathrm{i}_{\ell+1})} \bullet \mathcal{B}_{\max\{\mathrm{i}_{\ell+1},\mathrm{i}_\ell\},j} \bullet K_X^{(\mathrm{i}_\ell)} \bullet \cdots \\ \bullet \mathcal{B}_{\max\{\mathrm{i}_2,\mathrm{i}_1\},j} \bullet K_X^{(\mathrm{i}_1)}\big]_{i_1,i_2,i_3,i_4=0}(q+\tfrac{t}{2},q-\tfrac{t}{2},q'+\tfrac{t}{2},q'-\tfrac{t}{2})\big|_{t_0=0}, \end{aligned}$$

where $K_X^{(i)}$ is $F_S^{(i)} + \frac{1}{2}L^{(i)\,f}_{C\,\Sigma_i} - \frac{1}{2}L^{(i)\,f}_{S\,\Sigma_i}$ when $X = S$ and $F_C^{(i)} + \frac{1}{2}L^{(i)\,f}_{C\,\Sigma_i} + \frac{3}{2}L^{(i)\,f}_{S\,\Sigma_i}$ when $X = C$. By Cor. III.31 the differences

$$\mathcal{L}^{(j)}_{X,\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}(q,q',(t_0,\mathbf{0})) - A^{(j)}_{X\ell,\mathrm{i}_{\ell+1},\cdots,\mathrm{i}_1}(q,q',t_0)$$

and

$$\mathcal{L}^{(j)}_{X,\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}(q,q',(0,\mathbf{t})) - B^{(j)}_{X\ell,\mathrm{i}_{\ell+1},\cdots,\mathrm{i}_1}(q,q',\mathbf{t})$$

both converge to zero for all $t \ne 0$. The bounds on the $|K_X^{(\mathrm{i}_m)}|_{\mathrm{i}_m,\mathrm{i}_m}$'s required by Cor. III.31 are provided by (II.8) with $\delta = 0$.

That $\lim\limits_{j\to\infty} A^{(j)}_{X\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}(q,q',t_0)$ and $\lim\limits_{j\to\infty} B^{(j)}_{X\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}(q,q',\mathbf{t})$ exist and are continuous at $t = 0$ is proven using Lemma B.3 inductively on ℓ, with $I = F$, the full Fermi surface. For the induction step from $\ell - 1$ to ℓ, set $z = (q,q')$ and use

$$\begin{aligned} u_j(k,t_0,z) &= K_X^{(i_{\ell+1})}(q+\tfrac{t}{2},q-\tfrac{t}{2},k+\tfrac{t}{2},k-\tfrac{t}{2})\big|_{\mathbf{t}=0} \\ &\quad \times \nu^{(\ge i)}(k+t)\nu^{(\ge i)}(k)A^{(j)}_{\ell-1,\mathrm{i}_1,\cdots,\mathrm{i}_\ell}(k,q',t_0), \\ v_j(k,\mathbf{t},z) &= K_X^{(i_{\ell+1})}(q+\tfrac{t}{2},q-\tfrac{t}{2},k+\tfrac{t}{2},k-\tfrac{t}{2})\big|_{t_0=0} \\ &\quad \times \nu^{(\ge i)}(k+t)\nu^{(\ge i)}(k)B^{(j)}_{\ell-1,\mathrm{i}_1,\cdots,\mathrm{i}_\ell}(k,q',\mathbf{t}), \\ n_j(\omega) &= \nu_{i-1}(\omega)\big[1-\nu_j(\omega)^2\big], \\ i &= \max\{\mathrm{i}_{\ell+1},\mathrm{i}_\ell\}. \end{aligned}$$

Also fix some $0 < \aleph'' < \aleph$, and use $\tilde{\aleph} = \aleph'' + \frac{1}{2^\ell}(\aleph - \aleph'')$ and $\aleph' = \aleph'' + \frac{1}{2^{\ell+1}}(\aleph - \aleph'')$.
□

Proof of Theorem I.22. By the Lebesgue dominated convergence theorem and the uniform bounds of Lemma II.28, the existence of the limit $\lim\limits_{j\to\infty}$ and its continuity for $t \neq 0$, as well as the existence and continuity of the limits $\lim\limits_{t_0\to 0}\lim\limits_{j\to\infty}$ and $\lim\limits_{\mathbf{t}\to 0}\lim\limits_{j\to\infty}$ applied to $\mathcal{L}^{(j+1)}\Big|_{i_1,i_2,i_3,i_4=0}((q+\frac{t}{2},\sigma_1),(q-\frac{t}{2},\sigma_2),(q'+\frac{t}{2},\sigma_3),(q'-\frac{t}{2},\sigma_4))$ follow from the corresponding properties of $\mathfrak{L}^{(j)}_{\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}(q,q',t,\sigma_1,\cdots\sigma_4)$, for $\ell \geq 1$ and $\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1} \geq 2$. These were proven in Lemma II.29. $\square$

III. Bubbles

The bulk of this section is devoted to the proof of Theorem II.19. Parts b and c, reformulated as Theorem III.9, are relatively easy to prove. To do so, we fully decompose

$$\mathcal{C}^{[i,j]} = \sum_{m=i}^{j} \sum_{\substack{m_1,m_2\in\mathbb{N}_0 \\ \min\{m_1,m_2\}=m}} C_v^{(m_1)} \otimes C_v^{(m_2)\,t}, \tag{III.1}$$

and bound each term naively to achieve ordinary power counting. The factor $j-i+1 = \sum\limits_{m=i}^{j} 1$ in the first statement of part c is a reflection of the marginality of four–legged diagrams in naive power counting. When power counting bubbles with propagator $\mathrm{D}^{\beta}_{\mu,\mu'}\mathcal{C}^{[i,j]}$, $|\beta| \geq 1$, the sum $\sum\limits_{m=i}^{j} 1$ is replaced by $\sum\limits_{m=i}^{j} M^{|\beta|m} \leq \text{const}\, M^{|\beta|j}$, which is cancelled by the factors $\frac{1}{M^{|\beta|j}}$ on the left hand sides of parts b and c. In the $\beta = 0$ statement of part b, naive power counting gives $\sum\limits_{i_t=i}^{j}\sum\limits_{i_b>j} M^{-(i_b-i_t)} \leq \text{const}$.

The proof of Theorem II.19a, which follows Theorem III.15, relies on two distinct phenomena, volume improvement for large transfer momentum and a sign cancellation in momentum space for small transfer momentum. The mechanism underlying the sign cancellation has been illustrated in the model Lemma I.1 and is fully implemented in Theorem III.15.

We now sketch the idea behind volume improvement. To unravel the sector sums of the $\bullet$ product of Definition I.8, we define, for any translation invariant functions K on $\mathfrak{Y}^2_\Sigma \times (\mathbb{R}\times\mathbb{R}^2)$, K' on $(\mathbb{R}\times\mathbb{R}^2)\times\mathfrak{Y}^2_{\Sigma'}$ and bubble propagator P,

$$K \circ P(y_1,y_2,x_3,x_4) = \int dx_1 dx_2\; K(y_1,y_2,x_1,x_2)P(x_1,x_2,x_3,x_4),$$

$$P \circ K'(x_1,x_2,y_3,y_4) = \int dx_3 dx_4\; P(x_1,x_2,x_3,x_4)K'(x_3,x_4,y_3,y_4).$$

If at least one of y_1,y_2,y_3,y_4 is in $(\mathbb{R}\times\mathbb{R}^2)\times\Sigma$ or $(\mathbb{R}\times\mathbb{R}^2)\times\Sigma'$,

$$K \circ K'(y_1,y_2,y_3,y_4) = \int dx_1 dx_2\; K(y_1,y_2,x_1,x_2)K'(x_1,x_2,y_3,y_4).$$

On the other hand, if all of k_1, k_2, k_3, k_4 are in $\mathbb{M}$, $K \circ K'(k_1, k_2, k_3, k_4)$ is determined by

$$\begin{aligned} &K \circ K'(k_1, k_2, k_3, k_4)\ (2\pi)^3 \delta(k_1 - k_2 - k_3 + k_4) \\ &\quad = \int dx_1 dx_2\ K(k_1, k_2, x_1, x_2)\ K'(x_1, x_2, k_3, k_4) \end{aligned}$$

or equivalently, by

$$K \circ K'(k_1, k_2, k_3, k_4) = \int dx_n\ \ K(k_1, k_2, x_1, x_2)\ K'(x_1, x_2, k_3, k_4)\big|_{x_{3-n}=0}$$

for $n \in \{1, 2\}$. Then, for the functions g and h of the theorem,

$$\begin{aligned} &\big(g \bullet \mathcal{C}^{[i,j]} \bullet h\big)(y_1,y_2,y_3,y_4) \\ &\quad = \sum_{\substack{u_1,u_2 \in \Sigma_i \\ v_1,v_2 \in \Sigma_i}} g(y_1,y_2,(\,\cdot\,,u_1),(\,\cdot\,,u_2)) \circ \mathcal{C}^{[i,j]} \circ h((\,\cdot\,,v_1),(\,\cdot\,,v_2),y_3,y_4). \end{aligned} \tag{III.2}$$

Consider the case in which all external arguments $y_1, \cdots, y_4$ are momenta $k_1, \cdots, k_4$. Then

$$\begin{aligned} \big(g \bullet \mathcal{C}^{[i,j]} \bullet h\big)(k_1,k_2,k_3,k_4) = \tfrac{1}{(2\pi)^3} \int d^3p d^3k \sum_{\substack{u_1,u_2 \in \Sigma_i \\ v_1,v_2 \in \Sigma_i}} \delta(k_1 - k_2 - p + k)\ \check{g}(k_1,k_2,(p,u_1),(k,u_2)) \\ \times \mathcal{C}^{[i,j]}(p, k)\ \check{h}((p,v_1),(k,v_2),k_3,k_4), \end{aligned} \tag{III.3}$$

where

$$\mathcal{C}^{[i,j]}(p, k) = \sum_{m=i}^{j} \sum_{\substack{m_1,m_2 \in \mathbb{N}_0 \\ \min\{m_1,m_2\}=m}} C_v^{(m_1)}(p) C_v^{(m_2)}(k).$$

In order for $\mathcal{C}^{[i,j]}(p, k)$ to be nonzero, one must have p and k in the i^{th} neighbourhood. In particular, $\mathbf{p}$ and $\mathbf{k}$ must lie within a distance $\frac{\text{const}}{M^i}$ of the Fermi curve F. Furthermore, by conservation of momentum at the vertex g, the "transfer momentum"

$$t = k_1 - k_2$$

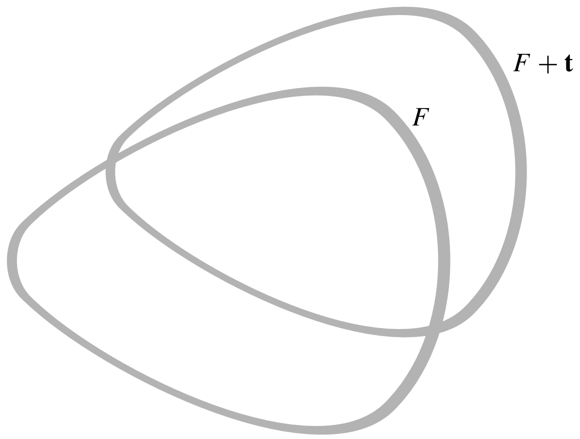
is equal to $p - k$. Thus, the set of pairs (p, k) for which the integrand of (III.3) does not vanish is contained in

$$\big\{\ (k, p) \in \big(\operatorname{supp} C^{(\geq i)}\big)^2\ \big|\ p - k = t\ \big\}.$$

For each fixed large $\mathbf{t}$, the volume of

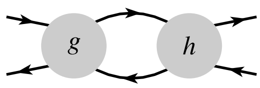
$$\big\{\ \mathbf{p} \in \operatorname{supp} C^{(\geq i)}\ \big|\ \mathbf{p} - \mathbf{t} \in \operatorname{supp} C^{(\geq i)}\ \big\} = \operatorname{supp} C^{(\geq i)} \cap \big(\mathbf{t} + \operatorname{supp} C^{(\geq i)}\big) \tag{III.4}$$

is very small compared to the volume of $\operatorname{supp} C^{(\geq i)}$, as the following figure illustrates.



In naive power counting, the volume of the set (III.4) is bounded by the volume of $\operatorname{supp} C^{(\geq i)}$, yielding a relatively loose bound. There is a similar volume improvement, when, for example, the external arguments $y_1 = (x_1, s_1)$ and $y_2 = (x_2, s_2)$ lie $\big(\mathbb{R} \times \mathbb{R}^2\big) \times \Sigma_\ell$ and the sectors s_1 and s_2 are widely separated. For a more detailed discussion of this volume improvement in perturbation theory see [FKLT2].

A critical factor in determining the size of the L^1–L^∞ norm (in position space) of the bubble



are the decay rates of its various components. We have arranged that the kernels, g and h, have very high decay rates and can be thought of as local objects. Consequently the overall decay rate of the bubble is determined by the higher of the decay rates of the two lines. That is, the lower of the scales of the two lines. As a result, we can control at once all contributions to the bubble with the higher scale running all the way from the lower scale to infinity.

We now give a somewhat more detailed technical outline of the contents of this section. By sector counting and relatively simple propagator estimates, the volume improvement effect can be implemented for all summands

$$\mathcal{C}^{(m)} = \sum_{\substack{m_1, m_2 \in \mathbb{N}_0 \\ \min\{m_1, m_2\} = m}} C_v^{(m_1)} \otimes C_v^{(m_2)\,t}$$

of (III.1) for which $\frac{1}{M^m}$ is small compared to the transfer momentum. Sector counting is made precise in Remark III.12.ii and Lemma C.2. The basic propagator estimates are stated in Appendix A and are adapted to the present situation in Lemma III.14. Lemma III.11 shows how one can combine sector counting and propagator estimates on quantities like $g \bullet \mathcal{C}^{(m)} \bullet h$. The resulting estimates turn out to be summable over those m's for which $\frac{1}{M^m}$ is smaller than the transfer momentum. This is used to prove parts (b) and (c) of Theorem II.19 (which are reformulated as Theorem III.9) and to reduce the statement of part (a) of Theorem II.19 to the situation of transfer momentum smaller than $\mathfrak{l}_j$.

The situation of small transfer momentum is treated in Theorem III.15. To estimate $g \bullet \mathcal{C}^{[i,j]} \bullet h$ when the transfer momentum is small compared to $\mathfrak{l}_j$, we replace $\mathcal{C}^{[i,j]}$ with a model bubble propagator $\mathcal{M}$ with a factorized cutoff similar to that of Lemma I.1. In Prop. III.27, we use a position space bound on $\mathcal{M}$ (which is proven in Appendix B) to estimate $g \bullet \mathcal{M} \bullet h$. Propositions III.19, III.22 and III.24 use sector counting and simple propagator estimates as above to bound $g \bullet (\mathcal{C}^{[i,j]} - \mathcal{M}) \bullet h$.

The results Lemma III.28 through Corollary III.31, of the final two subsections, are used in the proof, in Lemma II.29, of the existence and continuity properties of the infrared limit of ladders.

Before we implement the program outlined above, we introduce some notation, prove some utility lemmata and reformulate Theorem II.19 in terms of the new notation.

Let

$$\mathfrak{Y} = \mathbb{M} \,\dot\cup \big(\mathbb{R} \times \mathbb{R}^2\big)$$

be the disjoint union of the set, $\mathbb{M}$, of all possible momenta and the set, $\mathbb{R} \times \mathbb{R}^2$, of all possible positions. We consider $\mathfrak{Y}$ as the special case of the space $\mathfrak{Y}_\Sigma$ of the introduction, with the set of sectors $\Sigma = \Sigma_0$, where Σ_0 contains only a single element, namely all momentum space, $\mathbb{M}$. In particular, as in (I.2), $\mathfrak{Y}^4$ is the disjoint union

$$\mathfrak{Y}^4 = \dot{\bigcup_{i_1,i_2,i_3,i_4 \in \{0,1\}}} \mathfrak{Y}_{i_1} \times \mathfrak{Y}_{i_2} \times \mathfrak{Y}_{i_3} \times \mathfrak{Y}_{i_4},$$

where $\mathfrak{Y}_0 = \mathbb{M}$ and $\mathfrak{Y}_1 = \mathbb{R} \times \mathbb{R}^2$. For a translation invariant function f on $\mathfrak{Y}^4$, we define

$$|||f||| = \big|f\big|^{(0,0,0)}_{\Sigma_0,\Sigma_0}$$

using the norm $\big|\ \cdot\ \big|^{(0,0,0)}_{\Sigma,\Sigma'}$ of Def. I.14. Concretely,

$$|||f||| = \sum_{i_1,i_2,i_3,i_4 \in \{0,1\}} \sup_{\substack{k_\nu \in \mathbb{M} \\ \nu=1,2,3,4 \\ \text{with } i_\nu = 0}} \Big|\Big|\Big| f\big|_{(i_1,\cdots,i_4)} \Big|\Big|\Big|_{1,\infty}.$$

Here, the ν^{th} argument of f is k_ν when $i_\nu = 0$ and x_ν when $i_\nu = 1$. The $|||\ \cdot\ |||_{1,\infty}$ norm of Def. I.11 is applied to all spatial arguments of $f\big|_{(i_1,\cdots,i_4)}$.

Definition III.1. *We define the bubble operator norm of any translation invariant bubble propagator* $P(x_1, x_2, x_3, x_4)$ *by*

$$\|P\|_{\text{bubble}} = \sup_{G,H} \frac{|||G \circ P \circ H|||}{|||G|||\ |||H|||},$$

where the sup *is over nonzero, translation invariant functions on* $\mathfrak{Y}^4$.

This is much like the definition of the norm of a linear operator on $L^p(\mathbb{R}^n)$. We now show that this bubble norm can be bounded by a norm that is reminiscent of the $L^1 - L^\infty$ norm on the kernel of an integral operator on $L^p(\mathbb{R}^n)$.

Lemma III.2. *Let P be a translation invariant bubble propagator. Then*

$$\|P\|_{\text{bubble}} \le \min\Big\{ \min_{n=1,2} \sup_{x_1,x_2} \int dy_n \sup_{y_{\bar n}} |P(x_1,x_2,y_1,y_2)|,$$
$$\min_{n=1,2} \sup_{y_1,y_2} \int dx_n \sup_{x_{\bar n}} |P(x_1,x_2,y_1,y_2)| \Big\},$$

where $\bar n = 2$ if $n = 1$ and $\bar n = 1$ if $n = 2$.

Proof. Let c_P be the right hand side of the claim. We must prove that

$$|||G \circ P \circ H||| \le c_P\ |||G|||\ |||H|||$$

for all translation invariant functions, $G,\ H$ on $\mathfrak{Y}^4$. It suffices to consider G and H obeying

$$G = G\big|_{(i_1,i_2,1,1)}, \qquad H = H\big|_{(1,1,i_3,i_4)}$$

for some $i_1, i_2, i_3, i_4 \in \{0, 1\}$.

First consider the case $i_1 = i_2 = i_3 = i_4 = 0$. By definition,

$$\begin{aligned}
&|G \circ P \circ H(k_1, k_2, k_3, k_4)| \\
&\quad \le \int d^3x_2\, d^3y_1\, d^3y_2\ |G(k_1, k_2, 0, x_2)|\, |P(0, x_2, y_1, y_2)|\, |H(y_1, y_2, k_3, k_4)| \\
&\quad \le |||G|||\ \sup_{x_2} \int dy_1\, dy_2\ |P(0, x_2, y_1, y_2)|\, |H(y_1, y_2, k_3, k_4)| \\
&\quad \le |||G|||\ \sup_{x_2} \int dy_n\ \Big[\sup_{y_{\bar n}} |P(0, x_2, y_1, y_2)|\Big] \int dy_{\bar n}\ |H(y_1, y_2, k_3, k_4)| \\
&\quad \le |||G|||\ |||H|||\ \sup_{x_2} \int dy_n\ \sup_{y_{\bar n}} |P(0, x_2, y_1, y_2)|. && \text{(III.5)}
\end{aligned}$$

The other bound is achieved in a similar fashion, starting from

$$\begin{aligned}
&|G \circ P \circ H(k_1, k_2, k_3, k_4)| \\
&\quad \le \int d^3x_1\, d^3x_2\, d^3y_2\ |G(k_1, k_2, x_1, x_2)|\, |P(x_1, x_2, 0, y_2)|\, |H(0, y_2, k_3, k_4)|.
\end{aligned}$$

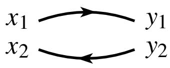
Now consider the case in which at least one of $i_1,\ i_2,\ i_3,\ i_4$ is one. Pick any $\ell \in \{1, 2, 3, 4\}$ with $i_\ell = 1$. Then, by translation invariance,

$$\begin{aligned}
&\sup_{y_\ell} \sup_{\substack{y_\nu \in \mathbb{M} \\ \nu=1,2,3,4 \\ \text{with } i_\nu=0}} \int \prod_{\substack{\nu=1,2,3,4 \\ \text{with } i_\nu=1 \\ \text{and } \nu\ne\ell}} dy_\nu\ \big|G \circ P \circ H(y_1, y_2, y_3, y_4)\big| \\
&\quad \le \sup_{y_\ell} \sup_{\substack{y_\nu \in \mathbb{M} \\ \nu=1,2,3,4 \\ \text{with } i_\nu=0}} \int \prod_{\substack{\nu=1,2,3,4 \\ \text{with } i_\nu=1 \\ \text{and } \nu\ne\ell}} dy_\nu \prod_{\nu=1,2,3,4} dx_\nu \big|G(y_1, y_2, x_1, x_2) P(x_1, x_2, x_3, x_4) H(x_3, x_4, y_3, y_4)\big| \\
&\quad = \sup_{\substack{y_\nu \in \mathbb{M} \\ \nu=1,2,3,4 \\ \text{with } i_\nu=0}} \int \prod_{\substack{\nu=1,2,3,4 \\ \text{with } i_\nu=1 \\ \text{and } \nu\ne\ell}} dy_\nu \prod_{\nu=1,2,3,4} dx_\nu\ \big|G(y_1, y_2, x_1, x_2) P(x_1, x_2, x_3, x_4) H(x_3, x_4, y_3, y_4)\big|_{y_\ell=0} \\
&\quad = \sup_{\substack{y_\nu \in \mathbb{M} \\ \nu=1,2,3,4 \\ \text{with } i_\nu=0}} \int \prod_{\substack{\nu=1,2,3,4 \\ \text{with } i_\nu=1 \\ \text{and } \nu\ne\ell}} dy_\nu \prod_{\nu=1,2,3,4} dx_\nu\ \big|G(y_1, y_2, 0, x_2) P(0, x_2, x_3, x_4) H(x_3, x_4, y_3, y_4)\big|_{y_\ell=-x_1} \\
&\quad = \sup_{\substack{y_\nu \in \mathbb{M} \\ \nu=1,2,3,4 \\ \text{with } i_\nu=0}} \int \prod_{\substack{\nu=1,2,3,4 \\ \text{with } i_\nu=1}} dy_\nu \prod_{\nu=2,3,4} dx_\nu\ \big|G(y_1, y_2, 0, x_2) P(0, x_2, x_3, x_4) H(x_3, x_4, y_3, y_4)\big| \\
&\quad \le |||G|||\ \sup_{\substack{y_\nu \in \mathbb{M} \\ \nu=3,4 \\ \text{with } i_\nu=0}} \sup_{x_2} \int \prod_{\substack{\nu=3,4 \\ \text{with } i_\nu=1}} dy_\nu \prod_{\nu=3,4} dx_\nu\ \big|P(0, x_2, x_3, x_4) H(x_3, x_4, y_3, y_4)\big| \\
&\quad \le |||G|||\ \sup_{\substack{y_\nu \in \mathbb{M} \\ \nu=3,4 \\ \text{with } i_\nu=0}} \sup_{x_2} \int dx_3\, dx_4\, \big|P(0, x_2, x_3, x_4)\big| \int \prod_{\substack{\nu=3,4 \\ \text{with } i_\nu=1}} dy_\nu\ \big|H(x_3, x_4, y_3, y_4)\big|. && \text{(III.6)}
\end{aligned}$$

For the second equality, we made the change of variables $y_\nu \to y_\nu + x_1$, for each $\nu \neq \ell$ with $i_\nu = 1$ and the change of variables $x_\nu \to x_\nu + x_1$, for each $\nu = 2, 3, 4$, and then used translation invariance of the three kernels. This replaces "$y_\ell = 0$" by "$y_\ell = -x_1$". For the third equality, we made the change of variables $x_1 \to -y_\ell$. Now we may continue as in the case $i_1 = i_2 = i_3 = i_4 = 0$. □

Our bubble propagators are typically of the form $P = A \otimes B^t$ with translation invariant propagators A and B. If A is a translation invariant propagator, we write $A(y-x)$ in place of $A(x, y)$. With this convention the L^1–L^∞ norm of Definition I.11 reduces to the L^1 norm $\|A\|_{L^1} = \int |A(y)|\, d^3y$. If $P = A \otimes B^t$, then

$$P(x_1, x_2, y_1, y_2) = A(y_1 - x_1)B(x_2 - y_2) =$$

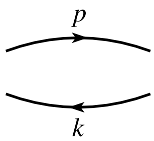


and, by Lemma III.2,

$$\|P\|_{\text{bubble}} \le \min\left\{\|A\|_{L^\infty}\|B\|_{L^1},\ \|A\|_{L^1}\|B\|_{L^\infty}\right\}. \tag{III.7}$$

Given any function $W(p, k)$ on $\mathbb{M}^2$, we associate to it the particle–hole bubble propagator

$$\begin{aligned} W(x_1, x_2, y_1, y_2) &= \int \tfrac{d^3p}{(2\pi)^3} \tfrac{d^3k}{(2\pi)^3}\ W(p, k)\, e^{\imath\langle p, x_1 - y_1\rangle_-} e^{\imath\langle k, y_2 - x_2\rangle_-} \\ &= \int \tfrac{d^3t}{(2\pi)^3} \tfrac{d^3k}{(2\pi)^3}\ W(k + t, k) e^{\imath\langle k, x_1 - y_1 + y_2 - x_2\rangle_-} e^{\imath\langle t, x_1 - y_1\rangle_-} \end{aligned} \tag{III.8}$$



Here k is the loop momentum and $t = p - k$ is the transfer momentum.

Motivated by the introduction to this section, we often treat small and large transfer momenta differently. To isolate a specific set of transfer momenta, we use a function $R(t)$ on $\mathbb{M}$ that is supported there.

Definition III.3. *For any function $W(x_1, x_2, y_1, y_2)$ and any function $R(t)$, with Fourier transform $\hat{R}(z)$, we set*

$$W_R(x_1, x_2, y_1, y_2) = \int dz\ W(x_1, x_2, y_1 - z, y_2 - z)\, \hat{R}(z).$$

If $W(x_1, x_2, y_1, y_2)$ is associated with $W(p, k)$ as in (III.8), then $W_R(x_1, x_2, y_1, y_2)$ is associated with

$$W_R(p, k) = W(p, k)R(p - k).$$

Lemma III.4. *Let A and B be translation invariant propagators and $R(t)$ a function on $\mathbb{M}$. Then*

$$\|(A \otimes B^t)_R\|_{\text{bubble}} \le \|\hat{R}(x)\|_{L^1} \min\left\{\|A(x)\|_{L^\infty}\|B(x)\|_{L^1},\ \|A(x)\|_{L^1}\|B(x)\|_{L^\infty}\right\}.$$

Proof. By Def. III.3,

$$(A \otimes B^t)_R(x_1, x_2, y_1, y_2) = \int dz\; A(y_1 - x_1 - z)B(x_2 - y_2 + z)\,\hat{R}(z).$$

By Lemma III.2,

$$\|(A \otimes B^t)_R\|_{\text{bubble}} \le \min_{n=1,2} \sup_{x_1,x_2} \int dy_n \sup_{y_{\bar n}} |(A \otimes B^t)_R(x_1, x_2, y_1, y_2)|.$$

We treat $n = 1$. The other case is similar.

$$\begin{aligned}
&\sup_{x_1,x_2} \int dy_1 \sup_{y_2} |(A \otimes B^t)_R(x_1, x_2, y_1, y_2)| \\
&\quad \le \int dy_1 \sup_{y_2} \int dz\; |A(y_1 - z)B(-y_2 - z)\hat{R}(z)| \\
&\quad \le \|B(x)\|_{L^\infty} \int dy_1\, dz\; |A(y_1 - z)\hat{R}(z)| \\
&\quad = \|B(x)\|_{L^\infty} \|A(x)\|_{L^1} \|\hat{R}(z)\|_{L^1}. \quad \square
\end{aligned}$$

Remark III.5. Define, for any function $\hat{R}(x)$, the bubble operator

$$O_R(x_1, x_2, y_1, y_2) = \hat{R}(y_1 - x_1)\delta(x_2 - y_2 + y_1 - x_1).$$

Then, for any bubble propagator W,

$$W \circ O_R = W_R.$$

Replacing g by $\frac{1}{M^{\ell|\delta_l|}} \mathrm{D}^{\delta_l}_{1,2} g$ and h by $\frac{1}{M^{j|\delta_r|}} \mathrm{D}^{\delta_r}_{3,4} h$ in Theorem II.19 reduces consideration of the norm $\left|g \bullet \mathcal{C}^{[i,j]} \bullet h\right|^{[\delta_l,0,\delta_r]}_{\ell,j}$ to a $\left|\;\cdot\;\right|^{[0,0,0]}_{\ell,j}$ norm. Therefore, we introduce the short hand notation

Definition III.6. *For f a function on $\mathfrak{Y}_{i_l,i_r}$, set*

$$\left|f\right|_{i_l,i_r} = \left|f\right|^{[0,0,0]}_{i_l,i_r}.$$

With the reduction to $\delta_l = \delta_r = 0$, indicated above, Theorem II.19 becomes bounds on the $\left|\;\cdot\;\right|_{\ell,j}$ norm of quantities like $g \bullet \mathcal{C}^{[i,j]} \bullet h$. For the rest of this section, we fix $\ell \ge 1$ and consider, more generally, $\left|\;\cdot\;\right|_{\ell,r}$ norms with $r \ge j$. The $\left|\;\cdot\;\right|_{\ell,r}$ norm of a function f is obtained by fixing all arguments that lie in $\mathbb{M}$ and the sectors of all arguments that lie in $\left(\mathbb{R} \times \mathbb{R}^2\right) \times \Sigma_\ell$ or $\left(\mathbb{R} \times \mathbb{R}^2\right) \times \Sigma_r$ and taking the $|||\;\cdot\;|||_{1,\infty}$ of the result. The transfer momentum t is determined by the momenta and sectors of the last two arguments of f. This motivates the following

Definition III.7.

i) *Let $\mathfrak{K}_r = \mathbb{M} \cup \Sigma_r$ be the disjoint union of the set $\mathbb{M}$ of external momenta and the set Σ_r of sectors of scale r.*

ii) *Let* $\kappa_1, \kappa_2 \in \mathfrak{K}_r$*. The subset* $\kappa_1 - \kappa_2$ *of* $\mathbb{M}$ *is defined by*

$$\kappa_1 - \kappa_2 = \begin{cases} \{\kappa_1 - \kappa_2\} & \text{if } \kappa_1, \kappa_2 \in \mathbb{M} \\ \left\{ \kappa_1 - k_2 \,\middle|\, k_2 \in \kappa_2 \right\} & \text{if } \kappa_1 \in \mathbb{M}, \kappa_2 \in \Sigma_r \\ \left\{ k_1 - \kappa_2 \,\middle|\, k_1 \in \kappa_1 \right\} & \text{if } \kappa_1 \in \Sigma_r, \kappa_2 \in \mathbb{M} \\ \left\{ k_1 - k_2 \,\middle|\, k_1 \in \kappa_1, k_2 \in \kappa_2 \right\} & \text{if } \kappa_1, \kappa_2 \in \Sigma_r \end{cases}.$$

iii) *Let* f *be a function on* $\mathfrak{Y}_{\ell,r}$ *and* $\kappa_1, \kappa_2 \in \mathfrak{K}_r$*. Then*

$$\|f\|_{\kappa_1,\kappa_2} = \begin{cases} \sum\limits_{i_1,i_2\in\{0,1\}} \max\limits_{\substack{s_\nu\in\Sigma_\ell \\ \text{if } i_\nu=1}} \sup\limits_{\substack{k_\nu\in\mathbb{M} \\ \text{if } i_\nu=0}} \left\| f\big|_{(i_1,i_2,0,0)}(y_1,y_2,\kappa_1,\kappa_2) \right\|_{1,\infty} \\ \qquad\qquad \text{if } \kappa_1, \kappa_2 \in \mathbb{M} \\ \sum\limits_{i_1,i_2\in\{0,1\}} \max\limits_{\substack{s_\nu\in\Sigma_\ell \\ \text{if } i_\nu=1}} \sup\limits_{\substack{k_\nu\in\mathbb{M} \\ \text{if } i_\nu=0}} \left\| f\big|_{(i_1,i_2,0,1)}(y_1,y_2,\kappa_1,(x_4,\kappa_2)) \right\|_{1,\infty} \\ \qquad\qquad \text{if } \kappa_1 \in \mathbb{M}, \kappa_2 \in \Sigma_r \\ \sum\limits_{i_1,i_2\in\{0,1\}} \max\limits_{\substack{s_\nu\in\Sigma_\ell \\ \text{if } i_\nu=1}} \sup\limits_{\substack{k_\nu\in\mathbb{M} \\ \text{if } i_\nu=0}} \left\| f\big|_{(i_1,i_2,1,0)}(y_1,y_2,(x_3,\kappa_1),\kappa_2) \right\|_{1,\infty} \\ \qquad\qquad \text{if } \kappa_1 \in \Sigma_r, \kappa_2 \in \mathbb{M} \\ \sum\limits_{i_1,i_2\in\{0,1\}} \max\limits_{\substack{s_\nu\in\Sigma_\ell \\ \text{if } i_\nu=1}} \sup\limits_{\substack{k_\nu\in\mathbb{M} \\ \text{if } i_\nu=0}} \left\| f\big|_{(i_1,i_2,1,1)}(y_1,y_2,(x_3,\kappa_1),(x_4,\kappa_2)) \right\|_{1,\infty} \\ \qquad\qquad \text{if } \kappa_1, \kappa_2 \in \Sigma_r \end{cases}.$$

Here, we use the decomposition of (I.2) and, for $\nu = 1, 2$*,* $y_\nu = k_\nu$ *if* $i_\nu = 0$ *and* $y_\nu = (x_\nu, s_\nu)$ *if* $i_\nu = 1$.

Remark III.8. For a function f on $\mathfrak{Y}_{\ell,r}$,

$$|f|_{\ell,r} \le 4\Big\{ \sup_{\kappa_1,\kappa_2\in\mathbb{M}} \|f\|_{\kappa_1,\kappa_2} + \sup_{\substack{k_1\in\mathbb{M} \\ \sigma_2\in\Sigma_r}} \|f\|_{\kappa_1,\sigma_2} + \sup_{\substack{\sigma_1\in\Sigma_r \\ k_2\in\mathbb{M}}} \|f\|_{\sigma_1,\kappa_2} + \sup_{\sigma_1,\sigma_2\in\mathbb{M}} \|f\|_{\sigma_1,\sigma_2} \Big\}.$$

We now state the reformulation of parts (b) and (c) of Theorem II.19. Recall the decomposition

$$\mathcal{C}^{[i,j]} = \mathcal{C}^{[i,j]}_{\text{top}} + \mathcal{C}^{[i,j]}_{\text{mid}} + \mathcal{C}^{[i,j]}_{\text{bot}}$$

of the particle–hole bubble propagator $\mathcal{C}^{[i,j]}$ with

$$\mathcal{C}^{[i,j]}_{\text{top}} = \sum_{\substack{i\le i_t\le j \\ i_b>j}} C_v^{(i_t)} \otimes C_v^{(i_b)\,t}, \quad \mathcal{C}^{[i,j]}_{\text{mid}} = \sum_{\substack{i\le i_t\le j \\ i\le i_b\le j}} C_v^{(i_t)} \otimes C_v^{(i_b)\,t}, \quad \mathcal{C}^{[i,j]}_{\text{bot}} = \sum_{\substack{i_t>j \\ i\le i_b\le j}} C_v^{(i_t)} \otimes C_v^{(i_b)\,t},$$

and recall from (II.1) that

$$\Delta = \left\{ \delta \in \mathbb{N}_0 \times \mathbb{N}_0^2 \,\middle|\, \delta_0 \le r_0,\ \delta_1 + \delta_2 \le r_e \right\},$$

where $r_e + 3$ is the degree of differentiability of the dispersion relation $e(\mathbf{k})$ and r_0 is the number of k_0 derivatives that we wish to control. The following theorem collects together a number of bounds that can be proven by relatively simple power counting arguments, without exploiting cancellations between scales. These will suffice to prove parts (b) and (c) of Theorem II.19.

Theorem III.9. *Let $1 \le i \le j \le r$ and $\ell \ge 1$ and let g and h be sectorized, translation invariant functions on $\mathfrak{Y}_{\ell,i}$ and $\mathfrak{Y}_{i,r}$ respectively. Let $\kappa_1, \kappa_2 \in \mathfrak{K}_r$.*

i) For any $\beta \in \Delta$,

$$\frac{1}{M^{|\beta|j}} \big\| g \bullet \mathrm{D}^{\beta}_{1,3} \mathcal{C}^{[i,j]}_{\mathrm{top}} \bullet h \big\|_{\kappa_1,\kappa_2}, \le \mathrm{const}\, \big|g\big|_{\ell,i} \big|h\big|_{i,r},$$
$$\frac{1}{M^{|\beta|j}} \big\| g \bullet \mathrm{D}^{\beta}_{2,4} \mathcal{C}^{[i,j]}_{\mathrm{bot}} \bullet h \big\|_{\kappa_1,\kappa_2} \le \mathrm{const}\, \big|g\big|_{\ell,i} \big|h\big|_{i,r}.$$

ii)

$$\big\| g \bullet \mathcal{C}^{[i,j]}_{\mathrm{mid}} \bullet h \big\|_{\kappa_1,\kappa_2} \le \mathrm{const}\, |j-i+1|\, \big|g\big|_{\ell,i} \big|h\big|_{i,r},$$

and for any $\beta \in \Delta$ with $|\beta| \ge 1$ and $(\mu, \mu') = (1,3), (2,4)$,

$$\frac{1}{M^{|\beta|j}} \big\| g \bullet \mathrm{D}^{\beta}_{\mu,\mu'} \mathcal{C}^{[i,j]}_{\mathrm{mid}} \bullet h \big\|_{\kappa_1,\kappa_2} \le \mathrm{const}\, \big|g\big|_{\ell,i} \big|h\big|_{i,r}.$$

The constant const *depends on $e(\mathbf{k})$, M and Δ, but not on $i, \ell, j, r, g, h, \kappa_1$ or κ_2.*

The proof of Theorem III.9 follows Lemma III.14.

Proof of Theorem II.19b,c (assuming Theorem III.9). As pointed out above, we may assume without loss of generality that $\delta_{\mathrm{l}} = \delta_{\mathrm{r}} = 0$. Then parts (b) and (c) of Theorem II.19 follow directly from Remark III.8 and parts (i) and (ii) of Theorem III.9, with $r = j$, respectively. □

We shall decompose bubble propagators $W = \sum_{m=i}^{j} \sum_{s_1,s_2 \in \Sigma_m} W^{(m)}_{s_1,s_2}$ with m being the smaller of the scales of the two lines of the bubble propagator and s_1, s_2 being sectors on the two lines. The following lemma provides an estimate on the norm of W, with some external momenta specified, using a term–by–term bound on this expansion.

Definition III.10. *For any subset $d \subset \mathbb{M}$, let $\mathcal{R}(d)$ be the set of all functions $R(t)$ that are identically one on d.*

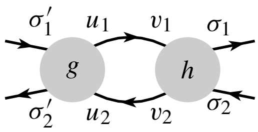
Lemma III.11. *Let $1 \le i \le j$ and $\ell, r \ge 1$. Let $\kappa_1, \kappa_2 \in \mathfrak{K}_r$ and g and h be sectorized, translation invariant functions on $\mathfrak{Y}_{\ell,i}$ and $\mathfrak{Y}_{i,r}$ respectively. Let W be a particle–hole bubble propagator whose total Fourier transform is of the form*

$$\check{W}(p_1,k_1,p_2,k_2) = \sum_{m=i}^{j} \sum_{s_1,s_2 \in \Sigma_m} W^{(m)}_{s_1,s_2}(p_1,k_1,p_2,k_2) \qquad \text{if } p_2 - k_2 \in \kappa_1 - \kappa_2$$

with $W^{(m)}_{s_1,s_2}(p_1,k_1,p_2,k_2)$ vanishing unless $p_1, p_2 \in s_1$ and $k_1, k_2 \in s_2$. Then

$$\begin{aligned} \|g \bullet W \bullet h\|_{\kappa_1,\kappa_2} &\le 81 |g|_{\ell,i} |h|_{i,r} \sum_{m=i}^{j} \sum_{\substack{s_1,s_2 \in \Sigma_m \\ (s_1-s_2)\cap(\kappa_1-\kappa_2)\ne\emptyset}} \inf_{R \in \mathcal{R}(\kappa_1-\kappa_2)} \big\| W^{(m)}_{s_1,s_2,R} \big\|_{\mathrm{bubble}} \\ &\le 81 |g|_{\ell,i} |h|_{i,r} \sum_{m=i}^{j} \sum_{\substack{s_1,s_2 \in \Sigma_m \\ (s_1-s_2)\cap(\kappa_1-\kappa_2)\ne\emptyset}} \big\| W^{(m)}_{s_1,s_2} \big\|_{\mathrm{bubble}}. \end{aligned}$$

Proof. Consider the case in which all of the external arguments of $g \bullet W \bullet h$ are (position, sector)'s. Fix (external) sectors $\sigma_1', \sigma_2' \in \Sigma_\ell$ and call $\sigma_1 = \kappa_1, \sigma_2 = \kappa_2 \in \Sigma_r$ and $d = \sigma_1 - \sigma_2$. With the sector names



we have

$$g \bullet W \bullet h = \sum_{m=i}^{j} \sum_{\substack{u_1,v_1\in\Sigma_i\\ u_2,v_2\in\Sigma_i\\ s_1,s_2\in\Sigma_m}} g\big((\cdot,\sigma_1'),(\cdot,\sigma_2'),(\cdot,u_1),(\cdot,u_2)\big) \circ W^{(m)}_{s_1,s_2} \circ h\big((\cdot,v_1),(\cdot,v_2),(\cdot,\sigma_1),(\cdot,\sigma_2)\big).$$

For each choice of $u_1, v_1, u_2, v_2, s_1, s_2$, by conservation of momentum at the vertex h,

$$\begin{aligned} &g\big((\cdot,\sigma_1'),(\cdot,\sigma_2'),(\cdot,u_1),(\cdot,u_2)\big) \circ W^{(m)}_{s_1,s_2} \circ h\big((\cdot,v_1),(\cdot,v_2),(\cdot,\sigma_1),(\cdot,\sigma_2)\big) \\ &\quad = g\big((\cdot,\sigma_1'),(\cdot,\sigma_2'),(\cdot,u_1),(\cdot,u_2)\big) \circ W^{(m)}_{s_1,s_2,R} \circ h\big((\cdot,v_1),(\cdot,v_2),(\cdot,\sigma_1),(\cdot,\sigma_2)\big) \end{aligned}$$

for all $R \in \mathcal{R}(d)$ and the convolution vanishes identically unless $(s_1 - s_2) \cap d \neq \emptyset$. The convolution also vanishes identically unless

$$\begin{aligned} u_1 \cap s_1 \neq \emptyset \quad s_1 \cap v_1 \neq \emptyset, \\ u_2 \cap s_2 \neq \emptyset \quad s_2 \cap v_2 \neq \emptyset. \end{aligned}$$

For each fixed s_1, s_2 there are only 81 quadruples (u_1, u_2, v_1, v_2) satisfying these conditions. The same is true, by a similar argument, if one or more of the external arguments of $g \bullet W \bullet h$ are momenta. Just replace, for example, σ_1' by $\{k'\}$. Hence

$$\|g \bullet W \bullet h\|_{\kappa_1,\kappa_2} \le 81|g|_{\ell,i}|h|_{i,r} \sum_{m=i}^{j} \sum_{\substack{s_1,s_2\in\Sigma_m\\ (s_1-s_2)\cap d\neq\emptyset}} \inf_{R\in\mathcal{R}(d)} \big\| W^{(m)}_{s_1,s_2,R} \big\|_{\text{bubble}}.$$

The second inequality follows by choosing an $R(t)$ that is identically one on a large enough ball. □

Remark III.12. Let $\kappa_1, \kappa_2 \in \mathfrak{K}_r$.

i) The set $\kappa_1 - \kappa_2$ is contained in a ball of radius $2\mathfrak{l}_r$.

ii) Let $m \le r$. Then,

$$\#\big\{ (s_1, s_2) \in \Sigma_m \times \Sigma_m \;\big|\; (s_1 - s_2) \cap (\kappa_1 - \kappa_2) \neq \emptyset \big\} \le \tfrac{\text{const}}{\mathfrak{l}_m}.$$

iii) The set $\big\{ t_0 \in \mathbb{R} \;\big|\; (t_0, \mathbf{t}) \in \kappa_1 - \kappa_2 \text{ for some } \mathbf{t} \in \mathbb{R}^2 \big\}$ is contained in an interval of length $\frac{4\sqrt{2M}}{M^r}$.

Proof. Part (i) is an immediate consequence of the facts that κ_1 and κ_2 are each contained in a ball of radius $\mathfrak{l}_r$. Given any fixed $s_1 \in \Sigma_m$, $(s_1 - s_2) \cap (\kappa_1 - \kappa_2) \neq \emptyset$ only if s_2 intersects $s_1 - \kappa_1 + \kappa_2$. As $s_1 - \kappa_1 + \kappa_2$ is contained in a ball of radius at most $3\mathfrak{l}_m$, there are at most eight sectors $s_2 \in \Sigma_m$ that intersect it. This proves part (ii). Part (iii) follows from the fact that, for $\nu = 1, 2$, $\big\{ k_0 \in \mathbb{R} \;\big|\; (k_0, \mathbf{k}) \in \kappa_\nu \text{ for some } \mathbf{k} \in \mathbb{R}^2 \big\}$ is contained in an interval of length $\frac{2\sqrt{2M}}{M^r}$. □

Remark III.13. Let $\boldsymbol{\pi} : k = (k_0, \mathbf{k}) \mapsto \mathbf{k}$ be the projection of $\mathbb{M} = \mathbb{R}\times\mathbb{R}^2$ onto its second factor. If we retain all of the hypotheses of Lemma III.11, except that we only require $\check{W}^{(m)}_{s_1,s_2}(p_1, k_1, p_2, k_2)$ to vanish unless $\boldsymbol{\pi}(p_1), \boldsymbol{\pi}(p_2) \in \boldsymbol{\pi}(s_1)$ and $\boldsymbol{\pi}(k_1), \boldsymbol{\pi}(k_2) \in \boldsymbol{\pi}(s_2)$, then we still have

$$\begin{aligned}\|g \bullet W \bullet h\|_{\kappa_1,\kappa_2} &\le 81|g|_{\ell,i}|h|_{i,r} \sum_{m=i}^{j} \sum_{\substack{s_1,s_2\in\Sigma_m \\ \boldsymbol{\pi}(s_1-s_2)\cap\boldsymbol{\pi}(\kappa_1-\kappa_2)\neq\emptyset}} \inf_{R\in\mathcal{R}(\kappa_1-\kappa_2)} \left\| W^{(m)}_{s_1,s_2,R}\right\|_{\rm bubble} \\ &\le 81|g|_{\ell,i}|h|_{i,r} \sum_{m=i}^{j} \sum_{\substack{s_1,s_2\in\Sigma_m \\ \boldsymbol{\pi}(s_1-s_2)\cap\boldsymbol{\pi}(\kappa_1-\kappa_2)\neq\emptyset}} \left\| W^{(m)}_{s_1,s_2}\right\|_{\rm bubble}.\end{aligned}$$

We are particularly interested in the particle–hole bubble propagator

$$\mathcal{C}^{[i,j]}(p,k) = \mathcal{C}^{[i,j]}_{\rm top}(p,k) + \mathcal{C}^{[i,j]}_{\rm mid}(p,k) + \mathcal{C}^{[i,j]}_{\rm bot}(p,k),$$

where

$$\begin{aligned}\mathcal{C}^{[i,j]}_{\rm top}(p,k) &= \sum_{\substack{i\le m_1\le j\\ m_2>j}} C_v^{(m_1)}(p)C_v^{(m_2)}(k),\\ \mathcal{C}^{[i,j]}_{\rm mid}(p,k) &= \sum_{\substack{i\le m_1\le j\\ i\le m_2\le j}} C_v^{(m_1)}(p)C_v^{(m_2)}(k),\\ \mathcal{C}^{[i,j]}_{\rm bot}(p,k) &= \sum_{\substack{m_1>j\\ i\le m_2\le j}} C_v^{(m_1)}(p)C_v^{(m_2)}(k).\end{aligned}$$

We split $\mathcal{C}^{[i,j]}_{\rm top}$, $\mathcal{C}^{[i,j]}_{\rm mid}$ and $\mathcal{C}^{[i,j]}_{\rm bot}$ into scales and we split each scale contribution into pieces with additional sector restrictions on the momenta p and k and the transfer momentum $p-k$. Recall, from just before Def. I.17, that $\sum_{s\in\Sigma_m}\chi_s(k)$ is a partition of unity of the $m^{\rm th}$ neighbourhood subordinate to Σ_m. For any scale $i\le m\le j$ and sectors $s_1, s_2 \in \Sigma_m$, set

$$\begin{aligned}\mathcal{C}^{(m)}_{{\rm top},j,s_1,s_2}(p,k) &= \sum_{m_2>j} C_v^{(m)}(p)\chi_{s_1}(p)C_v^{(m_2)}(k)\chi_{s_2}(k),\\ \mathcal{C}^{(m)}_{{\rm mid},j,s_1,s_2}(p,k) &= \sum_{\substack{m_1,m_2\le j\\ \min(m_1,m_2)=m}} C_v^{(m_1)}(p)\chi_{s_1}(p)C_v^{(m_2)}(k)\chi_{s_2}(k),\\ \mathcal{C}^{(m)}_{{\rm bot},j,s_1,s_2}(p,k) &= \sum_{m_1>j} C_v^{(m_1)}(p)\chi_{s_1}(p)C_v^{(m)}(k)\chi_{s_2}(k).\end{aligned}$$

Then, for each of loc = top, mid, bot,

$$\mathcal{C}^{[i,j]}_{\rm loc} = \sum_{m=i}^{j}\sum_{s_1,s_2\in\Sigma_m}\mathcal{C}^{(m)}_{{\rm loc},j,s_1,s_2}.$$

Lemma III.14. *Let* $1 \le m \le j$ *and* $s_1, s_2 \in \Sigma_m$*. If* $\beta \in \Delta$ *and* $(\mu, \mu') \in \{(1,3), (2,4)\}$*, then*

$$\|D^{\beta}_{1,3}\mathcal{C}^{(m)}_{\mathrm{top},j,s_1,s_2}\|_{\mathrm{bubble}} \le \mathrm{const}\, \mathfrak{l}_m \tfrac{M^m}{M^j} M^{|\beta| m},$$

$$\|D^{\beta}_{\mu,\mu'}\mathcal{C}^{(m)}_{\mathrm{mid},j,s_1,s_2}\|_{\mathrm{bubble}} \le \mathrm{const}\, \mathfrak{l}_m \begin{cases} M^m M^{(|\beta|-1)j} & \text{if } |\beta| \ge 2 \\ M^m (j-m+1) & \text{if } |\beta| = 1 \\ 1 & \text{if } |\beta| = 0 \end{cases},$$

$$\|D^{\beta}_{2,4}\mathcal{C}^{(m)}_{\mathrm{bot},j,s_1,s_2}\|_{\mathrm{bubble}} \le \mathrm{const}\, \mathfrak{l}_m \tfrac{M^m}{M^j} M^{|\beta| m}.$$

Proof. Set, for $s \in \Sigma_m$ and $n \ge 1$,

$$c^{(n)}_s(k) = C^{(n)}_v(k)\chi_s(k), \tag{III.9}$$

and denote by $c^{(n)}_s(x)$ its Fourier transform. By Lemma A.2, for all $\beta \in \Delta$,

$$\big\| x^\beta c^{(m)}_s(x)\big\|_{L^1} \le \mathrm{const}\, M^{(1+|\beta|)m}, \tag{III.10}$$

$$\big\| c^{(n)}_s(x)\big\|_{L^\infty} \le \mathrm{const}\, \tfrac{\mathfrak{l}_m}{M^n}, \tag{III.11}$$

$$\big\| x^\beta c^{(n)}_s(x)\big\|_{L^\infty} \le \mathrm{const}\, \mathfrak{l}_m M^{(|\beta|-1)n} \qquad \text{if } n \ge m. \tag{III.12}$$

Recall that

$$D^{\beta}_{1,3}\mathcal{C}^{(m)}_{\mathrm{top},j,s_1,s_2}(x_1,x_2,y_1,y_2) = \sum_{n>j} (y_1-x_1)^\beta c^{(m)}_{s_1}(y_1-x_1)\, c^{(n)}_{s_2}(x_2-y_2).$$

Hence, by the triangle inequality and (III.7),

$$\begin{aligned} \big\|D^{\beta}_{1,3}\mathcal{C}^{(m)}_{\mathrm{top},j,s_1,s_2}\big\|_{\mathrm{bubble}} &\le \sum_{n>j} \big\|x^\beta c^{(m)}_{s_1}\big\|_{L^1} \big\|c^{(n)}_{s_2}\big\|_{L^\infty} \\ &\le \sum_{n>j} \mathrm{const}\, M^{(1+|\beta|)m} \tfrac{\mathfrak{l}_m}{M^n} \\ &\le \mathrm{const}\, \tfrac{M^m}{M^j} \mathfrak{l}_m M^{|\beta| m}. \end{aligned}$$

The bound on $\big\|D^{\beta}_{2,4}\mathcal{C}^{(m)}_{\mathrm{bot},j,s_1,s_2}\big\|_{\mathrm{bubble}}$ is proven similarly. As

$$\begin{aligned} D^{\beta}_{1,3}\mathcal{C}^{(m)}_{\mathrm{mid},j,s_1,s_2}(x_1,x_2,y_1,y_2) = &\sum_{m\le n\le j} (y_1-x_1)^\beta c^{(m)}_{s_1}(y_1-x_1)\, c^{(n)}_{s_2}(x_2-y_2) \\ &+ \sum_{m< n\le j} (y_1-x_1)^\beta c^{(n)}_{s_1}(y_1-x_1)\, c^{(m)}_{s_2}(x_2-y_2), \end{aligned}$$

we have

$$\begin{aligned} \big\|D^{\beta}_{1,3}\mathcal{C}^{(m)}_{\mathrm{mid},j,s_1,s_2}\big\|_{\mathrm{bubble}} &\le \sum_{m\le n\le j} \big\|x^\beta c^{(m)}_{s_1}\big\|_{L^1} \big\|c^{(n)}_{s_2}\big\|_{L^\infty} + \sum_{m<n\le j} \big\|x^\beta c^{(n)}_{s_1}\big\|_{L^\infty} \big\|c^{(m)}_{s_2}\big\|_{L^1} \\ &\le \sum_{m\le n\le j} \mathrm{const}\, M^{(1+|\beta|)m} \tfrac{\mathfrak{l}_m}{M^n} + \sum_{m<n\le j} \mathrm{const}\, \tfrac{\mathfrak{l}_m}{M^n} M^{|\beta| n} M^m \\ &\le \mathrm{const}\, \mathfrak{l}_m \sum_{m\le n\le j} \tfrac{M^m}{M^n} M^{|\beta| n}. \end{aligned}$$

To bound $\big\|x^\beta c^{(n)}_{s_1}\big\|_{L^\infty}$, we used (III.12). □

Proof of Theorem III.9.i. We prove the bound for $\mathcal{C}_{\rm top}^{[i,j]}$. The proof for $\mathcal{C}_{\rm bot}^{[i,j]}$ is virtually identical. By Lemma III.11, Remark III.12.ii and Lemma III.14,

$$\begin{aligned}\big\|g\bullet \mathrm{D}_{1,3}^{\beta}\mathcal{C}_{\rm top}^{[i,j]}\bullet h\big\|_{\kappa_1,\kappa_2} &\le \mathrm{const}\,\big|g\big|_{\ell,i}\big|h\big|_{i,r}\sum_{m=i}^{j}\tfrac{\mathrm{const}}{\mathfrak{l}_m}\max_{s_1,s_2\in\Sigma_m}\big\|\mathrm{D}_{1,3}^{\beta}\mathcal{C}_{\mathrm{top},j,s_1,s_2}^{(m)}\big\|_{\rm bubble}\\ &\le \mathrm{const}\,\big|g\big|_{\ell,i}\big|h\big|_{i,r}\sum_{m=i}^{j}\tfrac{\mathrm{const}}{\mathfrak{l}_m}\mathfrak{l}_m\tfrac{M^m}{M^j}M^{|\beta|m}\\ &\le \mathrm{const}\,M^{|\beta|j}\big|g\big|_{\ell,i}\big|h\big|_{i,r}.\quad\Box\end{aligned}$$

Proof of Theorem III.9.ii. By Lemma III.11, followed by Lemma III.14 and Remark III.12.ii, we have

$$\begin{aligned}\big\|g\bullet \mathcal{C}_{\rm mid}^{[i,j]}\bullet h\big\|_{\kappa_1,\kappa_2} &\le \mathrm{const}\,\big|g\big|_{\ell,i}\big|h\big|_{i,r}\sum_{m=i}^{j}\tfrac{\mathrm{const}}{\mathfrak{l}_m}\max_{s_1,s_2\in\Sigma_m}\big\|\mathcal{C}_{\mathrm{mid},s_1,s_2}^{(m)}\big\|_{\rm bubble}\\ &\le \mathrm{const}\,\big|g\big|_{\ell,i}\big|h\big|_{i,r}\sum_{m=i}^{j}\tfrac{\mathrm{const}}{\mathfrak{l}_m}\mathfrak{l}_m\\ &\le \mathrm{const}\,|j-i+1|\big|g\big|_{\ell,i}\big|h\big|_{i,r}.\end{aligned}$$

For $|\beta|\ge 1$ and $(\mu,\mu')=(1,3),(2,4)$, by Lemma III.14,

$$\begin{aligned}\big\|g\bullet \mathrm{D}_{\mu,\mu'}^{\beta}\mathcal{C}_{\rm mid}^{[i,j]}\bullet h\big\|_{\kappa_1,\kappa_2} &\le \mathrm{const}\,\big|g\big|_{\ell,i}\big|h\big|_{i,r}\sum_{m=i}^{j}\tfrac{\mathrm{const}}{\mathfrak{l}_m}\max_{s_1,s_2\in\Sigma_m}\big\|\mathrm{D}_{\mu,\mu'}^{\beta}\mathcal{C}_{\mathrm{mid},j,s_1,s_2}^{(m)}\big\|_{\rm bubble}\\ &\le \mathrm{const}\,\big|g\big|_{\ell,i}\big|h\big|_{i,r}\sum_{m=i}^{j}\tfrac{1}{\mathfrak{l}_m}\mathfrak{l}_m M^m\begin{cases}M^{(|\beta|-1)j} & |\beta|\ge 2\\ (j-m+1) & |\beta|=1\end{cases}\\ &\le \mathrm{const}\,M^{|\beta|j}\big|g\big|_{\ell,i}\big|h\big|_{i,r},\end{aligned}$$

since, for $|\beta|\ge 2$,

$$\sum_{m=i}^{j}M^m M^{(|\beta|-1)j}\le \mathrm{const}\,M^{|\beta|j}$$

and, for $|\beta|=1$,

$$\sum_{m=i}^{j}M^m(j-m+1)=M^j\sum_{m=i}^{j}M^{-(j-m)}(j-m+1)\le \mathrm{const}\,M^j.\quad\Box$$

We now start the proof of part (a) of Theorem II.19. This is the delicate part of Theorem II.19. For small transfer momentum, we shall need to exploit cancellations between scales. We shall prove, at the end of this subsection, the following bound on the small transfer momentum contributions to $g\bullet\mathcal{C}^{[i,j]}\bullet h$.

Theorem III.15. *Let $1\le i\le j\le r$ and $\ell\ge 1$ and let $\kappa_1,\kappa_2\in\mathfrak{K}_r$. Set $d=\kappa_1-\kappa_2$ and denote by* $\mathbf{d}$ *the projection of d onto $\{0\}\times\mathbb{R}^2$ identified with $\mathbb{R}^2$. By Remark III.12,* *the set* $\mathbf{d}$ *is contained in a disc of radius* $2\mathfrak{l}_r$. *Fix such a disk and denote by* $\boldsymbol{\tau}$ *its centre. Furthermore, set* $\tau_0 = \inf\left\{\ |t_0|\ \middle|\ (t_0, \mathbf{t}) \in d \text{ for some } \mathbf{t} \in \mathbb{R}^2\ \right\}$. *Assume that*

$$\tau_0 \le \tfrac{1}{M^{j-1}}, \qquad |\boldsymbol{\tau}| \le \max\left\{\tfrac{1}{M^j}, r^3\mathfrak{l}_r\right\}, \qquad M^i \le \mathfrak{l}_j M^j.$$

Also assume that $p^{(i)}$ *vanishes for all* $i > j+1$. *For any sectorized, translation invariant functions* g *and* h *on* $\mathfrak{Y}_{\ell,i}$ *and* $\mathfrak{Y}_{i,r}$ *respectively,*

$$\left\| g \bullet C^{[i,j]} \bullet h \right\|_{\kappa_1,\kappa_2} \le \text{const} \max_{\substack{\alpha_r,\alpha_l \in \mathbb{N}_0\times\mathbb{N}_0^2 \\ |\alpha_r|+|\alpha_l|\le 3}} |g|_{\ell,i}^{[0,0,\alpha_r]} |h|_{i,r}^{[\alpha_l,0,0]}.$$

The constant const *depends on* $e(\mathbf{k})$, M *and* Δ, *but not on* $i, \ell, j, r, g, h, \kappa_1$ *or* κ_2.

Theorem III.15 is proven following Prop. III.27.

Proof of Theorem II.19a (assuming Theorem III.15). As pointed out above, we may assume without loss of generality that $\delta_l = \delta_r = 0$. Fix $0 \le i, \ell \le j$ and sectorized, translation invariant functions g and h on $\mathfrak{Y}_{\ell,i}$ and $\mathfrak{Y}_{i,j}$ as in Theorem II.19. By Remark III.8, it suffices to prove that

$$\left\| g \bullet C^{[i,j]} \bullet h \right\|_{\kappa_1,\kappa_2} \le \text{const } i \max_{\substack{\alpha_r,\alpha_l \in \mathbb{N}_0\times\mathbb{N}_0^2 \\ |\alpha_r|+|\alpha_l|\le 3}} |g|_{\ell,i}^{[0,0,\alpha_r]} |h|_{i,j}^{[\alpha_l,0,0]} \tag{III.13}$$

for all $\kappa_1, \kappa_2 \in \mathfrak{K}_j$. Fix $\kappa_1, \kappa_2 \in \mathfrak{K}_j$. Set $d = \kappa_1 - \kappa_2$ and denote by $\mathbf{d}$ the projection of d onto $\{0\} \times \mathbb{R}^2$ identified with $\mathbb{R}^2$. By Remark III.12, the set $\mathbf{d}$ is contained in a disc of radius $2\mathfrak{l}_j$. We fix such a disk and denote by $\boldsymbol{\tau}$ its centre. Furthermore, we define $\tau_0 = \inf\left\{\ |t_0|\ \middle|\ (t_0, \mathbf{t}) \in d \text{ for some } \mathbf{t} \in \mathbb{R}^2\ \right\}$. Define

$$j_0 = \begin{cases} \max\left\{ n \in \mathbb{N}_0 \ \middle|\ \tau_0 \le \frac{1}{M^{n-1}} \right\} & \text{if } 0 < \tau_0 \le M \\ 0 & \text{if } \tau_0 \ge M \\ \infty & \text{if } \tau_0 = 0 \end{cases},$$

$$j_1 = \begin{cases} \max\left\{ n \in \mathbb{N}_0 \ \middle|\ |\boldsymbol{\tau}| \le \frac{1}{M^{n}} \right\} & \text{if } j^3\mathfrak{l}_j < |\boldsymbol{\tau}| \le 1 \\ 0 & \text{if } |\boldsymbol{\tau}| \ge 1 \\ \infty & \text{if } |\boldsymbol{\tau}| \le j^3\mathfrak{l}_j \end{cases},$$

$$\bar{j} = \max\Big\{ i-1, \min\{j, j_0, j_1\} \Big\}.$$

The transfer momentum $\kappa_1 - \kappa_2$ effectively imposes an infrared cutoff at scale $\bar{j}$ on the bubble, in the sense that contributions at scales higher than $\bar{j}$ can be controlled relatively easily using power counting arguments like in the proof of parts (b) and (c) of Theorem II.19. □

Proposition III.16 (Large transfer momentum).

$$\left\| g \bullet C^{[\bar{j}+1,j]} \bullet h \right\|_{\kappa_1,\kappa_2} \le \text{const } |g|_{\ell,i} |h|_{i,j}.$$

Proof. If $\min\{j, j_0, j_1\} = j$, then $\bar{j} = j$ and $C^{[\bar{j}+1,j]} = 0$ so that there is nothing to prove. So we may assume that $\min\{j_0, j_1\} < j$.

Case 1. $j_0 \le j_1$. In this case, $\|g \bullet \mathcal{C}^{[\bar{j}+1,j]} \bullet h\|_{\kappa_1,\kappa_2} = 0$, because $\mathcal{C}^{[\bar{j}+1,j]}(p,k)$ vanishes unless $|p_0|, |k_0| \le \frac{\sqrt{2M}}{M^{\bar{j}+1}}$ and hence unless $|p_0 - k_0| \le \frac{2\sqrt{2M}}{M^{\bar{j}+1}} < \frac{1}{M^{\bar{j}}} < \tau_0$, while $|t_0| \ge \tau_0$ for all $t \in d$.

Case 2. $j_1 < j_0$. In this case $|\boldsymbol{\tau}| > j^3 \mathfrak{l}_j$. Let δ_F be the constant of Lemma C.2. By Lemma C.2.a, with $\epsilon = 2\mathfrak{l}_j$ and $m \le j$,

$$\#\big\{ (s_1, s_2) \in \Sigma_m \times \Sigma_m \ \big|\ (s_1 - s_2) \cap d \ne \emptyset \big\} \le \text{const} \begin{cases} \frac{1}{\sqrt{\mathfrak{l}_m}} & \text{if } |\boldsymbol{\tau}| \ge \delta_F \\ 1 + \frac{1}{|\boldsymbol{\tau}|\mathfrak{l}_m}\big(\frac{1}{M^m} + \mathfrak{l}_j\big) & \text{otherwise} \end{cases}.$$

Hence, by Lemma III.11 and Lemma III.14,

$$\begin{aligned} \big\|g \bullet \mathcal{C}^{[\bar{j}+1,j]} \bullet h\big\|_{\kappa_1,\kappa_2} &\le \text{const}\, |g|_{\ell,i} |h|_{i,j} \sum_{m=\bar{j}+1}^{j} \mathfrak{l}_m \, \#\big\{ (s_1, s_2) \in \Sigma_m \\ &\qquad \times \Sigma_m \ \big|\ (s_1 - s_2) \cap d \ne \emptyset \big\} \\ &\le \text{const}\, |g|_{\ell,i} |h|_{i,j} \begin{cases} \sum\limits_{m=\bar{j}+1}^{j} \sqrt{\mathfrak{l}_m} & \text{if } |\boldsymbol{\tau}| \ge \delta_F \\ 1 + \frac{1}{|\boldsymbol{\tau}|} \sum\limits_{m=j_1+1}^{j} \big(\frac{1}{M^m} + \mathfrak{l}_j\big) & \text{otherwise} \end{cases} \\ &\le \text{const}\, |g|_{\ell,i} |h|_{i,j} \end{aligned}$$

since, by the definition of j_1,

$$\frac{1}{|\boldsymbol{\tau}|} \sum_{m=j_1+1}^{j} \big(\frac{1}{M^m} + \mathfrak{l}_j\big) \le \frac{1}{|\boldsymbol{\tau}|}\big(\frac{1}{M^{j_1}} + j\mathfrak{l}_j\big) \le \text{const}. \quad \square$$

Continuation of the proof of Theorem II.19a (assuming Theorem III.15). When $M^i \ge \mathfrak{l}_{\bar{j}} M^{\bar{j}} = M^{(1-\aleph)\bar{j}}$, we have $|\bar{j} - i + 1| \le \text{const}\, i$. In this case Theorem III.9, with $r = j$ and $j = \bar{j}$, gives

$$\big\|g \bullet \mathcal{C}^{[i,\bar{j}]} \bullet h\big\|_{\kappa_1,\kappa_2} \le \text{const}\, i\, \big|g\big|_{\ell,i} \big|h\big|_{i,j}.$$

This together with Prop. III.16 yields (III.13). Therefore, we may assume that

$$M^i \le \mathfrak{l}_{\bar{j}} M^{\bar{j}}. \tag{III.14}$$

Furthermore, if $\bar{j} = i - 1$, $\mathcal{C}^{[i,\bar{j}]} = 0$ and there is nothing more to prove. So we may also assume that $j_0, j_1 \ge i$ and $\bar{j} \le j, j_0, j_1$.

Set $v' = \sum_{i=2}^{\bar{j}+1} p^{(i)}$. Recall that $\mathcal{C}^{[i,\bar{j}]} = \mathcal{C}^{[i,\bar{j}]}_{\text{top}} + \mathcal{C}^{[i,\bar{j}]}_{\text{mid}} + \mathcal{C}^{[i,\bar{j}]}_{\text{bot}}$ with

$$\mathcal{C}^{[i,\bar{j}]}_{\text{top}} = \sum_{\substack{i \le i_t \le \bar{j} \\ i_b > \bar{j}}} C_v^{(i_t)} \otimes C_v^{(i_b)\,t}, \quad \mathcal{C}^{[i,\bar{j}]}_{\text{mid}} = \sum_{\substack{i \le i_t \le \bar{j} \\ i \le i_b \le \bar{j}}} C_v^{(i_t)} \otimes C_v^{(i_b)\,t}, \quad \mathcal{C}^{[i,\bar{j}]}_{\text{bot}} = \sum_{\substack{i_t > \bar{j} \\ i \le i_b \le \bar{j}}} C_v^{(i_t)} \otimes C_v^{(i_b)\,t},$$

and set $\mathcal{C}'^{[i,\bar{j}]} = \mathcal{C}'^{[i,\bar{j}]}_{\text{top}} + \mathcal{C}'^{[i,\bar{j}]}_{\text{mid}} + \mathcal{C}'^{[i,\bar{j}]}_{\text{bot}}$ with

$$\mathcal{C}'^{[i,\bar{j}]}_{\text{top}} = \sum_{\substack{i \le i_t \le \bar{j} \\ i_b > \bar{j}}} C_{v'}^{(i_t)} \otimes C_{v'}^{(i_b)\,t}, \quad \mathcal{C}'^{[i,\bar{j}]}_{\text{mid}} = \sum_{\substack{i \le i_t \le \bar{j} \\ i \le i_b \le \bar{j}}} C_{v'}^{(i_t)} \otimes C_{v'}^{(i_b)\,t}, \quad \mathcal{C}'^{[i,\bar{j}]}_{\text{bot}} = \sum_{\substack{i_t > \bar{j} \\ i \le i_b \le \bar{j}}} C_{v'}^{(i_t)} \otimes C_{v'}^{(i_b)\,t}.$$

As $v - v'$ is supported on the $(\bar{j}+2)^{\rm nd}$ extended neighbourhood, $\mathcal{C}_{\rm mid}^{[i,\bar{j}]} = \mathcal{C}_{\rm mid}^{'[i,\bar{j}]}$. Hence, by Theorem III.9.i, with $\beta = 0$, $r = j$ and $j = \bar{j}$,

$$\big\| g \bullet \big[\mathcal{C}^{[i,\bar{j}]} - \mathcal{C}^{'[i,\bar{j}]}\big] \bullet h \big\|_{\kappa_1,\kappa_2} \le \text{const} \, \big|g\big|_{\ell,i} \big|h\big|_{i,j}. \tag{III.15}$$

By (III.14) and the definitions of $\bar{j}$ and $\mathcal{C}^{'[i,\bar{j}]}$, the hypotheses of Theorem III.15, with $r = j$ and $j = \bar{j}$, apply to $g \bullet \mathcal{C}^{'[i,\bar{j}]} \bullet h$. Hence

$$\big\| g \bullet \mathcal{C}^{'[i,\bar{j}]} \bullet h \big\|_{\kappa_1,\kappa_2} \le \text{const} \max_{\substack{\alpha_{\rm r},\alpha_{\rm l} \in \mathbb{N}_0 \times \mathbb{N}_0^2 \\ |\alpha_{\rm r}| + |\alpha_{\rm l}| \le 3}} \big|g\big|_{\ell,i}^{[0,0,\alpha_{\rm r}]} \big|h\big|_{i,j}^{[\alpha_{\rm l},0,0]}.$$

This together with (III.15) and Prop. III.16 yields (III.13). This completes the proof that Theorem III.15 implies Theorem II.19.a. □

The rest of this subsection is devoted to the proof of Theorem III.15. So we fix $\ell \ge 1$, $1 \le i \le j \le r$ and sectorized, translation invariant functions, g and h, on $\mathfrak{Y}_{\ell,i}$ and $\mathfrak{Y}_{i,j}$ respectively. We also fix $\kappa_1, \kappa_2 \in \mathfrak{K}_r$ and assume that

$$\tau_0 \le \tfrac{1}{M^{j-1}} \qquad |\boldsymbol{\tau}| \le \max\big\{\tfrac{1}{M^j}, r^3 \mathfrak{l}_r\big\} \qquad M^i \le \mathfrak{l}_j M^j \tag{III.16}$$

and that $p^{(i)}$ vanishes for all $i > j+1$.

We shall not need to decompose $\mathcal{C}^{[i,j]} = \mathcal{C}_{\rm top}^{[i,j]} + \mathcal{C}_{\rm mid}^{[i,j]} + \mathcal{C}_{\rm bot}^{[i,j]}$ but we still split $\mathcal{C}^{[i,j]}$ into scales and split each scale contribution into pieces with additional sector restrictions. For any scale $i \le m \le j$ and sectors $s_1, s_2 \in \Sigma_m$, set

$$\mathcal{C}_{s_1,s_2}^{(m)}(p,k) = \sum_{\substack{m_1,m_2 \ge 0 \\ \min(m_1,m_2)=m}} c_{s_1}^{(m_1)}(p)\, c_{s_2}^{(m_2)}(k),$$

where $c_s^{(n)}$ was defined in (III.9). Then

$$\mathcal{C}^{[i,j]} = \sum_{m=i}^{j} \sum_{s_1,s_2 \in \Sigma_m} \mathcal{C}_{s_1,s_2}^{(m)}.$$

By Lemmas III.4 and III.14,

$$\begin{aligned} \|\mathcal{C}_{s_1,s_2,R}^{(m)}\|_{\rm bubble} &\le \text{const}\, \mathfrak{l}_m \|\hat{R}(x)\|_{L^1}, \\ \|\mathcal{C}_{s_1,s_2}^{(m)}\|_{\rm bubble} &\le \text{const}\, \mathfrak{l}_m. \end{aligned} \tag{III.17}$$

Reduction to the Model Bubble Propagator. The above argument for large transfer momentum implicitly exploited the fact that the particle–hole bubble is Hölder continuous in the transfer momentum t when t is nonzero. As was pointed out in the introduction, this is false for $t = 0$. However, if one restricts to transfer momenta with $t_0 = 0$ then, at least for the delta function interaction, C^∞ dispersion relation and a model propagator with suitable cutoff procedure, the particle–hole bubble is in fact C^∞ for $\mathbf{t}$ near zero. This was seen in Lemma I.1.

Lemma I.1 applied to the particle–hole bubble with a delta function interaction and choice of cutoff different from that used in this paper. In the present situation, we have general interaction kernels g and h rather than delta functions and cutoffs that do not treat k_0 and $e(\mathbf{k})$ independently. Furthermore, the time component t_0 of the transfer momentum need not be zero. We now perform three reduction steps leading to a situation similar to that of Lemma I.1.

Step 1 (Decoupling of the k_0 integral). Define the zero component localization operator

$$\mathcal{Z}(x_1, x_2, y_1, y_2) = \delta(x_1 - y_1)\delta(\mathbf{x}_2 - \mathbf{y}_2)\delta(y_{1,0} - y_{2,0}), \qquad \text{(III.18)}$$

where $x_i = (x_{i,0}, \mathbf{x}_i)$ and $y_i = (y_{i,0}, \mathbf{y}_i)$. The transpose of this operator has kernel

$$\mathcal{Z}^t(x_1, x_2, y_1, y_2) = \delta(x_1 - y_1)\delta(\mathbf{x}_2 - \mathbf{y}_2)\delta(x_{1,0} - x_{2,0}).$$

Remark III.17. If $W(x_1, x_2, y_1, y_2)$ is a particle–hole propagator

$$\big(\mathcal{Z} \circ W \circ \mathcal{Z}^t\big)(x_1, x_2, y_1, y_2) = W\big(x_1, (x_{1,0}, \mathbf{x}_2), y_1, (y_{1,0}, \mathbf{y}_2)\big).$$

If $W(x_1, x_2, y_1, y_2)$ is associated to $W(p, k)$ as in (III.8), then

$$\begin{aligned}\big(\mathcal{Z} \circ W \circ \mathcal{Z}^t\big)(x_1, x_2, y_1, y_2) = \int \tfrac{d^3t}{(2\pi)^3} \tfrac{d^2\mathbf{k}}{(2\pi)^3}\, e^{\imath \mathbf{k}\cdot(\mathbf{x}_1 - \mathbf{y}_1 + \mathbf{y}_2 - \mathbf{x}_2)} e^{\imath <t, x_1 - y_1>_-} \\ \times \int dk_0\, W(k + t, k).\end{aligned}$$

That is, $\big(\mathcal{Z} \circ W \circ \mathcal{Z}^t\big)$ is associated to $\delta(k_0) \int d\omega\, W\big((\omega, \mathbf{0}) + p, (\omega, \mathbf{k})\big)$.

Lemma III.18. *Let W be a particle–hole bubble propagator.*

i) Let $R(t)$ be any cutoff function for the transfer momentum. Then,

$$\big(\mathcal{Z} \circ W \circ \mathcal{Z}^t\big)_R = \mathcal{Z} \circ W_R \circ \mathcal{Z}^t.$$

ii) For any translation invariant kernels G on $\mathfrak{Y}^2 \times \big(\mathbb{R} \times \mathbb{R}^2\big)^2$ and H on $\big(\mathbb{R} \times \mathbb{R}^2\big)^2 \times \mathfrak{Y}^2$,

$$|||G \circ \mathcal{Z}||| \le |||G||| \qquad \text{and} \qquad |||\mathcal{Z}^t \circ H||| \le |||H|||.$$

iii)

$$\|\mathcal{Z} \circ W\|_{\text{bubble}} \le \|W\|_{\text{bubble}} \qquad \textit{and} \qquad \|W \circ \mathcal{Z}^t\|_{\text{bubble}} \le \|W\|_{\text{bubble}}.$$

iv)

$$\begin{aligned}&\|\mathcal{Z} \circ W \circ \mathcal{Z}^t\|_{\text{bubble}} \\ &\quad \le \min\Big\{ \min_{n=1,2} \sup_{x_1, x_2} \int d\mathbf{y}_n dy_{1,0} \sup_{\mathbf{y}_{\bar n}} \big|W\big(x_1, (x_{1,0}, \mathbf{x}_2), y_1, (y_{1,0}, \mathbf{y}_2)\big)\big|, \\ &\qquad \times \min_{n=1,2} \sup_{y_1, y_2} \int d\mathbf{x}_n dx_{1,0} \sup_{\mathbf{x}_{\bar n}} \big|W\big(x_1, (x_{1,0}, \mathbf{x}_2), y_1, (y_{1,0}, \mathbf{y}_2)\big)\big| \Big\},\end{aligned}$$

where $\bar n = 2$ if $n = 1$ and $\bar n = 1$ if $n = 2$.

Proof. i) This is obvious from Remark III.5, since $\mathcal{Z}^t \circ O_R = O_R \circ \mathcal{Z}^t$.

ii) This is obvious since

$$\begin{aligned}(G \circ \mathcal{Z})(\,\cdot\,, \,\cdot\,, x_3, x_4) &= \delta(x_{3,0} - x_{4,0}) \int d\omega\, G\big(\,\cdot\,, \,\cdot\,, x_3, (\omega, \mathbf{x}_4)\big), \\ (\mathcal{Z}^t \circ H)(x_1, x_2, \,\cdot\,, \,\cdot\,) &= \delta(x_{1,0} - x_{2,0}) \int d\omega\, H\big(x_1, (\omega, \mathbf{x}_2), \,\cdot\,, \,\cdot\,\big).\end{aligned}$$

iii) By part (ii), for any translation invariant G, H,

$$|||G \circ \mathcal{Z} \circ W \circ H||| \le |||G \circ \mathcal{Z}|||\ \|W\|_{\text{bubble}}\ |||H||| \le |||G|||\ \|W\|_{\text{bubble}}\ |||H|||$$

and similarly for $|||G \circ W \circ \mathcal{Z}^t \circ H|||$.

iv) The bounds with $n = 1, \bar{n} = 2$ are direct consequences of Remark III.17 and Lemma III.2. We prove

$$\|\mathcal{Z} \circ W \circ \mathcal{Z}^t\|_{\text{bubble}} \le \sup_{x_1,x_2} \int d\mathbf{y}_2\, dy_{1,0} \sup_{\mathbf{y}_1} \big|W\big(x_1, (x_{1,0}, \mathbf{x}_2), y_1, (y_{1,0}, \mathbf{y}_2)\big)\big|.$$

The remaining case is similar. Let $G(y_1, y_2, u_1, u_2)$ and $H(v_1, v_2, y_3, y_4)$ be translation invariant four–legged kernels obeying

$$G = G\big|_{(i_1,i_2,1,1)}, \qquad H = H\big|_{(1,1,i_3,i_4)}$$

for some $i_1, i_2, i_3, i_4 \in \{0, 1\}$. By (III.5) and (III.6), with P replaced by $\mathcal{Z} \circ W \circ \mathcal{Z}^t$,

$$\begin{aligned}
&|||G \circ \mathcal{Z} \circ W \circ \mathcal{Z}^t \circ H|||\\
&\quad\le |||G||| \sup_{\substack{y_\nu \in \mathbb{M}\\ \nu=3,4\\ \text{with } i_\nu=0}} \sup_{u_1,u_2} \int \prod_{\substack{\nu=3,4\\ \text{with } i_\nu=1}} dy_\nu\, dv_1 dv_2\\
&\qquad\times \big|W\big(u_1, (u_{1,0}, \mathbf{u}_2), v_1, (v_{1,0}, \mathbf{v}_2)\big) H(v_1, v_2, y_3, y_4)\big|\\
&\quad\le |||G||| \sup_{\substack{y_\nu \in \mathbb{M}\\ \nu=3,4\\ \text{with } i_\nu=0}} \sup_{u_1,u_2} \int dv_{1,0}\, d\mathbf{v}_2 \Big\{ \sup_{\mathbf{v}_1} \big|W\big(u_1, (u_{1,0}, \mathbf{u}_2), v_1, (v_{1,0}, \mathbf{v}_2)\big)\big|\\
&\qquad\times \int \prod_{\substack{\nu=3,4\\ \text{with } i_\nu=1}} dy_\nu\, dv_{2,0}\, d\mathbf{v}_1 \big|H(v_1, v_2, y_3, y_4)\big| \Big\}\\
&\quad\le |||G||| \Big[\sup_{u_1,u_2} \int dv_{1,0} d\mathbf{v}_2 \sup_{\mathbf{v}_1} \big|W\big(u_1, (u_{1,0}, \mathbf{u}_2), v_1, (v_{1,0}, \mathbf{v}_2)\big)\big| \Big]\\
&\qquad\times \sup_{\substack{y_\nu \in \mathbb{M}\\ \nu=3,4\\ \text{with } i_\nu=0}} \sup_{v_{1,0},\mathbf{v}_2} \int \prod_{\substack{\nu=3,4\\ \text{with } i_\nu=1}} dy_\nu\, dv_{2,0}\, d\mathbf{v}_1 \big|H(v_1, v_2, y_3, y_4)\big|.
\end{aligned}$$

By translation invariance

$$\begin{aligned}
&\sup_{v_{1,0},\mathbf{v}_2} \int \prod_{\substack{\nu=3,4\\ \text{with } i_\nu=1}} dy_\nu\, dv_{2,0}\, d\mathbf{v}_1 \big|H(v_1, v_2, y_3, y_4)\big|\\
&\quad= \sup_{v_1} \int \prod_{\substack{\nu=3,4\\ \text{with } i_\nu=1}} dy_\nu\, dv_2\, \big|H\big((v_{1,0}, \mathbf{v}_2), (v_{2,0}, \mathbf{v}_1), y_3, y_4\big)\big|\\
&\quad= \sup_{v_1} \int \prod_{\substack{\nu=3,4\\ \text{with } i_\nu=1}} dy_\nu\, dv_2\, \big|H\big(v_1, (v_{2,0}, 2\mathbf{v}_1 - \mathbf{v}_2), y_3, y_4\big)\big|\\
&\quad= \sup_{v_1} \int \prod_{\substack{\nu=3,4\\ \text{with } i_\nu=1}} dy_\nu\, dv_2\, \big|H(v_1, v_2, y_3, y_4)\big| \le |||H|||. \qquad \square
\end{aligned}$$

Proposition III.19.

$$\big\|g \bullet \big(\mathcal{C}^{[i,j]} - \mathcal{Z} \bullet \mathcal{C}^{[i,j]} \bullet \mathcal{Z}^t\big) \bullet h\big\|_{\kappa_1,\kappa_2} \le \text{const} \max_{\substack{\alpha_l,\alpha_r \in \Delta\\ \alpha_l+\alpha_r=(1,0,0)}} |g|_{\ell,i}^{[0,0,\alpha_r]} |h|_{i,r}^{[\alpha_l,0,0]}.$$

In preparation for the proof, which follows Lemma III.21, we define

$$
\begin{aligned}
g_{\rm r}\big((x_1,\sigma_1'),(x_2,\sigma_2'),(x_3,u_1),(x_4,u_2)\big) &= (x_{4,0}-x_{3,0})\,g\big((x_1,\sigma_1'),(x_2,\sigma_2'),(x_3,u_1),(x_4,u_2)\big),\\
h_{\rm l}\big((x_1,v_1),(x_2,v_2),(x_3,\sigma_1),(x_4,\sigma_2)\big) &= (x_{2,0}-x_{1,0})\,h\big((x_1,v_1),(x_2,v_2),(x_3,\sigma_1),(x_4,\sigma_2)\big).
\end{aligned}
$$

For a particle–hole bubble propagator $W(x_1,x_2,y_1,y_2)$ set

$$
\begin{aligned}
(D_{\rm l}W)(x_1,x_2,y_1,y_2) &= \int_0^1 d\omega\ \tfrac{\partial\ W}{\partial x_{2,0}}\big(x_1,(\omega x_{2,0}+(1-\omega)x_{1,0},\mathbf{x}_2),y_1,y_2\big),\\
(D_{\rm r}W)(x_1,x_2,y_1,y_2) &= \int_0^1 d\omega\ \tfrac{\partial\ W}{\partial y_{2,0}}\big(x_1,x_2,y_1,(\omega y_{2,0}+(1-\omega)y_{1,0},\mathbf{y}_2)\big).
\end{aligned}
$$

Lemma III.20.

$$
g\bullet(W-\mathcal{Z}W\mathcal{Z}^t)\bullet h = g_{\rm r}\bullet D_{\rm l}W\bullet h + g\circ\mathcal{Z}\bullet D_{\rm r}W\bullet h_{\rm l}.
$$

Proof. By Remark III.17,

$$
\begin{aligned}
g\bullet(W-\mathcal{Z}W\mathcal{Z}^t)\bullet h &= g\bullet\big\{W(x_1,x_2,y_1,y_2)-W\big(x_1,(x_{1,0},\mathbf{x}_2),y_1,(y_{1,0},\mathbf{y}_2)\big)\big\}\bullet h\\
&= g_{\rm r}\bullet\Big[\tfrac{1}{x_{2,0}-x_{1,0}}\big\{W(x_1,x_2,y_1,y_2)-W\big(x_1,(x_{1,0},\mathbf{x}_2),y_1,y_2\big)\big\}\Big]\bullet h\\
&\quad+g\bullet\Big[\big\{W\big(x_1,(x_{1,0},\mathbf{x}_2),y_1,y_2\big)-W\big(x_1,(x_{1,0},\mathbf{x}_2),y_1,(y_{1,0},\mathbf{y}_2)\big)\big\}\tfrac{1}{y_{2,0}-y_{1,0}}\Big]\bullet h_{\rm l}\\
&= g_{\rm r}\bullet D_{\rm l}W\bullet h + g\circ\mathcal{Z}\bullet D_{\rm r}W\bullet h_{\rm l}
\end{aligned}
$$

by the Fundamental Theorem of Calculus. □

Lemma III.21. *Let $i\le m\le j$ and $s_1,s_2\in\Sigma_m$. Then*

$$
\begin{aligned}
\|D_{\rm l}\mathcal{C}^{(m)}_{s_1,s_2}\|_{\rm bubble} &\le {\rm const}\ \tfrac{\mathfrak{l}_m}{M^m},\\
\|D_{\rm r}\mathcal{C}^{(m)}_{s_1,s_2}\|_{\rm bubble} &\le {\rm const}\ \tfrac{\mathfrak{l}_m}{M^m}.
\end{aligned}
$$

Proof. We treat $D_{\rm l}\mathcal{C}^{(m)}_{s_1,s_2}$. The other case is similar. For each fixed $0\le\omega\le 1$,

$$
\begin{aligned}
&\Big(\tfrac{\partial}{\partial x_{2,0}}\mathcal{C}^{(m)}_{s_1,s_2}\Big)\big(x_1,(\omega x_{2,0}+(1-\omega)x_{1,0},\mathbf{x}_2),y_1,y_2\big)\\
&\quad=\sum_{\substack{m_1,m_2\ge 0\\ \min\{m_1,m_2\}=m}} c^{(m_1)}_{s_1}(y_1-x_1)\Big(\tfrac{\partial}{\partial x_{2,0}}c^{(m_2)}_{s_2}\Big)\big((\omega x_{2,0}+(1-\omega)x_{1,0}-y_{2,0},\mathbf{x}_2-\mathbf{y}_2)\big).
\end{aligned}
$$

We bound the bubble norm of each term separately. For $m_1\ge m_2=m$, by Lemma III.2,

$$
\begin{aligned}
&\Big\|c^{(m_1)}_{s_1}\Big(\tfrac{\partial}{\partial x_{2,0}}c^{(m_2)}_{s_2}\Big)\Big\|_{\rm bubble}\\
&\quad\le\big\|c^{(m_1)}_{s_1}\big\|_{L^\infty}\sup_{x_1,x_2}\int dy_2\ \Big|\Big(\tfrac{\partial}{\partial x_{2,0}}c^{(m_2)}_{s_2}\Big)\big((\omega x_{2,0}+(1-\omega)x_{1,0}-y_{2,0},\mathbf{x}_2-\mathbf{y}_2)\big)\Big|\\
&\quad=\big\|c^{(m_1)}_{s_1}\big\|_{L^\infty}\Big\|\tfrac{\partial}{\partial x_{2,0}}c^{(m_2)}_{s_2}\Big\|_{L^1}\\
&\quad\le{\rm const}\ \tfrac{\mathfrak{l}_m}{M^{m_1}}\tfrac{1}{M^{m_2}}M^{m_2}\le{\rm const}\ \tfrac{\mathfrak{l}_m}{M^{m_1}}
\end{aligned}
$$

by parts (iii) and (iv) of Lemma A.2. For $m=m_1\le m_2$,

$$
\Big\|c^{(m_1)}_{s_1}\Big(\tfrac{\partial}{\partial x_{2,0}}c^{(m_2)}_{s_2}\Big)\Big\|_{\rm bubble}\le\Big\|\tfrac{\partial}{\partial x_{2,0}}c^{(m_2)}_{s_2}\Big\|_{L^\infty}\big\|c^{(m_1)}_{s_1}\big\|_{L^1}\le{\rm const}\ \tfrac{\mathfrak{l}_m}{M^{2m_2}}M^{m_1}.
$$

Hence

$$
\|D_{\rm l}\mathcal{C}^{(m)}_{s_1,s_2}\|_{\rm bubble}\le{\rm const}\sum_{n=m}^{\infty}\Big[\tfrac{\mathfrak{l}_m}{M^n}+\tfrac{\mathfrak{l}_m}{M^{2n}}M^m\Big]\le{\rm const}\ \tfrac{\mathfrak{l}_m}{M^m}.\quad\square
$$

Proof of Proposition III.19. By Lemma III.20 followed by Remark III.13[3]

$$\begin{aligned}
&\big\|g \bullet \big(\mathcal{C}^{[i,j]} - \mathcal{Z} \bullet \mathcal{C}^{[i,j]} \bullet \mathcal{Z}^t\big) \bullet h\big\|_{\kappa_1,\kappa_2} \\
&\le \big\|g_{\rm r} \bullet D_{\rm l}\mathcal{C}^{[i,j]} \bullet h\big\|_{\kappa_1,\kappa_2} + \big\|g \circ \mathcal{Z} \bullet D_{\rm r}\mathcal{C}^{[i,j]} \bullet h_{\rm l}\big\|_{\kappa_1,\kappa_2} \\
&\le \text{const}\, |g_{\rm r}|_{\ell,i} |h|_{i,r} \sum_{m=i}^{j} \sum_{\substack{s_1,s_2\in\Sigma_m \\ \boldsymbol{\pi}(s_1-s_2)\cap\boldsymbol{\pi}(d)\ne\emptyset}} \big\| D_{\rm l}\mathcal{C}^{(m)}_{s_1,s_2}\big\|_{\rm bubble} \\
&\qquad + \text{const}\, |g\circ\mathcal{Z}|_{\ell,i} |h_{\rm l}|_{i,r} \sum_{m=i}^{j} \sum_{\substack{s_1,s_2\in\Sigma_m \\ \boldsymbol{\pi}(s_1-s_2)\cap\boldsymbol{\pi}(d)\ne\emptyset}} \big\| D_{\rm r}\mathcal{C}^{(m)}_{s_1,s_2}\big\|_{\rm bubble}.
\end{aligned}$$

By the spatial projection of Remark III.12.ii,

$$\#\big\{ (s_1,s_2)\in\Sigma_m\times\Sigma_m \;\big|\; \boldsymbol{\pi}(s_1-s_2)\cap\boldsymbol{\pi}(d)\ne\emptyset \big\} \le \tfrac{\text{const}}{\mathfrak{l}_m}. \tag{III.19}$$

Using this, Lemma III.21, Lemma III.18.ii and the definitions of $g_{\rm r}$, $h_{\rm l}$, we have

$$\begin{aligned}
&\big\|g \bullet \big(\mathcal{C}^{[i,j]} - \mathcal{Z} \bullet \mathcal{C}^{[i,j]} \bullet \mathcal{Z}^t\big) \bullet h\big\|_{\kappa_1,\kappa_2} \\
&\le \text{const}\, M^i |g|^{[0,0,(1,0,0)]}_{\ell,i} |h|_{i,r} \sum_{m=i}^{j} \tfrac{1}{\mathfrak{l}_m}\tfrac{\mathfrak{l}_m}{M^m} + \text{const}\, |g|_{\ell,i} M^i |h|^{[(1,0,0),0,0]}_{i,r} \sum_{m=i}^{j} \tfrac{1}{\mathfrak{l}_m}\tfrac{\mathfrak{l}_m}{M^m}. \qquad \square
\end{aligned}$$

Step 2 (Reduction to $t_0 = 0$). For any particle–hole bubble propagator $W(x_1,x_2,y_1,y_2)$ set

$$\widetilde{W}(x_1,x_2,y_1,y_2) = \delta(y_{1,0}-x_{1,0}) \int dz_0\; W\big(x_1,(x_{1,0},\mathbf{x}_2),(z_0,\mathbf{y}_1),(z_0,\mathbf{y}_2)\big). \tag{III.20}$$

If $W(x_1,x_2,y_1,y_2)$ is associated to $W(p,k)$ as in (III.8), then $\widetilde{W}(x_1,x_2,y_1,y_2)$ is associated to

$$\widetilde{W}(p,k) = \delta(k_0)\int d\omega\; W\big((\omega,\mathbf{p}),(\omega,\mathbf{k})\big).$$

By Remark III.17, $\widetilde{W} = \mathcal{Z}\circ\widetilde{W}\circ\mathcal{Z}^t$ for all particle–hole bubble propagators W.

Proposition III.22.

$$\big\|g \bullet \big(\mathcal{Z}\bullet\mathcal{C}^{[i,j]}\bullet\mathcal{Z}^t - \widetilde{\mathcal{C}}^{[i,j]}\big)\bullet h\big\|_{\kappa_1,\kappa_2} \le \text{const}\, |g|_{\ell,i}|h|_{i,r}.$$

Proof. Choose a C_0^∞ function $\phi(t_0)$ that takes values in $[0,1]$, is supported in the interval $|t_0| \le 2\frac{M+4\sqrt{2M}}{M^j}$, is identically one for $|t_0| \le \frac{M+4\sqrt{2M}}{M^j}$ and obeys $\big|\frac{d^n}{dt_0^n}\phi(t_0)\big| \le$ const M^{jn} for $n \le 2$. By Remark III.12.iii,

$$\big\{ |t_0| \;\big|\; (t_0,\mathbf{t})\in d \text{ for some } \mathbf{t}\in\mathbb{R}^2 \big\} \subset \Big[\tau_0, \tau_0 + \tfrac{4\sqrt{2M}}{M^j}\Big].$$

[3] The operators $D_{\rm l}$, $D_{\rm r}$ can enlarge supports in the k_0 direction. So we cannot apply Lemma III.11 directly.

Hence, by (III.16), ϕ is in $\mathcal{R}(d)$. By Remark III.13 and (III.19),

$$\begin{aligned}&\big\| g \bullet \big(\mathcal{Z}\circ \mathcal{C}^{[i,j]}\circ \mathcal{Z}^t - \widetilde{\mathcal{C}}^{[i,j]}\big)\bullet h\big\|_{\kappa_1,\kappa_2}\\ &\quad\le \text{const}\,|g|_{\ell,i}|h|_{i,r}\sum_{m=i}^{j}\tfrac{1}{\mathfrak{l}_m}\max_{s_1,s_2\in\Sigma_m}\big\|\mathcal{Z}\circ \mathcal{C}^{(m)}_{s_1,s_2,\phi}\circ\mathcal{Z}^t - \big(\widetilde{\mathcal{C}}^{(m)}_{s_1,s_2}\big)_\phi\big\|_{\rm bubble}.\end{aligned}$$

Here, we used $(\mathcal{Z}\circ W\circ\mathcal{Z}^t)_R = \mathcal{Z}\circ W_R\circ \mathcal{Z}^t$, which was proven in Lemma III.18.i. The proposition follows from the next lemma. $\square$

Lemma III.23. *Let $m\le j$ and $s_1,s_2\in\Sigma_m$. Then*

$$\big\|\mathcal{Z}\circ\mathcal{C}^{(m)}_{s_1,s_2,\phi}\circ\mathcal{Z}^t - \big(\widetilde{\mathcal{C}}^{(m)}_{s_1,s_2}\big)_\phi\big\|_{\rm bubble}\le \text{const}\,\mathfrak{l}_m\tfrac{M^m}{M^j}.$$

Proof. For any $m_1,m_2\ge 0$, set

$$\begin{aligned}W^{(m_1,m_2)} &= \mathcal{Z}\circ\big(c^{(m_1)}_{s_1}\otimes c^{(m_2)\,t}_{s_2}\big)\circ\mathcal{Z}^t - \big(c^{(m_1)}_{s_1}\otimes c^{(m_2)\,t}_{s_2}\big)^{\sim}\\ &= \mathcal{Z}\circ\big(c^{(m_1)}_{s_1}\otimes c^{(m_2)\,t}_{s_2}\big)\circ\mathcal{Z}^t - \mathcal{Z}\circ\big(c^{(m_1)}_{s_1}\otimes c^{(m_2)\,t}_{s_2}\big)^{\sim}\circ\mathcal{Z}^t.\end{aligned}$$

Observe that

$$\mathcal{Z}\circ\mathcal{C}^{(m)}_{s_1,s_2,\phi}\circ\mathcal{Z}^t - \big(\widetilde{\mathcal{C}}^{(m)}_{s_1,s_2}\big)_\phi = \sum_{\substack{m_1,m_2\in\mathbb{N}_0\\ \min\{m_1,m_2\}=m}} W^{(m_1,m_2)}_\phi. \tag{III.21}$$

We now fix any $m_1,m_2\ge 0$ with $\min\{m_1,m_2\}=m$ and bound $\big\|W^{(m_1,m_2)}_\phi\big\|_{\rm bubble}$. By definition

$$\begin{aligned}&W^{(m_1,m_2)}(x_1,x_2,y_1,y_2)\\ &\quad= c^{(m_1)}_{s_1}(y_1-x_1)c^{(m_2)}_{s_2}\big((x_{1,0}-y_{1,0},\mathbf{x}_2-\mathbf{y}_2)\big)\\ &\qquad\times\delta(y_{1,0}-x_{1,0})\int du\; c^{(m_1)}_{s_1}\big((u-x_{1,0},\mathbf{y}_1-\mathbf{x}_1)\big)c^{(m_2)}_{s_2}\big((x_{1,0}-u,\mathbf{x}_2-\mathbf{y}_2)\big)\end{aligned}$$

and

$$\begin{aligned}&W^{(m_1,m_2)}_\phi(x_1,x_2,y_1,y_2)\\ &\quad=\int dz_0\;\hat\phi(z_0)\Big[c^{(m_1)}_{s_1}((y_{1,0}-x_{1,0}-z_0,\mathbf{y}_1-\mathbf{x}_1))c^{(m_2)}_{s_2}((x_{1,0}-y_{1,0}+z_0,\mathbf{x}_2-\mathbf{y}_2))\\ &\qquad\times\delta(y_{1,0}-x_{1,0}-z_0)\int du\; c^{(m_1)}_{s_1}((u-x_{1,0},\mathbf{y}_1-\mathbf{x}_1))c^{(m_2)}_{s_2}((x_{1,0}-u,\mathbf{x}_2-\mathbf{y}_2))\Big]\\ &\quad=\int dz_0\; c^{(m_1)}_{s_1}((y_{1,0}-x_{1,0}-z_0,\mathbf{y}_1-\mathbf{x}_1))c^{(m_2)}_{s_2}((x_{1,0}-y_{1,0}+z_0,\mathbf{x}_2-\mathbf{y}_2))\hat\phi(z_0)\\ &\qquad\times\int du\; c^{(m_1)}_{s_1}((u-x_{1,0},\mathbf{y}_1-\mathbf{x}_1))c^{(m_2)}_{s_2}((x_{1,0}-u,\mathbf{x}_2-\mathbf{y}_2))\hat\phi(y_{1,0}-x_{1,0})\\ &\quad=\int dz_0\; c^{(m_1)}_{s_1}((z_0-x_{1,0},\mathbf{y}_1-\mathbf{x}_1))c^{(m_2)}_{s_2}((x_{1,0}-z_0,\mathbf{x}_2-\mathbf{y}_2))\big[\hat\phi(y_{1,0}-z_0)-\hat\phi(y_{1,0}-x_{1,0})\big].\end{aligned}$$

The last factor

$$\hat\phi(y_{1,0}-z_0)-\hat\phi(y_{1,0}-x_{1,0}) = (x_{1,0}-z_0)\int_0^1 dt\;\hat\phi'\big(y_{1,0}-x_{1,0}+t(x_{1,0}-z_0)\big).$$

Observe that

$$\int dy_{1,0}\int_0^1 dt\ \big|\hat\phi'\big(y_{1,0}-x_{1,0}+t(x_{1,0}-z_0)\big)\big| = \int dy_{1,0}\ |\hat\phi'(y_{1,0})| \le \tfrac{\rm const}{M^j},$$

since

$$|\hat\phi'(y_{1,0})| \le {\rm const}\ \tfrac{1/M^{2j}}{[1+|y_{1,0}/M^j|]^2}.$$

If $m_2 \ge m_1 = m$, we apply Lemma III.18.iv[4], (III.11) and (III.10), giving

$$\begin{aligned}
&\big\| W_\phi^{(m_1,m_2)}\big\|_{\rm bubble}\\
&\quad\le {\rm const}\ \sup_{x_1,x_2}\int dy_1 \sup_{\mathbf{y}_2}\int dz_0\ |x_{1,0}-z_0|\\
&\qquad\times\big|c_{s_1}^{(m_1)}((z_0-x_{1,0},\mathbf{y}_1-\mathbf{x}_1))\big|\ \big|c_{s_2}^{(m_2)}((x_{1,0}-z_0,\mathbf{x}_2-\mathbf{y}_2))\big|\\
&\qquad\times\int_0^1 dt\ \big|\hat\phi'(y_{1,0}-x_{1,0}+t(x_{1,0}-z_0))\big|\\
&\quad\le {\rm const}\ \big\|c_{s_2}^{(m_2)}\big\|_{L^\infty}\sup_{x_1}\int d\mathbf{y}_1 dz_0\ |x_{1,0}-z_0|\big|c_{s_1}^{(m_1)}((z_0-x_{1,0},\mathbf{y}_1-\mathbf{x}_1))\big|\\
&\qquad\times\int dy_{1,0}\int_0^1 dt\ \big|\hat\phi'(y_{1,0}-x_{1,0}+t(x_{1,0}-z_0))\big|\\
&\quad\le {\rm const}\ \tfrac{1}{M^j}\tfrac{\mathfrak{l}_m}{M^{m_2}}\Big[\sup_{x_1}\int d\mathbf{y}_1 dz_0\ |x_{1,0}-z_0|\big|c_{s_1}^{(m_1)}((z_0-x_{1,0},\mathbf{y}_1-\mathbf{x}_1))\big|\Big]\\
&\quad= {\rm const}\ \tfrac{1}{M^j}\tfrac{\mathfrak{l}_m}{M^{m_2}}\big\|x_0 c_{s_1}^{(m_1)}(x)\big\|_{L^1}\\
&\quad\le {\rm const}\ \tfrac{1}{M^j}\tfrac{\mathfrak{l}_m}{M^{m_2}}M^{2m}.
\end{aligned}$$

Similarly, if $m_1 \ge m_2 = m$,

$$\begin{aligned}
&\big\| W_\phi^{(m_1,m_2)}\big\|_{\rm bubble}\\
&\quad\le {\rm const}\ \sup_{x_1,x_2}\int dy_{1,0}d\mathbf{y}_2 \sup_{\mathbf{y}_1}\int dz_0\ |x_{1,0}-z_0|\\
&\qquad\times\big|c_{s_1}^{(m_1)}((z_0-x_{1,0},\mathbf{y}_1-\mathbf{x}_1))\big|\ \big|c_{s_2}^{(m_2)}((x_{1,0}-z_0,\mathbf{x}_2-\mathbf{y}_2))\big|\\
&\qquad\times\int_0^1 dt\ \big|\phi'(y_{1,0}-x_{1,0}+t(x_{1,0}-z_0))\big|\\
&\quad\le {\rm const}\ \tfrac{1}{M^j}\big\|c_{s_1}^{(m_1)}\big\|_{L^\infty}\sup_{x_{1,0},\mathbf{x}_2}\int d\mathbf{y}_2 dz_0\ |x_{1,0}-z_0|\big|c_{s_2}^{(m_2)}((x_{1,0}-z_0,\mathbf{x}_2-\mathbf{y}_2))\big|\\
&\quad\le {\rm const}\ \tfrac{1}{M^j}\tfrac{\mathfrak{l}_m}{M^{m_1}}M^{2m}.
\end{aligned}$$

Consequently, by (III.21),

$$\begin{aligned}
\big\|\mathcal{Z}\circ\mathcal{C}_{s_1,s_2,\phi}^{(m)}\circ\mathcal{Z}^t-\big(\widetilde{\mathcal{C}}_{s_1,s_2}^{(m)}\big)_\phi\big\|_{\rm bubble}
&\le {\rm const}\sum_{\substack{m_1,m_2\ge 0\\ \min\{m_1,m_2\}=m}}\tfrac{M^{2m}}{M^j}\mathfrak{l}_m\min\big\{\tfrac{1}{M^{m_1}},\tfrac{1}{M^{m_2}}\big\}\\
&\le {\rm const}\ \tfrac{M^{2m}}{M^j}\mathfrak{l}_m\sum_{m'\ge m}\tfrac{1}{M^{m'}}\\
&\le {\rm const}\ \tfrac{M^m}{M^j}\mathfrak{l}_m. \qquad \square
\end{aligned}$$

---

[4] Note that in the bound on the right hand side of Lemma III.18.iv, W only appears in the form $W(x_1,(x_{1,0},\mathbf{x}_2),y_1,(y_{1,0},\mathbf{y}_2)) = (\mathcal{Z}\circ W\circ\mathcal{Z}^t)(x_1,x_2,y_1,y_2)$.

Step 3 (Introduction of Factorized Cutoff). Define

$$\nu_0(\omega) = \sum_{m=i+1}^{j-1} \nu\big(M^{2m}\omega^2\big),$$
$$\nu_1(\mathbf{p},\mathbf{k}) = \Big[\sum_{m_1=i+1}^{\infty} \nu\big(M^{2m_1}e(\mathbf{p})^2\big)\Big]\Big[\sum_{m_2=i+1}^{\infty} \nu\big(M^{2m_2}e(\mathbf{k})^2\big)\Big],$$

where ν is the single scale cutoff introduced in Def. I.2. Recall that $\nu(x)$ is identically one on $\big[\frac{2}{M}, M\big]$ and is supported on $\big[\frac{1}{M}, 2M\big]$. Define

$$e'(k) = e(\mathbf{k}) - v(k)$$

and the model particle–hole bubble propagator

$$\mathcal{M}(p,k) = \delta(k_0)\int d\omega\, \frac{\nu_0(\omega)\nu_1(\mathbf{p},\mathbf{k})}{[i\omega - e'(\omega,\mathbf{p})][i\omega - e'(\omega,\mathbf{k})]}. \tag{III.22}$$

Observe that $\mathcal{Z}\circ\mathcal{M}\circ\mathcal{Z}^t = \mathcal{M}$.

Proposition III.24.

$$\big\| g \bullet \big(\widetilde{\mathcal{C}}^{[i,j]} - \mathcal{M}\big) \bullet h \big\|_{\kappa_1,\kappa_2} \le \text{const}\, |g|_{\ell,i}|h|_{i,r}.$$

Proof. The cutoff $\nu_0(\omega)\nu_1(\mathbf{p},\mathbf{k})$ is supported on $\Big\{(\omega,\mathbf{p},\mathbf{k})\,\Big|\, |\omega|, |e(\mathbf{p})|, |e(\mathbf{k})| \le \sqrt{\frac{2}{M}\frac{1}{M^i}}\Big\}$. Since $\nu^{(m)}(k)$ vanishes for all $|ik_0 - e(\mathbf{k})| \le \frac{1}{\sqrt{M}}\frac{1}{M^{i-1}}$ when $m \le i-1$,

$$\nu_0(\omega)\nu_1(\mathbf{p},\mathbf{k}) = \sum_{m_1,m_2\ge i} \nu_0(\omega)\nu_1(\mathbf{p},\mathbf{k})\nu^{(m_1)}((\omega,\mathbf{p}))\nu^{(m_2)}((\omega,\mathbf{k})),$$

if M is large enough. Since every k in the support of $\nu^{(m)}(k)$ obeys $|ik_0 - e(\mathbf{k})| \le \sqrt{2M}\frac{1}{M^{j+1}}$ for all $m \ge j+1$ and since $\nu_0(\omega)$ is supported on $|\omega| \ge \sqrt{M}\frac{1}{M^j}$,

$$\nu_0(\omega)\nu_1(\mathbf{p},\mathbf{k}) = \sum_{i\le \min\{m_1,m_2\}\le j} \nu_0(\omega)\nu_1(\mathbf{p},\mathbf{k})\nu^{(m_1)}((\omega,\mathbf{p}))\nu^{(m_2)}((\omega,\mathbf{k})).$$

Recall that

$$\begin{aligned}\widetilde{\mathcal{C}}^{[i,j]}(p,k) &= \delta(k_0)\int d\omega\, \mathcal{C}^{[i,j]}\big((\omega,\mathbf{p}),(\omega,\mathbf{k})\big)\\ &= \delta(k_0)\int d\omega\, \tfrac{\nu^{(\ge i)}((\omega,\mathbf{p}))\nu^{(\ge i)}((\omega,\mathbf{k})) - \nu^{(\ge j+1)}((\omega,\mathbf{p}))\nu^{(\ge j+1)}((\omega,\mathbf{k}))}{[i\omega - e'(\omega,\mathbf{p})][i\omega - e'(\omega,\mathbf{k})]}.\end{aligned}$$

The difference of cutoff functions

$$\begin{aligned}&\nu^{(\ge i)}((\omega,\mathbf{p}))\nu^{(\ge i)}((\omega,\mathbf{k})) - \nu^{(\ge j+1)}((\omega,\mathbf{p}))\nu^{(\ge j+1)}((\omega,\mathbf{k})) - \nu_0(\omega)\nu_1(\mathbf{p},\mathbf{k})\\ &\quad = \sum_{m=i}^{j} \sum_{\substack{m_1,m_2\ge 1\\ \min\{m_1,m_2\}=m}} \mu^{(m_1,m_2)}(\omega,\mathbf{k},\mathbf{p}),\end{aligned}$$

where

$$\mu^{(m_1,m_2)}(\omega,\mathbf{k},\mathbf{p}) = \Big[1-\nu_0(\omega)\nu_1(\mathbf{p},\mathbf{k})\Big]\nu^{(m_1)}((\omega,\mathbf{p}))\nu^{(m_2)}((\omega,\mathbf{k})).$$

Define, for $i \le m \le j$, $m_1, m_2 \ge m$ and $s_1, s_2 \in \Sigma_m$,

$$\mathcal{D}^{m_1,m_2}_{s_1,s_2}(p,k) = \delta(k_0)\phi(p_0)\Delta^{m_1,m_2}_{s_1,s_2}(\mathbf{p},\mathbf{k}),$$

where

$$\Delta^{m_1,m_2}_{s_1,s_2}(\mathbf{p},\mathbf{k}) = \int d\omega\, \frac{\mu^{(m_1,m_2)}(\omega,\mathbf{k},\mathbf{p})\chi_{s_1}(0,\mathbf{p})\chi_{s_2}(0,\mathbf{k})}{[i\omega - e'(\omega,\mathbf{p})][i\omega - e'(\omega,\mathbf{k})]},$$

and ϕ was defined at the beginning of the proof of Prop. III.22. Define

$$\mathcal{D}^{(m)}_{s_1,s_2}(p,k) = \sum_{\substack{m_1,m_2\ge 1\\ \min\{m_1,m_2\}=m}} \mathcal{D}^{m_1,m_2}_{s_1,s_2}(p,k). \tag{III.23}$$

As $\phi(p_0) = 1$ for all $|p_0| \le \frac{M+4\sqrt{2M}}{M^j}$,

$$\widetilde{\mathcal{C}}^{[i,j]} - \mathcal{M} = \sum_{m=i}^{j} \sum_{s_1,s_2\in\Sigma_m} \mathcal{D}^{(m)}_{s_1,s_2} \qquad \text{if } |p_0| \le \tfrac{M+4\sqrt{2M}}{M^j}.$$

Observe that the kernels of $\widetilde{\mathcal{C}}^{[i,j]}$, $\mathcal{M}$ and $\mathcal{D}^{(m)}_{s_1,s_2}(p,k)$ each contain a factor of $\delta(k_0)$. Hence, in the product $g \bullet \big(\widetilde{\mathcal{C}}^{[i,j]} - \mathcal{M} - \sum_{m=i}^{j}\mathcal{D}^{(m)}_{s_1,s_2}\big) \bullet h$, $\check{h}$ is restricted to $k_0 = 0$, so that $p_0 = t_0$, where t is the transfer momentum. But $t \in d$, so that, by Remark III.12 and (III.16), $|t_0| \le \tau_0 + \frac{4\sqrt{2M}}{M^r} \le \frac{M+4\sqrt{2M}}{M^j}$. Hence, by Remark III.13 and (III.19),

$$\big\| g \bullet \big(\widetilde{\mathcal{C}}^{[i,j]} - \mathcal{M}\big) \bullet h \big\|_{\kappa_1,\kappa_2} \le \text{const}\, |g|_{\ell,i} |h|_{i,r} \sum_{m=i}^{j} \tfrac{1}{\mathfrak{l}_m} \max_{s_1,s_2\in\Sigma_m} \big\|\mathcal{D}^{(m)}_{s_1,s_2}\big\|_{\text{bubble}}.$$

The proposition follows from Lemma III.25 below. □

Lemma III.25. *Let $i \le m \le j$ and $s_1, s_2 \in \Sigma_m$. Then*

$$\big\|\mathcal{D}^{(m)}_{s_1,s_2}\big\|_{\text{bubble}} \le \text{const} \begin{cases} \mathfrak{l}_m & \text{if } m = i, i+1 \\ (j-m+1)\frac{M^m}{M^j}\mathfrak{l}_m & \text{if } m \ge i+2 \end{cases}.$$

Proof. Fix any m_1, m_2 with $\min\{m_1, m_2\} = m$. If ω is in the support of $\mu^{(m_1,m_2)}(\omega,\mathbf{k},\mathbf{p})$ for some $\mathbf{k}$, $\mathbf{p}$ then $|\omega| \le \frac{\text{const}}{M^{\max\{m_1,m_2\}}}$. In the case when $m > i+1$, $|\omega|$ is restricted even farther. Then, in the support of $\mu^{(m_1,m_2)}(\omega,\mathbf{k},\mathbf{p})$, both $|i\omega - e(\mathbf{p})| \le \frac{\sqrt{2M}}{M^{m_1}} \le \frac{\sqrt{2M}}{M^{i+2}}$ and $|i\omega - e(\mathbf{k})| \le \frac{\sqrt{2M}}{M^{m_2}} \le \frac{\sqrt{2M}}{M^{i+2}}$, and hence $|\omega|$, $|e(\mathbf{p})|$, $|e(\mathbf{k})| \le \frac{\sqrt{2M}}{M^{i+2}}$. But $\nu_1(\mathbf{p},\mathbf{k}) = 1$ whenever $|e(\mathbf{p})|$, $|e(\mathbf{k})| \le \frac{1}{M^{i+1/2}}$. Hence on the support of $\mu^{(m_1,m_2)}(\omega,\mathbf{k},\mathbf{p})$, $|\omega| \le \frac{\sqrt{2M}}{M^{i+2}}$ and $\nu_0(\omega) \ne 1$. This forces $|\omega| \le \frac{\sqrt{2}}{M^{j-1/2}} \le \frac{1}{M^{j-3/4}}$. Set

$$b(m_1,m_2) = \begin{cases} \frac{\text{const}}{M^{\max\{m_1,m_2\}}} & \text{if } m = i, i+1 \\ \min\Big\{\frac{1}{M^{j-3/4}}, \frac{\text{const}}{M^{\max\{m_1,m_2\}}}\Big\} & \text{if } m \ge i+2 \end{cases}.$$

Thus

$$\Delta^{m_1,m_2}_{s_1,s_2}(\mathbf{p},\mathbf{k}) = \int_{-b(m_1,m_2)}^{b(m_1,m_2)} d\omega\, \frac{\mu^{(m_1,m_2)}(\omega,\mathbf{k},\mathbf{p})\chi_{s_1}(\mathbf{p})\chi_{s_2}(\mathbf{k})}{[i\omega - e'(\omega,\mathbf{p})][i\omega - e'(\omega,\mathbf{k})]}.$$

By Lemma A.3, the Fourier transform of $\Delta^{m_1,m_2}_{s_1,s_2}(\mathbf{p},\mathbf{k})$ obeys

$$\int d\mathbf{z}_1 \sup_{\mathbf{z}_2} \left|\hat{\Delta}^{m_1,m_2}_{s_1,s_2}(\mathbf{z}_1,\mathbf{z}_2)\right| \le \text{const}\, b(m_1,m_2)\, \mathfrak{l}_m^2 \tfrac{M^{m_1}}{\mathfrak{l}_{m_1}},$$
$$\int d\mathbf{z}_2 \sup_{\mathbf{z}_1} \left|\hat{\Delta}^{m_1,m_2}_{s_1,s_2}(\mathbf{z}_1,\mathbf{z}_2)\right| \le \text{const}\, b(m_1,m_2)\, \mathfrak{l}_m^2 \tfrac{M^{m_2}}{\mathfrak{l}_{m_2}}.$$

As the particle–hole bubble propagator associated to $\mathcal{D}^{(m)}_{s_1,s_2}(p,k)$ by (III.8), namely

$$\mathcal{D}^{m_1,m_2}_{s_1,s_2}(x_1,x_2,y_1,y_2) = \hat{\phi}(y_{1,0}-x_{1,0})\hat{\Delta}^{m_1,m_2}_{s_1,s_2}(\mathbf{y}_1-\mathbf{x}_1,\mathbf{x}_2-\mathbf{y}_2),$$

is independent of $x_{2,0}$ and $y_{2,0}$, $\mathcal{Z}\circ\mathcal{D}^{m_1,m_2}_{s_1,s_2}\circ\mathcal{Z}^t = \mathcal{D}^{m_1,m_2}_{s_1,s_2}$ and we have, by Lemma III.18.iv,

$$\begin{aligned}
\|\mathcal{D}^{m_1,m_2}_{s_1,s_2}\|_{\text{bubble}} &\le \text{const}\ \min\Big\{ \sup_{x_1,x_2} \int dy_1 \sup_{\mathbf{y}_2} \left|\hat{\phi}(y_{1,0}-x_{1,0})\hat{\Delta}^{m_1,m_2}_{s_1,s_2}(\mathbf{y}_1-\mathbf{x}_1,\mathbf{x}_2-\mathbf{y}_2)\right|,\\
&\qquad \times \sup_{x_1,x_2}\int dy_{1,0}d\mathbf{y}_2 \sup_{\mathbf{y}_1}\left|\hat{\phi}(y_{1,0}-x_{1,0})\hat{\Delta}^{m_1,m_2}_{s_1,s_2}(\mathbf{y}_1-\mathbf{x}_1,\mathbf{x}_2-\mathbf{y}_2)\right|\Big\}\\
&\le \text{const}\ \min\Big\{ \int d\mathbf{z}_1 \sup_{\mathbf{z}_2}\left|\hat{\Delta}^{m_1,m_2}_{s_1,s_2}(\mathbf{z}_1,\mathbf{z}_2)\right|,\\
&\qquad \times \int d\mathbf{z}_2 \sup_{\mathbf{z}_1}\left|\hat{\Delta}^{m_1,m_2}_{s_1,s_2}(\mathbf{z}_1,\mathbf{z}_2)\right|\Big\}\\
&\le \text{const}\, b(m_1,m_2)\mathfrak{l}_m^2 \min\left\{\tfrac{M^{m_1}}{\mathfrak{l}_{m_1}}, \tfrac{M^{m_2}}{\mathfrak{l}_{m_2}}\right\}\\
&= \text{const}\, b(m_1,m_2)\mathfrak{l}_m M^m.
\end{aligned}$$

If $m \in \{i, i+1\}$,

$$\left\|\mathcal{D}^{(m)}_{s_1,s_2}\right\|_{\text{bubble}} \le \sum_{\min\{m_1,m_2\}=m} \|\mathcal{D}^{m_1,m_2}_{s_1,s_2}\|_{\text{bubble}} \le \text{const} \sum_{m'\ge m} \tfrac{1}{M^{m'}}\mathfrak{l}_m M^m \le \text{const}\, \mathfrak{l}_m,$$

and if $m \ge i+2$,

$$\begin{aligned}
\left\|\mathcal{D}^{(m)}_{s_1,s_2}\right\|_{\text{bubble}} &\le \sum_{\min\{m_1,m_2\}=m} \|\mathcal{D}^{m_1,m_2}_{s_1,s_2}\|_{\text{bubble}}\\
&\le \text{const} \sum_{m'=m}^{j} \tfrac{1}{M^j}\mathfrak{l}_m M^m + \text{const}\sum_{m'>j}\tfrac{1}{M^{m'}}\mathfrak{l}_m M^m\\
&\le \text{const}\,(j-m+1)\tfrac{M^m}{M^j}\mathfrak{l}_m. \quad \square
\end{aligned}$$

Model Propagator in Position Space.

Lemma III.26. *The set* $\mathbf{d} = \big\{ \mathbf{p}_1 - \mathbf{p}_2 \,\big|\, \mathbf{p}_1 \in \boldsymbol{\kappa}_1,\ \mathbf{p}_2 \in \boldsymbol{\kappa}_2 \big\}$ *is contained in a rectangle with two sides of length* const $\mathfrak{l}_j$ *and two sides of length* const $\frac{1}{M^j}$.

Proof. For every sector $\sigma \in \Sigma_r$, let $\mathbf{k}_\sigma$ be the centre of $\sigma \cap F$, $\mathfrak{t}_\sigma$ a unit tangent vector to F at $\mathbf{k}_\sigma$ and

$$\boldsymbol{\sigma} = \big\{ \mathbf{k} \in \mathbb{R}^2 \,\big|\, (k_0, \mathbf{k}) \in \sigma \text{ for some } k_0 \in \mathbb{R} \big\}.$$

Then $\boldsymbol{\sigma}$ is contained in a rectangle R_σ centered at $\mathbf{k}_\sigma$ with two sides parallel to $\mathfrak{t}_\sigma$ of length const $\mathfrak{l}_r$ and two sides perpendicular to $\mathfrak{t}_\sigma$ of length const $\frac{1}{M^r}$. If at least one of κ_1 and κ_2 are in $\mathbb{M}$, the claim follows since $j \le r$. So assume that $\kappa_1, \kappa_2 \in \Sigma_r$. Then the distance between $\mathbf{k}_{\kappa_1}$ and $\mathbf{k}_{\kappa_2}$ is at most $|\boldsymbol{\tau}| + 5\mathfrak{l}_r$ and therefore the angle between $\mathfrak{t}_{\kappa_1}$ and $\mathfrak{t}_{\kappa_2}$ is at most const $\big(|\boldsymbol{\tau}| + \mathfrak{l}_r\big)$. Consequently $\mathbf{d}$ is contained in a rectangle with two sides parallel to $\mathfrak{t}_{\kappa_1}$ of length const $\mathfrak{l}_r$ and two sides perpendicular to $\mathfrak{t}_{\kappa_1}$ of length

$$\text{const}\left(\tfrac{1}{M^r} + \big(|\boldsymbol{\tau}| + \mathfrak{l}_r\big)\mathfrak{l}_r\right) \le \text{const}\left(\tfrac{1}{M^r} + |\boldsymbol{\tau}|\mathfrak{l}_r\right).$$

By (III.16), $|\boldsymbol{\tau}| \le \max\{\frac{1}{M^j}, r^3\mathfrak{l}_r\}$, so that $|\boldsymbol{\tau}|\mathfrak{l}_r \le \text{const}\,\frac{1}{M^j}$. □

Fix two mutually perpendicular unit vector $\mathfrak{t}$ and $\mathfrak{n}$ and a rectangle R, with two sides parallel to $\mathfrak{t}$ of length const $\mathfrak{l}_j$ and two sides parallel to $\mathfrak{n}$ of length const $\frac{1}{M^j}$, such that $\mathbf{d} \subset R$. By Lemma III.26, such a rectangle exists. Let $\rho(\mathbf{t})$ be identically one on R, be supported on a set of area twice that of R and obey

$$\Big|\big(\mathfrak{n}\cdot\partial_{\mathbf{t}}\big)^{\alpha_1}\big(\mathfrak{t}\cdot\partial_{\mathbf{t}}\big)^{\alpha_2}\rho(\mathbf{t})\Big| \le \text{const}\, M^{\alpha_1 j}\,\frac{1}{\mathfrak{l}_j^{\alpha_2}}$$

for all $\alpha_1, \alpha_2 \le 2$. Define $\mathcal{M}_\rho$ as in Def. III.3. Then

$$\big\|g \bullet \mathcal{M} \bullet h\big\|_{\kappa_1,\kappa_2} = \big\|g \bullet \mathcal{M}_\rho \bullet h\big\|_{\kappa_1,\kappa_2}. \tag{III.24}$$

Proposition III.27.

$$\big\|g \bullet \mathcal{M} \bullet h\big\|_{\kappa_1,\kappa_2} \le \text{const} \max_{\substack{\alpha_{\mathrm{r}},\alpha_1 \in \mathbb{N}_0\times\mathbb{N}_0^2 \\ |\alpha_{\mathrm{r}}|+|\alpha_1| \le 3}} \big|g\big|_{\ell,i}^{[0,0,\alpha_{\mathrm{r}}]}\big|h\big|_{i,r}^{[\alpha_1,0,0]}.$$

Proof. Write

$$\mathcal{M}_\rho(p,k) = \sum_{s_1,s_2 \in \Sigma_i} \mathcal{M}_{s_1,s_2}(p,k),$$

where, for $s_1, s_2 \in \Sigma_i$,

$$\mathcal{M}_{s_1,s_2}(p,k) = \mathcal{M}(p,k)\rho(\mathbf{p}-\mathbf{k})\chi_{s_1}(\mathbf{p})\chi_{s_2}(\mathbf{k}).$$

By (III.19),

$$\begin{aligned} \big\|g \bullet \mathcal{M}_\rho \bullet h\big\|_{\kappa_1,\kappa_2} &\le \sum_{\substack{s_1,s_2 \in \Sigma_i \\ (s_1-s_2)\cap d \ne \emptyset}} \big\|g \bullet \mathcal{M}_{s_1,s_2} \bullet h\big\|_{\kappa_1,\kappa_2} \\ &\le \frac{\text{const}}{\mathfrak{l}_i} \max_{s_1,s_2 \in \Sigma_i} \big\|g \bullet \mathcal{M}_{s_1,s_2} \bullet h\big\|_{\kappa_1,\kappa_2}. \end{aligned} \tag{III.25}$$

Define, for $v_1, v_2 \in \Sigma_i$,

$$h_{v_1,v_2}(x_1, x_2, y_3, y_4) = h\big((x_1, v_1), (x_2, v_2), y_3, y_4\big) \quad \text{where } y_\nu = \begin{cases} \kappa_{\nu-2} & \text{if } \kappa_{\nu-2} \in \mathbb{M} \\ (x_\nu, \kappa_{\nu-2}) & \text{if } \kappa_{\nu-2} \in \Sigma_r \end{cases}.$$

Observe that h_{v_1,v_2} is a function on $\big(\mathbb{R}\times\mathbb{R}^2\big)^{2+n_r}$ where $n_r = \#\big\{\ \nu \in \{1,2\}\ \big|\ \kappa_\nu \in \Sigma_r\ \big\}$. Also define, for $u_3, u_4 \in \Sigma_i$ and $\lambda_1, \lambda_2 \in \mathbb{M} \cup \Sigma_\ell$,

$$g_{\lambda_1,\lambda_2;u_3,u_4}(y_1, y_2, x_3, x_4) = g\big(y_1, y_2, (x_3, u_3), (x_4, u_4)\big) \quad \text{where } y_\nu = \begin{cases} \lambda_\nu & \text{if } \lambda_\nu \in \mathbb{M} \\ (x_\nu, \lambda_\nu) & \text{if } \lambda_\nu \in \Sigma_r \end{cases}.$$

Observe that $g_{\lambda_1,\lambda_2;u_3,u_4}$ is a function on $\big(\mathbb{R}\times\mathbb{R}^2\big)^{2+n_l}$ where $n_l = \#\big\{\ \nu \in \{1,2\}\ \big|\ \lambda_\nu \in \Sigma_\ell\ \big\}$. Then, for all $s_1, s_2 \in \Sigma_i$,

$$\big\|g \bullet \mathcal{M}_{s_1,s_2} \bullet h\big\|_{\kappa_1,\kappa_2} \le \sup_{\lambda_1,\lambda_2} \sum_{\substack{u_3,u_4\in\Sigma_i \\ v_1,v_2\in\Sigma_i}} \big|\big|\big| g_{\lambda_1,\lambda_2;u_3,u_4} \circ \mathcal{M}_{s_1,s_2} \circ h_{v_1,v_2} \big|\big|\big|_{1,\infty}. \tag{III.26}$$

By conservation of momentum, the convolution $g_{\lambda_1,\lambda_2;u_3,u_4} \circ \mathcal{M}_{s_1,s_2} \circ h_{v_1,v_2}$ vanishes identically unless

$$u_3 \cap s_1 \ne \emptyset, \quad s_1 \cap v_1 \ne \emptyset,$$
$$u_4 \cap s_2 \ne \emptyset, \quad s_2 \cap v_2 \ne \emptyset.$$

There only 3^4 quadruples (u_3, u_4, v_1, v_2) satisfying these conditions, so that, by (III.24), (III.25) and (III.26),

$$\big\|g \bullet \mathcal{M} \bullet h\big\|_{\kappa_1,\kappa_2} \le \text{const}\, \tfrac{1}{\mathfrak{l}_i} \sup_{\lambda_1,\lambda_2} \max_{\substack{s_1,s_2\in\Sigma_i \\ u_3,u_4\in\Sigma_i \\ v_1,v_2\in\Sigma_i}} \big|\big|\big| g_{\lambda_1,\lambda_2;u_3,u_4} \circ \mathcal{M}_{s_1,s_2} \circ h_{v_1,v_2} \big|\big|\big|_{1,\infty}. \tag{III.27}$$

Fix $\lambda_1, \lambda_2, s_1, s_2, u_3, u_4, v_1, v_2$ and denote $g' = g_{\lambda_1,\lambda_2;u_3,u_4}$ and $h' = h_{v_1,v_2}$. Write the convolution

$$\begin{aligned}
& g' \circ \mathcal{M}_{s_1,s_2} \circ h' \\
&= \int d^3z_1 d^3z_2 d^3y_1 d^3y_2 \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} d\omega\; e^{\imath<p,z_1-y_1>_-} e^{\imath<k,y_2-z_2>_-}\; g'(\,\cdot\,,\,\cdot\,, z_1, z_2) \\
&\quad \times \delta(k_0) \frac{\nu_0(\omega)\nu_1(\mathbf{p},\mathbf{k})\chi_{s_1}(\mathbf{p})\chi_{s_2}(\mathbf{k})}{[i\omega - e'(\omega,\mathbf{p})][i\omega - e'(\omega,\mathbf{k})]} \rho(\mathbf{p}-\mathbf{k}) h'(y_1, y_2, \,\cdot\,,\,\cdot\,) \\
&= \int d^3z_1 d^3z_2 d^2\mathbf{y}_1 d^3y_2 \frac{d^2\mathbf{p}}{(2\pi)^2} \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{d\omega}{2\pi}\; e^{\imath\mathbf{p}\cdot(\mathbf{z}_1-\mathbf{y}_1)} e^{\imath\mathbf{k}\cdot(\mathbf{y}_2-\mathbf{z}_2)}\; g'(\,\cdot\,,\,\cdot\,, z_1, z_2) \\
&\quad \times \frac{\nu_0(\omega)\nu_1(\mathbf{p},\mathbf{k})\chi_{s_1}(\mathbf{p})\chi_{s_2}(\mathbf{k})}{[i\omega - e'(\omega,\mathbf{p})][i\omega - e'(\omega,\mathbf{k})]} \rho(\mathbf{p}-\mathbf{k}) h'\big((z_{1,0}, \mathbf{y}_1), y_2, \,\cdot\,,\,\cdot\,\big) \\
&= \int d^3z_1 d^3z_2 d^2\mathbf{y}_1 d^3y_2 \frac{d^2\mathbf{t}}{(2\pi)^2} \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{d\omega}{2\pi}\; e^{\imath\mathbf{t}\cdot(\mathbf{z}_1-\mathbf{y}_1)} e^{\imath\mathbf{k}\cdot(\mathbf{z}_1-\mathbf{z}_2+\mathbf{y}_2-\mathbf{y}_1)}\; g'(\,\cdot\,,\,\cdot\,, z_1, z_2) \\
&\quad \times \frac{\nu_0(\omega)\nu_1(\mathbf{k}+\mathbf{t},\mathbf{k})\chi_{s_1}(\mathbf{k}+\mathbf{t})\chi_{s_2}(\mathbf{k})}{[i\omega - e'(\omega,\mathbf{k}+\mathbf{t})][i\omega - e'(\omega,\mathbf{k})]} \rho(\mathbf{t}) h'\big((z_{1,0}, \mathbf{y}_1), y_2, \,\cdot\,,\,\cdot\,\big) \\
&= \int d^3z_1 d^3z_2 d^2\mathbf{y}_1 d^3y_2\; g'(\,\cdot\,,\,\cdot\,, z_1, z_2)\; \widehat{B}_{s_1,s_2}(\mathbf{z}_1, \mathbf{z}_2, \mathbf{y}_1, \mathbf{y}_2)\; h'\big((z_{1,0}, \mathbf{y}_1), y_2, \,\cdot\,,\,\cdot\,\big),
\end{aligned}$$

where

$$\widehat{B}_{s_1,s_2}(\mathbf{z}_1,\mathbf{z}_2,\mathbf{y}_1,\mathbf{y}_2)=\int \tfrac{d^2\mathbf{t}}{(2\pi)^2}\, e^{\imath \mathbf{t}\cdot(\mathbf{z}_1-\mathbf{y}_1)} B_{s_1,s_2}(\mathbf{t},\mathbf{z}_1-\mathbf{z}_2+\mathbf{y}_2-\mathbf{y}_1)\,\rho(\mathbf{t})$$

with

$$B_{s_1,s_2}(\mathbf{t},\mathbf{w})=\int \frac{d^2\mathbf{k}}{(2\pi)^2}\frac{d\omega}{2\pi}\, e^{\imath \mathbf{k}\cdot\mathbf{w}}\frac{\nu_0(\omega)\nu_1(\mathbf{k}+\mathbf{t},\mathbf{k})\chi_{s_1}(\mathbf{k}+\mathbf{t})\chi_{s_2}(\mathbf{k})}{[i\omega-e'(\omega,\mathbf{k}+\mathbf{t})][i\omega-e'(\omega,\mathbf{k})]}.$$

Recall that $M^i \le \mathfrak{l}_j M^j$ and that $p^{(i)}=0$ for $i>j+1$. By Theorem B.2, with $u(\mathbf{k},\mathbf{t})=e^{\imath\mathbf{k}\cdot\mathbf{w}}\nu_1(\mathbf{k}+\mathbf{t},\mathbf{k})\chi_{s_1}(\mathbf{k}+\mathbf{t})\chi_{s_2}(\mathbf{k})$, there is an $a>1$ such that

$$\big|\widehat{B}_{s_1,s_2}(\mathbf{z}_1,\mathbf{z}_2,\mathbf{y}_1,\mathbf{y}_2)\big| \le \text{const}\, \tfrac{\mathfrak{l}_i\mathfrak{l}_j}{M^j}\tfrac{1+|\mathbf{z}_2-\mathbf{z}_1+\mathbf{y}_1-\mathbf{y}_2|^3/M^{3i}}{[1+(\mathbf{n}\cdot(\mathbf{z}_1-\mathbf{y}_1)/M^j)^{3/2}][1+|\mathfrak{l}_j\,\mathbf{t}\cdot(\mathbf{z}_1-\mathbf{y}_1)|^a]}.$$

Since $\sup_{\mathbf{z}_1}\int d^2\mathbf{y}_1\ \tfrac{\mathfrak{l}_j}{M^j}\tfrac{1}{[1+(\mathbf{n}\cdot(\mathbf{z}_1-\mathbf{y}_1)/M^j)^{3/2}][1+|\mathfrak{l}_j\,\mathbf{t}\cdot(\mathbf{z}_1-\mathbf{y}_1)|^a]}\le \text{const}$, we have

$$\big|\!\big|\!\big| g'\circ \mathcal{M}_{s_1,s_2}\circ h' \big|\!\big|\!\big|_{1,\infty}\le \text{const}\,\mathfrak{l}_i \max_{\substack{\alpha_{\rm r},\alpha_{\rm l}\in\mathbb{N}_0\times\mathbb{N}_0^2\\ |\alpha_{\rm r}|+|\alpha_{\rm l}|\le 3}} \big|g\big|_{\ell,i}^{[0,0,\alpha_{\rm r}]}\big|h\big|_{i,r}^{[\alpha_{\rm l},0,0]}, \tag{III.28}$$

and the proposition follows by (III.27). □

Proof of Theorem III.15. By Props. III.19, III.22, III.24 and III.27,

$$\begin{aligned}\big\|g\bullet\mathcal{C}^{[i,j]}\bullet h\big\|_{\kappa_1,\kappa_2} &\le \big\|g\bullet\big(\mathcal{C}^{[i,j]}-\mathcal{Z}\bullet\mathcal{C}^{[i,j]}\bullet\mathcal{Z}^t\big)\bullet h\big\|_{\kappa_1,\kappa_2}\\ &\quad+\big\|g\bullet\big(\mathcal{Z}\bullet\mathcal{C}^{[i,j]}\bullet\mathcal{Z}^t-\widetilde{\mathcal{C}}^{[i,j]}\big)\bullet h\big\|_{\kappa_1,\kappa_2}\\ &\quad+\big\|g\bullet\big(\widetilde{\mathcal{C}}^{[i,j]}-\mathcal{M}\big)\bullet h\big\|_{\kappa_1,\kappa_2}+\big\|g\bullet\mathcal{M}\bullet h\big\|_{\kappa_1,\kappa_2}\\ &\le \text{const}\max_{\substack{\alpha_{\rm r},\alpha_{\rm l}\in\mathbb{N}_0\times\mathbb{N}_0^2\\ |\alpha_{\rm r}|+|\alpha_{\rm l}|\le 3}}\big|g\big|_{\ell,i}^{[0,0,\alpha_{\rm r}]}\big|h\big|_{i,r}^{[\alpha_{\rm l},0,0]},\end{aligned}$$

as desired. □

1. The Infrared Limit – Nonzero Transfer Momentum. The rest of this section is devoted to the infrared limit of compound particle hole ladders. The results contained here are used in the proof of Theorem I.22.

Lemma III.28. *Let $\kappa_1,\kappa_2\in\mathbb{M}$ and g and h be sectorized, translation invariant functions on $\mathfrak{Y}_{\Sigma_j}^4$ and $\mathfrak{Y}_{\Sigma_j}^2\times\mathbb{M}^2$, respectively. Then*

$$\|g\bullet\mathcal{C}^{(j)}\bullet h\|_{\kappa_1,\kappa_2}\le \text{const}\,|g|_{j,j}\,\|h\|_{\kappa_1,\kappa_2}\Big\{\sqrt{\mathfrak{l}_j}+\min\big\{1,\tfrac{1}{|\kappa_1-\kappa_2|M^j}\big\}\Big\}.$$

Furthermore, for $(\mathrm{loc},\mu,\mu')\in\big\{(\mathrm{top},1,3),(\mathrm{bot},2,4),(\mathrm{mid},1,3),(\mathrm{mid},2,4)\big\}$ *and* $\beta\in\Delta$,

$$\|g\bullet \mathrm{D}_{\mu,\mu'}^{\beta}\mathcal{C}_{\rm loc}^{[j,j]}\bullet h\|_{\kappa_1,\kappa_2}\le \text{const}\,|g|_{j,j}\,\|h\|_{\kappa_1,\kappa_2}\,M^{|\beta|j}\Big\{\sqrt{\mathfrak{l}_j}+\min\big\{1,\tfrac{1}{|\kappa_1-\kappa_2|M^j}\big\}\Big\}.$$

Proof. Write $\mathcal{C}^{(j)} = \sum_{s_1,s_2\in\Sigma_j} \big[\mathcal{C}^{(j)}_{\mathrm{top},j,s_1,s_2} + \mathcal{C}^{(j)}_{\mathrm{mid},j,s_1,s_2} + \mathcal{C}^{(j)}_{\mathrm{bot},j,s_1,s_2}\big]$. As in Lemma III.11,

$$\|g \bullet \mathrm{D}^{\beta}_{\mu,\mu'}\mathcal{C}^{[j,j]}_{\mathrm{loc}} \bullet h\|_{\kappa_1,\kappa_2} \le \mathrm{const}\, |g|_{j,j}\, \|h\|_{\kappa_1,\kappa_2} \sum_{\substack{s_1,s_2\in\Sigma_j \\ \kappa_1-\kappa_2\in s_1-s_2}} \big\|\mathrm{D}^{\beta}_{\mu,\mu'}\mathcal{C}^{(j)}_{\mathrm{loc},j,s_1,s_2}\big\|_{\mathrm{bubble}}.$$

By Lemma III.14, for $(\mathrm{loc},\mu,\mu') \in \big\{(\mathrm{top},1,3), (\mathrm{bot},2,4), (\mathrm{mid},1,3), (\mathrm{mid},2,4)\big\}$ and $\beta \in \Delta$,

$$\big\|\mathrm{D}^{\beta}_{\mu,\mu'}\mathcal{C}^{(j)}_{\mathrm{loc},j,s_1,s_2}\big\|_{\mathrm{bubble}} \le \mathrm{const}\, \mathfrak{l}_j M^{|\beta| j},$$

and by Lemma C.2,

$$\#\big\{ (s_1,s_2)\in\Sigma_j \ \big|\ \kappa_1-\kappa_2\in s_1-s_2 \big\} \le \mathrm{const}\, \Big\{ \tfrac{1}{\sqrt{\mathfrak{l}_j}} + \tfrac{1}{\mathfrak{l}_j} \min\big\{1, \tfrac{1}{|\kappa_1-\kappa_2| M^j}\big\}\Big\},$$

so that the desired bounds follow. □

Recall, from just before Lemma II.28, that

$$\begin{aligned}
&\mathfrak{L}^{(j)}_{\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}(q,q',t,\sigma_1,\cdots\sigma_4) \\
&\quad = \Big[\big(F^{(\mathrm{i}_1)} + L^{(\mathrm{i}_1)} f\big) \bullet \mathcal{C}^{[\max\{\mathrm{i}_1,\mathrm{i}_2\},j]} \bullet \big(F^{(\mathrm{i}_2)} + L^{(\mathrm{i}_2)} f\big) \bullet \mathcal{C}^{[\max\{\mathrm{i}_\ell,\mathrm{i}_{\ell+1}\},j]} \bullet \cdots \\
&\qquad \cdots \bullet \big(F^{(\mathrm{i}_{\ell+1})} + L^{(\mathrm{i}_{\ell+1})} f\big)\Big]_{i_1,i_2,i_3,i_4=0} \big((q+\tfrac{t}{2},\sigma_1),(q-\tfrac{t}{2},\sigma_2),(q'+\tfrac{t}{2},\sigma_3),(q'-\tfrac{t}{2},\sigma_4)\big).
\end{aligned}$$

Lemma III.29. *For $t \ne 0$, the limit*

$$\mathfrak{L}_{\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}(q,q',t,\sigma_1,\cdots\sigma_4) = \lim_{j\to\infty} \mathfrak{L}^{(j)}_{\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}(q,q',t,\sigma_1,\cdots\sigma_4)$$

exists. The limit is continuous in (q,q',t) for $t \ne 0$.

Proof. It suffices to consider separately the spin and charge parts, in the sense of Lemma II.8, of $\mathfrak{L}^{(j)}_{\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}(q,q',t,\sigma_1,\cdots\sigma_4)$. We denote them $\mathfrak{L}^{(j)}_{X,\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}(q,q',t)$ with $X = S, C$. Define $L_{X,j_1,\cdots,j_\ell}(q,q',t)$ to be the X part of

$$\begin{aligned}
&\Big[\big(F^{(\mathrm{i}_1)} + L^{(\mathrm{i}_1)} f\big) \bullet \mathcal{C}^{(j_1)} \bullet \big(F^{(\mathrm{i}_2)} + L^{(\mathrm{i}_2)} f\big) \bullet \cdots \\
&\quad \cdots \bullet \mathcal{C}^{(j_\ell)} \bullet \big(F^{(\mathrm{i}_{\ell+1})} + L^{(\mathrm{i}_{\ell+1})} f\big)\Big]_{i_1,i_2,i_3,i_4=0} \big((q+\tfrac{t}{2},\sigma_1),(q-\tfrac{t}{2},\sigma_2),(q'+\tfrac{t}{2},\sigma_3),(q'-\tfrac{t}{2},\sigma_4)\big)
\end{aligned}$$

and

$$\Delta\mathfrak{L}^{(p)}_X(q,q',t) = \sum_{\substack{\max\{\mathrm{i}_1,\mathrm{i}_2\}\le j_1\le j \\ \vdots \\ \max\{\mathrm{i}_\ell,\mathrm{i}_{\ell+1}\}\le j_\ell\le j \\ \max\{j_1,\cdots,j_\ell\}=p}} L_{X,j_1,\cdots,j_\ell}(q,q',t).$$

Then

$$\mathfrak{L}^{(j)}_{X,\ell,\mathrm{i}_1,\cdots,\mathrm{i}_{\ell+1}}(q,q',t,\sigma_1,\cdots\sigma_4) = \sum_{p=2}^{j} \Delta\mathfrak{L}^{(p)}_X(q,q',t,\sigma_1,\cdots\sigma_4).$$

By repeated use of Lemma III.28, (II.8) and its $X = C$ analog,

$$\|L_{X,j_1,\cdots,j_\ell}\|_{\kappa_1,\kappa_2} \le \text{const} \prod_{m=1}^{\ell} \Big\{\sqrt{\mathfrak{l}_{j_m}} + \min\big\{1, \tfrac{1}{|\kappa_1-\kappa_2|M^{j_m}}\big\}\Big\},$$

with the constant depending on ℓ and $i_1, \cdots, i_{\ell+1}$. Summing this bound over $j_1, \cdots, j_\ell$ with $\max\{j_1, \cdots, j_\ell\} = p$,

$$\|\Delta\mathfrak{L}_X^{(p)}\|_{\kappa_1,\kappa_2} \le \text{const} \Big\{\sqrt{\mathfrak{l}_p} + \min\big\{1, \tfrac{1}{|\kappa_1-\kappa_2|M^p}\big\}\Big\}.$$

The constant now depends on $|\kappa_1 - \kappa_2|$ as well but remains finite as long as $\kappa_1 \ne \kappa_2$. The existence of the limit $j \to \infty$ is now a consequence of the Lebesgue dominated convergence theorem. Similarly, if D is any first order differential operator in q, q' or t,

$$\|\mathrm{D}L_{X,j_1,\cdots,j_\ell}\|_{\kappa_1,\kappa_2} \le \text{const}\, M^{\max\{j_1,\cdots,j_\ell\}} \prod_{m=1}^{\ell} \Big\{\sqrt{\mathfrak{l}_{j_m}} + \min\big\{1, \tfrac{1}{|\kappa_1-\kappa_2|M^{j_m}}\big\}\Big\},$$

$$\|\mathrm{D}\Delta\mathfrak{L}_X^{(p)}\|_{\kappa_1,\kappa_2} \le \text{const}\, M^p\Big\{\sqrt{\mathfrak{l}_p} + \min\big\{1, \tfrac{1}{|\kappa_1-\kappa_2|M^p}\big\}\Big\}.$$

Continuity now follows from

$$\tfrac{|g(x)-g(y)|}{|x-y|^\epsilon} = |g(x) - g(y)|^{1-\epsilon} \big[\tfrac{|g(x)-g(y)|}{|x-y|}\big]^\epsilon \le \big[2 \sup |g(x)|\big]^{1-\epsilon} \big[\sup |\nabla g(x)|\big]^\epsilon,$$

and the observation that

$$\sum_{p=1}^{\infty} M^{\epsilon p}\Big\{\sqrt{\mathfrak{l}_p} + \min\big\{1, \tfrac{1}{|\kappa_1-\kappa_2|M^p}\big\}\Big\} < \infty \qquad \text{if } \epsilon < \tfrac{\aleph}{2} \text{ and } \kappa_1 \ne \kappa_2. \quad \square$$

2. The Infrared Limit – Reduction to Factorized Cutoffs. Recall from (II.13) that

$$\mathcal{C}^{[i,j]}(p,k) = \tfrac{\nu^{(\ge i)}(p)\nu^{(\ge i)}(k)-\nu^{(\ge j+1)}(p)\nu^{(\ge j+1)}(k)}{[ip_0-e'(p)][ik_0-e'(k)]},$$
$$\mathcal{A}_{i,j}(p,k) = \tfrac{\nu^{(\ge i)}(p)\nu^{(\ge i)}(k)[1-\nu_j(e(\mathbf{p}))\nu_j(e(\mathbf{k}))]}{[ip_0-e'(p)][ik_0-e'(k)]},$$
$$\mathcal{B}_{i,j}(p,k) = \tfrac{\nu^{(\ge i)}(p)\nu^{(\ge i)}(k)[1-\nu_j(p_0)\nu_j(k_0)]}{[ip_0-e'(p)][ik_0-e'(k)]},$$

where $e'(k) = e(\mathbf{k}) - v(k)$ and

$$\nu_j(\omega) = \sum_{m=j}^{\infty} \nu\big(M^{2m}\omega^2\big),$$

with ν being the single scale cutoff introduced in Def. I.2.

Proposition III.30. *Let $1 \le i \le j$ and $\ell, r \ge 1$. Let $\kappa_1, \kappa_2 \in \mathbb{M}$, and g and h be sectorized, translation invariant functions on $\mathfrak{Y}_{\ell,i}$ and $\mathfrak{Y}_{i,r}$ respectively. If $|\kappa_1-\kappa_2| > 0$, then*

$$\big\|g \bullet \big(\mathcal{C}^{[i,j]} - \mathcal{A}_{i,j}\big) \bullet h\big\|_{\kappa_1,\kappa_2}, \big\|g \bullet \big(\mathcal{C}^{[i,j]} - \mathcal{B}_{i,j}\big) \bullet h\big\|_{\kappa_1,\kappa_2}$$
$$\le \text{const}\, |g|_{\ell,i} \|h\|_{\kappa_1,\kappa_2}\, j \,\big\{\sqrt{\mathfrak{l}_j} + \tfrac{1}{|\kappa_1-\kappa_2|M^j\mathfrak{l}_j}\big\}.$$

Proof. Let

$$\mathcal{D}(p,k)=\tfrac{\nu^{(\ge j+1)}(p)\nu^{(\ge j+1)}(k)}{[ip_0-e'(p)][ik_0-e'(k)]}\ \text{or}\ \tfrac{\nu^{(\ge i)}(p)\nu^{(\ge i)}(k)\nu_j(e(\mathbf{p}))\nu_j(e(\mathbf{k}))}{[ip_0-e'(p)][ik_0-e'(k)]}\ \text{or}\ \tfrac{\nu^{(\ge i)}(p)\nu^{(\ge i)}(k)\nu_j(p_0)\nu_j(k_0)}{[ip_0-e'(p)][ik_0-e'(k)]}.$$

It suffices to prove that

$$\left\| g\bullet\mathcal{D}\bullet h\right\|_{\kappa_1,\kappa_2}\le \text{const}\,|g|_{\ell,i}\|h\|_{\kappa_1,\kappa_2}\, j\left\{\sqrt{\mathfrak{l}_j}+\tfrac{1}{|\kappa_1-\kappa_2|M^j\mathfrak{l}_j}\right\}.$$

Define, for $m\ge i$, $m_1,m_2\ge m$ and $s_1,s_2\in\Sigma_m$,

$$\mathcal{D}^{m_1,m_2}_{s_1,s_2}(p,k)=\tfrac{\nu^{(m_1)}(p)\nu^{(m_2)}(k)\chi_{s_1}(p)\chi_{s_2}(k)}{[ip_0-e'(p)][ik_0-e'(k)]}\begin{cases}\nu^{(\ge j+1)}(p)\nu^{(\ge j+1)}(k) & \text{or}\\ \nu_j(e(\mathbf{p}))\nu_j(e(\mathbf{k})) & \text{or}\\ \nu_j(p_0)\nu_j(k_0)\end{cases}$$

and

$$\mathcal{D}^{(m)}_{s_1,s_2}(p,k)=\sum_{\substack{m_1,m_2\ge 1\\ \min\{m_1,m_2\}=m}}\mathcal{D}^{m_1,m_2}_{s_1,s_2}(p,k).$$

We have

$$\mathcal{D}=\sum_{m=i}^{\infty}\sum_{s_1,s_2\in\Sigma_m}\mathcal{D}^{(m)}_{s_1,s_2}.$$

Hence, as in Lemma III.11,

$$\left\| g\bullet\mathcal{D}\bullet h\right\|_{\kappa_1,\kappa_2}\le \text{const}\,|g|_{\ell,i}\|h\|_{\kappa_1,\kappa_2}\sum_{m=i}^{\infty}\sum_{\substack{s_1,s_2\in\Sigma_m\\ \kappa_1-\kappa_2\in s_1-s_2}}\max_{s_1,s_2\in\Sigma_m}\left\|\mathcal{D}^{(m)}_{s_1,s_2}\right\|_{\text{bubble}}.$$

To bound $\left\|\mathcal{D}^{(m)}_{s_1,s_2}\right\|_{\text{bubble}}$, set, for $s\in\Sigma_m$ and $n\ge m$,

$$c_s^{(n,j)}(k)=\tfrac{\nu^{(n)}(k)\chi_s(k)}{ik_0-e'(k)}\begin{cases}\nu^{(\ge j+1)}(k) & \text{or}\\ \nu_j(e(\mathbf{k})) & \text{or}\\ \nu_j(k_0)\end{cases}$$

and denote by $c_s^{(n,j)}(x)$ its Fourier transform. As in Lemma A.2,

$$\begin{aligned}\left\|c_s^{(m,j)}(x)\right\|_{L^1}&\le \text{const}\,M^m\tfrac{\mathfrak{l}_m}{\mathfrak{l}_{\max\{m,j\}}},\\ \left\|c_s^{(n,j)}(x)\right\|_{L^\infty}&\le \text{const}\,\mathfrak{l}_m M^n\tfrac{1}{M^n}\tfrac{1}{M^{\max\{n,j\}}}\le \text{const}\,\tfrac{\mathfrak{l}_m}{M^{\max\{n,j\}}}.\end{aligned}$$

The factor $\frac{\mathfrak{l}_m}{M^{\max\{n,j\}}}$ in the first inequality arises, when $j>m$, from splitting s up into sectors of length $\mathfrak{l}_j$. Hence, by the triangle inequality and (III.7),

$$\begin{aligned}\left\|\mathcal{D}^{(m)}_{s_1,s_2}\right\|_{\text{bubble}}&\le\sum_{n\ge m}\left[\left\|c_{s_1}^{(m,j)}\right\|_{L^1}\left\|c_{s_2}^{(n,j)}\right\|_{L^\infty}+\left\|c_{s_1}^{(n,j)}\right\|_{L^\infty}\left\|c_{s_2}^{(m,j)}\right\|_{L^1}\right]\\ &\le\sum_{n>m}\text{const}\,M^m\tfrac{\mathfrak{l}_m}{\mathfrak{l}_{\max\{m,j\}}}\tfrac{\mathfrak{l}_m}{M^{\max\{n,j\}}}\end{aligned}$$

$$\le \text{const}\, \tfrac{M^m \mathfrak{l}_m^2}{\mathfrak{l}_{\max\{m,j\}}} \sum_{n>m} \tfrac{1}{M^{\max\{n,j\}}}$$
$$\le \text{const}\, \tfrac{M^m \mathfrak{l}_m^2}{\mathfrak{l}_{\max\{m,j\}}} \tfrac{j}{M^{\max\{m,j\}}}\,.$$

By Lemma C.2,

$$\#\big\{ (s_1,s_2)\in\Sigma_m \,\big|\, \kappa_1-\kappa_2\in s_1-s_2 \big\} \le \text{const}\left\{ \tfrac{1}{\sqrt{\mathfrak{l}_m}} + \tfrac{1}{\mathfrak{l}_m}\min\big\{1, \tfrac{1}{|\kappa_1-\kappa_2|M^m}\big\}\right\}$$

so that

$$\begin{aligned}\|g\bullet\mathcal{D}\bullet h\|_{\kappa_1,\kappa_2} &\le \text{const}\,|g|_{\ell,i}\|h\|_{\kappa_1,\kappa_2}\sum_{m=i}^{\infty}\left\{\tfrac{1}{\sqrt{\mathfrak{l}_m}}+\tfrac{1}{\mathfrak{l}_m}\min\big\{1,\tfrac{1}{|\kappa_1-\kappa_2|M^m}\big\}\right\}\\ &\quad\times\tfrac{M^m\mathfrak{l}_m^2}{\mathfrak{l}_{\max\{m,j\}}}\tfrac{j}{M^{\max\{m,j\}}}\\ &\le \text{const}\,|g|_{\ell,i}\|h\|_{\kappa_1,\kappa_2}\sum_{m=1}^{j}\left\{\tfrac{1}{\sqrt{\mathfrak{l}_m}}+\tfrac{1}{|\kappa_1-\kappa_2|M^m\mathfrak{l}_m}\right\}\tfrac{M^m\mathfrak{l}_m^2}{\mathfrak{l}_j}\tfrac{j}{M^j}\\ &\quad+\text{const}\,|g|_{\ell,i}\|h\|_{\kappa_1,\kappa_2}\; j\sum_{m=j}^{\infty}\left\{\tfrac{1}{\sqrt{\mathfrak{l}_m}}+\tfrac{1}{|\kappa_1-\kappa_2|M^m\mathfrak{l}_m}\right\}\mathfrak{l}_m\\ &\le \text{const}\,|g|_{\ell,i}\|h\|_{\kappa_1,\kappa_2}\; j\left\{\tfrac{1}{M^j\mathfrak{l}_j}M^j\mathfrak{l}_j^{3/2}+\tfrac{1}{M^j\mathfrak{l}_j}\tfrac{1}{|\kappa_1-\kappa_2|}\right.\\ &\quad\left.+\sqrt{\mathfrak{l}_j}+\tfrac{1}{|\kappa_1-\kappa_2|M^j}\right\}\\ &\le \text{const}\,|g|_{\ell,i}\|h\|_{\kappa_1,\kappa_2}\; j\left\{\sqrt{\mathfrak{l}_j}+\tfrac{1}{|\kappa_1-\kappa_2|M^j\mathfrak{l}_j}\right\}. \quad\square\end{aligned}$$

Corollary III.31. *Let $\ell\ge 1$, $i_1,\cdots,i_{\ell+1}\ge 2$, $j\ge\max\{i_1,\cdots,i_{\ell+1}\}$ and $|\kappa_1-\kappa_2|>0$. Define $I_0=i_1$, $I_{\ell+1}=i_{\ell+1}$ and, for $1\le m\le\ell$, $I_m=\max\{i_m,i_{m+1}\}$. For each $1\le m\le\ell+1$, let g_m be a sectorized, translation invariant function on $\mathfrak{Y}_{I_{m-1},I_m}$ with $|g_m|_{I_{m-1},I_m}<\infty$. Then*

$$\big\|g_1\bullet\mathcal{C}^{[I_1,j]}\bullet g_2\bullet\cdots\bullet\mathcal{C}^{[I_\ell,j]}\bullet g_{\ell+1}-g_1\bullet\mathcal{A}_{I_1,j}\bullet g_2\bullet\cdots\bullet\mathcal{A}_{I_\ell,j}\bullet g_{\ell+1}\big\|_{\kappa_1,\kappa_2}$$

and

$$\big\|g_1\bullet\mathcal{C}^{[I_1,j]}\bullet g_2\bullet\cdots\bullet\mathcal{C}^{[I_\ell,j]}\bullet g_{\ell+1}-g_1\bullet\mathcal{B}_{I_1,j}\bullet g_2\bullet\cdots\bullet\mathcal{B}_{I_\ell,j}\bullet g_{\ell+1}\big\|_{\kappa_1,\kappa_2}$$

both converge to zero as $j\to\infty$.

Proof. Use Prop. III.30 and the bounds

$$\big\|g\bullet\mathcal{A}_{i,j}\bullet h\big\|_{\kappa_1,\kappa_2},\ \big\|g\bullet\mathcal{B}_{i,j}\bullet h\big\|_{\kappa_1,\kappa_2},\ \big\|g\bullet\mathcal{C}^{i,j}\bullet h\big\|_{\kappa_1,\kappa_2}\le\text{const}\,|j-i+1|\,\big|g\big|_{\ell,i}\big\|h\big\|_{\kappa_1,\kappa_2}$$

which follow from Prop. III.30 and Theorem III.9, with $\beta=0$, to prove that

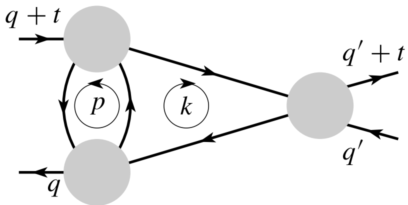
$$\begin{aligned}&\big\|g_1\bullet\mathcal{A}_{I_1,j}\bullet g_2\bullet\cdots\bullet\mathcal{A}_{I_{m-1},j}\bullet g_m\bullet\big[\mathcal{C}^{[I_m,j]}-\mathcal{A}_{I_m,j}\big]\bullet g_{m+1}\bullet\mathcal{C}^{[I_{m+1},j]}\bullet\cdots\bullet\mathcal{C}^{[I_\ell,j]}\\ &\qquad\bullet g_{\ell+1}\big\|_{\kappa_1,\kappa_2}\,,\\ &\big\|g_1\bullet\mathcal{B}_{I_1,j}\bullet g_2\bullet\cdots\bullet\mathcal{B}_{I_{m-1},j}\bullet g_m\bullet\big[\mathcal{C}^{[I_m,j]}-\mathcal{B}_{I_m,j}\big]\bullet g_{m+1}\bullet\mathcal{C}^{[I_{m+1},j]}\bullet\cdots\bullet\mathcal{C}^{[I_\ell,j]}\\ &\qquad\bullet g_{\ell+1}\big\|_{\kappa_1,\kappa_2}\\ &\quad\le\text{const}^\ell\,|j|^\ell\big\{\sqrt{\mathfrak{l}_j}+\tfrac{1}{|\kappa_1-\kappa_2|M^j\mathfrak{l}_j}\big\}\prod_{m=1}^{\ell+1}|g_m|_{I_{m-1},I_m}. \quad\square\end{aligned}$$

IV. Double Bubbles

In this section we prove the "double bubble bound", Theorem II.20. The techniques we use are essentially those of §III with one additional wrinkle – volume improvement due to overlapping loops.

To illustrate the effect of overlapping loops we consider one double bubble, namely

$D(q, q', t) \;=$



with the kernels of all vertices being identically one in momentum space and all lines having propagator $C^{(j)}$. By the Feynman rules

$$D(q, q', t) = \int dk\, dp\;\; C^{(j)}(p+q)\, C^{(j)}(k+p)\, C^{(j)}(k+t)\, C^{(j)}(k).$$

The naive power counting bound is, for each $q, q', t \in \mathbb{R} \times \mathbb{R}^2$,

$$\begin{aligned}
|D(q, q', t)| &\le \int dk\, dp\;\; |C^{(j)}(p+q)|\, |C^{(j)}(k+p)|\, |C^{(j)}(k+t)|\, |C^{(j)}(k)| \\
&\le \|C^{(j)}\|_\infty^2 \int dk\, dp\;\; |C^{(j)}(k)|\, |C^{(j)}(p+q)| \\
&= \|C^{(j)}\|_\infty^2\, \|C^{(j)}\|_1^2 \\
&\le \text{const}\,,
\end{aligned}$$

since, denoting the j^{th} shell by S_j (see Def. I.2),

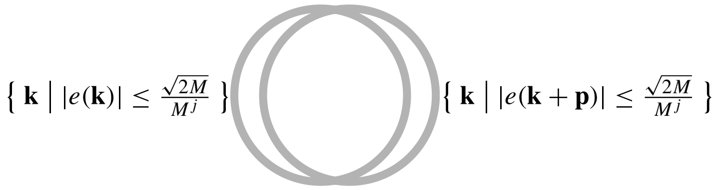
$$\begin{aligned}
\|C^{(j)}\|_\infty &= \sup_{k \in S_j} \tfrac{1}{|\imath k_o - e(\mathbf{k})|} \le \sqrt{M}\, M^j \\
\|C^{(j)}\|_1 &\le \|C^{(j)}\|_\infty \big(\text{volume of } S_j\big) \le \text{const}\, M^j\, \tfrac{1}{M^{2j}} = \tfrac{\text{const}}{M^j}.
\end{aligned}$$

In the naive bound, we ignored the constraint that $|e(\mathbf{k}+\mathbf{p})| \le \frac{\sqrt{2M}}{M^j}$. Taking it into account, one has the better estimate

$$\begin{aligned}
&|D(q, q', t)| \\
&\quad \le \int dk\, dp\;\; |C^{(j)}(p+q)|\, |C^{(j)}(k+p)|\, |C^{(j)}(k+t)|\, |C^{(j)}(k)| \\
&\quad \le \text{const}\, M^{4j} \int dk\, dp\; \nu^{(j)}(p+q)\; \nu^{(j)}(k+q)\; \nu^{(j)}(k) \\
&\quad \le \text{const}\, M^{4j}\; \text{vol}\Big\{(k, p) \in \big(\mathbb{R} \times \mathbb{R}^2\big)^2 \;\Big|\; |\imath(p_0+q_0) - e(\mathbf{p}+\mathbf{q})| \\
&\quad \le \tfrac{\sqrt{2M}}{M^j},\, |\imath(k_0+p_0) - e(\mathbf{k}+\mathbf{p})| \le \tfrac{\sqrt{2M}}{M^j},\, |\imath k_0 - e(\mathbf{k})| \le \tfrac{\sqrt{2M}}{M^j}\Big\} \\
&\quad \le \text{const}\, M^{4j}\, \tfrac{2M}{M^{2j}}\; \text{vol}\Big\{(\mathbf{k}, \mathbf{p}) \in \mathbb{R}^2 \times \mathbb{R}^2 \;\Big|\; |e(\mathbf{k})|, |e(\mathbf{k}+\mathbf{p})|, |e(\mathbf{p}+\mathbf{q})| \le \tfrac{\sqrt{2M}}{M^j}\Big\} \\
&\quad \le \text{const}\, M^{2j} \int_{|e(\mathbf{p}+\mathbf{q})| \le \frac{\sqrt{2M}}{M^j}} d\mathbf{p}\;\; \text{vol}\Big\{\mathbf{k} \in \mathbb{R}^2 \;\Big|\; |e(\mathbf{k})|, |e(\mathbf{k}+\mathbf{p})| \le \tfrac{\sqrt{2M}}{M^j}\Big\}.
\end{aligned}$$

There is $\varepsilon > 0$ such that for $\mathbf{p}$ outside a ball of radius $\frac{\text{const}}{M^{j(1-\varepsilon)}}$ around the origin

$$\mathrm{vol}\Big(\big\{\mathbf{k}\in\mathbb{R}^2\ \big|\ |e(\mathbf{k})|\le \tfrac{\sqrt{2M}}{M^j}\big\}\cap\big\{\mathbf{k}\in\mathbb{R}^2\ \big|\ |e(\mathbf{k}+\mathbf{p})|\le \tfrac{\sqrt{2M}}{M^j}\big\}\Big)\ \le\ \tfrac{const}{M^{(1+\varepsilon)j}}$$



because, roughly speaking, $\big\{\mathbf{k}\in\mathbb{R}^2\ \big|\ |e(\mathbf{k})|\le\frac{\sqrt{2M}}{M^j}\big\}$ and $\big\{\mathbf{k}\in\mathbb{R}^2\ \big|\ |e(\mathbf{k}+\mathbf{p})|\le\frac{\sqrt{2M}}{M^j}\big\}$ cross at an angle of about const $|\mathbf{p}|\ge\frac{\text{const}}{M^{j(1-\varepsilon)}}$. Therefore

$$\begin{aligned}\|D\|_\infty &\le \text{const}\, M^{2j}\big(\tfrac{1}{M^{2j(1-\varepsilon)}}\tfrac{\sqrt{2M}}{M^j}+\tfrac{\sqrt{2M}}{M^j}\tfrac{1}{M^{(1+\varepsilon)j}}\big)\\ &\le \text{const}\,\tfrac{1}{M^{\varepsilon j}},\end{aligned}$$

This "volume improvement" is realized in terms of sector counting in Lemma C.2. Sector counting and simple propagator estimates (Lemma III.14 and Lemma IV.2) are combined using Corollary IV.3 (an analog of Lemma III.11) to prove Theorem IV.4 (which is essentially a reformulation of Theorem II.20 parts b and c in terms of the $\|\cdot\|_{\kappa_1,\kappa_2}$ norm of Definition III.7) and to treat the large transfer momentum part of the reformulation, Theorem IV.5, of Theorem II.20a (Prop. IV.6). Theorem II.20 parts b and c are proven following Theorem IV.4. The treatment of the small transfer momentum part of Theorem IV.5 closely parallels the corresponding argument in §III. Theorem II.20a is proven following Theorem IV.5.

We first prove a general bound on $(g_1\bullet\mathcal{D}\bullet g_2)^f\bullet W\bullet h$ similar to Lemma III.11.

Lemma IV.1. *Let $1\le\ell\le i\le j$ and $r\ge 1$. Let $\kappa_1,\kappa_2\in\mathfrak{K}_r$ and g_1, g_2 and h be sectorized, translation invariant functions on $\mathfrak{Y}_{\ell,\ell}$, $\mathfrak{Y}_{\ell,\ell}$ and $\mathfrak{Y}_{i,r}$ respectively. Let W be a particle–hole bubble propagator whose total Fourier transform is of the form*

$$\check{W}(p_1,k_1,p_2,k_2)=\sum_{m=i}^{j}\ \sum_{s_1,s_2\in\Sigma_m}W^{(m)}_{s_1,s_2}(p_1,k_1,p_2,k_2)\qquad \textit{if}\ \ p_2-k_2\in\kappa_1-\kappa_2$$

with $W^{(m)}_{s_1,s_2}(p_1,k_1,p_2,k_2)$ vanishing unless $\boldsymbol{\pi}(p_1),\boldsymbol{\pi}(p_2)\in\boldsymbol{\pi}(s_1)$ and $\boldsymbol{\pi}(k_1),\boldsymbol{\pi}(k_2)\in\boldsymbol{\pi}(s_2)$. Here $\boldsymbol{\pi}: k=(k_0,\mathbf{k})\mapsto\mathbf{k}$ is the projection of $\mathbb{M}=\mathbb{R}\times\mathbb{R}^2$ onto its second factor. Let $\mathcal{V}$ be another particle–hole bubble propagator with

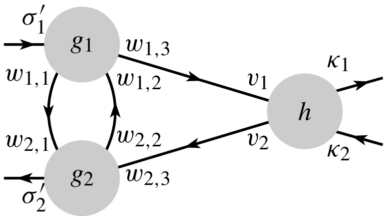
$$\check{\mathcal{V}}(p_1,k_1,p_2,k_2)=\sum_{u_1,u_2\in\Sigma_\ell}\mathcal{V}_{u_1,u_2}(p_1,k_1,p_2,k_2)$$

and $\mathcal{V}_{u_1,u_2}(p_1,k_1,p_2,k_2)=0$ unless $\boldsymbol{\pi}(p_1),\boldsymbol{\pi}(p_2)\in\boldsymbol{\pi}(u_1)$ and $\boldsymbol{\pi}(k_1),\boldsymbol{\pi}(k_2)\in\boldsymbol{\pi}(u_2)$. Then

$$\begin{aligned}&\big\|(g_1\bullet\mathcal{V}\bullet g_2)^f\bullet W\bullet h\big\|_{\kappa_1,\kappa_2}\\ &\quad\le 3^8\big|g_1\big|_{\ell,\ell}\big|g_2\big|_{\ell,\ell}\big|h\big|_{i,r}\max_{(u_1,u_2)\in\Sigma_\ell}\|\mathcal{V}_{u_1,u_2}\|_{\text{bubble}}\end{aligned}$$

$$\times \sup_{\kappa'\in\mathfrak{K}_\ell} \sum_{m=i}^{j} \sum_{\substack{s_1,s_2\in\Sigma_m \\ \boldsymbol{\pi}(s_1-s_2)\cap\boldsymbol{\pi}(\kappa_1-\kappa_2)\neq\emptyset}} \inf_{R\in\mathcal{R}(\kappa_1-\kappa_2)} \big\| W^{(m)}_{s_1,s_2,R}\big\|_{\rm bubble}$$
$$\times\#\big\{ (u_1,u_2)\in\Sigma_\ell\times\Sigma_\ell \,\big|\, \boldsymbol{\pi}(u_1-u_2)\cap\boldsymbol{\pi}(\kappa'-s_1)\neq\emptyset \big\}.$$

Proof. Consider the case in which all of the external arguments of $(g_1\bullet\mathcal{V}\bullet g_2)^f\bullet W\bullet h$ are (position, sector)'s. Set $\sigma_1'=\kappa'$ and fix an external sector $\sigma_2'\in\Sigma_\ell$. With the sector names



we have

$$\begin{aligned}&(g_1\bullet\mathcal{V}\bullet g_2)^f\bullet W\bullet h\\ &\quad=\sum_{\substack{u_1,u_2\in\Sigma_\ell\\ w_{1,1},w_{1,2}\in\Sigma_\ell\\ w_{2,1},w_{2,2}\in\Sigma_\ell}}\ \sum_{m=i}^{j}\ \sum_{\substack{w_{1,3},w_{2,3}\in\Sigma_\ell\\ s_1,s_2\in\Sigma_m\\ v_1,v_2\in\Sigma_i}}\\ &\quad\Big(g_1\big((\cdot,\sigma_1'),(\cdot,w_{1,3}),(\cdot,w_{1,1}),(\cdot,w_{1,2})\big)\circ\mathcal{V}_{u_1,u_2}\circ g_2\big((\cdot,w_{2,1}),(\cdot,w_{2,2}),(\cdot,\sigma_2'),(\cdot,w_{2,3})\big)\Big)^f\\ &\quad\times\circ W^{(m)}_{s_1,s_2}\circ h\big((\cdot,v_1),(\cdot,v_2),(\cdot,\kappa_1),(\cdot,\kappa_2)\big).\end{aligned}$$

For each choice of sectors, by conservation of momentum at the vertex h, we may replace the $W^{(m)}_{s_1,s_2}$ above by $W^{(m)}_{s_1,s_2,R}$ with any $R\in\mathcal{R}(\kappa_1-\kappa_2)$. Furthermore the multiple convolution vanishes unless

$$\boldsymbol{\pi}(s_1-s_2)\cap\boldsymbol{\pi}(\kappa_1-\kappa_2)\neq\emptyset\qquad \boldsymbol{\pi}(u_1-u_2)\cap\boldsymbol{\pi}(\sigma_1'-s_1)\neq\emptyset \tag{IV.1}$$

and

$$\begin{aligned}&\boldsymbol{\pi}(w_{1,1})\cap\boldsymbol{\pi}(u_1)\neq\emptyset\quad \boldsymbol{\pi}(w_{2,1})\cap\boldsymbol{\pi}(u_1)\neq\emptyset,\\ &\boldsymbol{\pi}(w_{1,2})\cap\boldsymbol{\pi}(u_2)\neq\emptyset\quad \boldsymbol{\pi}(w_{2,2})\cap\boldsymbol{\pi}(u_2)\neq\emptyset,\\ &\boldsymbol{\pi}(w_{1,3})\cap\boldsymbol{\pi}(s_1)\neq\emptyset\quad \boldsymbol{\pi}(v_1)\cap\boldsymbol{\pi}(s_1)\neq\emptyset,\\ &\boldsymbol{\pi}(w_{2,3})\cap\boldsymbol{\pi}(s_2)\neq\emptyset\quad \boldsymbol{\pi}(v_2)\cap\boldsymbol{\pi}(s_2)\neq\emptyset.\end{aligned} \tag{IV.2}$$

For each fixed (u_1,u_2,s_1,s_2), at most 3^8 8–tuples $(w_{1,1},w_{2,1},w_{1,2},w_{2,2},w_{1,3},v_1,w_{2,3},v_2)$ can satisfy (IV.2). Hence

$$\begin{aligned}&\big\|(g_1\bullet\mathcal{V}\bullet g_2)^f\bullet W\bullet h\big\|_{\kappa_1,\kappa_2}\\ &\le 3^8|h|_{i,r}\max_{\sigma_1'\in\Sigma_\ell}\sum_{m=i}^{j}\sum_{\substack{s_1,s_2\in\Sigma_m\\ \boldsymbol{\pi}(s_1-s_2)\cap\boldsymbol{\pi}(\kappa_1-\kappa_2)\neq\emptyset}}\inf_{R\in\mathcal{R}(\kappa_1-\kappa_2)}\big\|W^{(m)}_{s_1,s_2,R}\big\|_{\rm bubble}\\ &\times\sum_{\substack{u_1,u_2\in\Sigma_\ell\\ \boldsymbol{\pi}(u_1-u_2)\cap\boldsymbol{\pi}(\sigma_1'-s_1)\neq\emptyset}}\big|(g_1\bullet\mathcal{V}_{u_1,u_2}\bullet g_2)^f\big|_{\ell,\ell}.\end{aligned}$$

By definition of the $\|\cdot\|_{\rm bubble}$ norm,

$$\big|(g_1\bullet\mathcal{V}_{u_1,u_2}\bullet g_2)^f\big|_{\ell,\ell} \le \big|g_1\big|_{\ell,\ell}\big|g_2\big|_{\ell,\ell}\|\mathcal{V}_{u_1,u_2}\|_{\rm bubble}.$$

Inserting this gives the lemma. □

Recall, for the statement of Theorem II.20, that, for $\nu\in\mathbb{N}_0\times\mathbb{N}_0^2$,

$$\begin{aligned}\mathcal{D}^{(\ell)}_{\nu,\rm up}(x_1,x_2,x_3,x_4) &= \tfrac{1}{M^{|\nu|\ell}}\sum_{m=\ell}^{\infty}\mathrm{D}^{\nu}_{1;3}C^{(\ell)}_v(x_1,x_3)C^{(m)}_v(x_4,x_2),\\ \mathcal{D}^{(\ell)}_{\nu,\rm dn}(x_1,x_2,x_3,x_4) &= \tfrac{1}{M^{|\nu|\ell}}\sum_{m=\ell+1}^{\infty}C^{(m)}_v(x_1,x_3)\mathrm{D}^{\nu}_{2;4}C^{(\ell)}_v(x_4,x_2).\end{aligned}\tag{IV.3}$$

Express $\mathcal{D}^{(\ell)}_{\nu,\rm up}=\sum_{u_1,u_2\in\Sigma_\ell}\mathcal{D}^{(\ell)}_{\nu,{\rm up},u_1,u_2}$ with

$$\mathcal{D}^{(\ell)}_{\nu,{\rm up},u_1,u_2}(x_1,x_2,x_3,x_4) = \tfrac{1}{M^{|\nu|\ell}}\sum_{m\ge\ell}(x_1-x_3)^{\nu}c^{(\ell)}_{u_1}(x_3-x_1)c^{(m)}_{u_2}(x_2-x_4),$$

where $c^{(n)}_u(x)$ was defined in (III.9). Do the same for $\mathcal{D}^{(\ell)}_{\nu,\rm up}$.

Lemma IV.2. *Let $\ell\ge1$, $u_1,u_2\in\Sigma_\ell$ and $\nu\in\Delta$. Then*

$$\begin{aligned}\|\mathcal{D}^{(\ell)}_{\nu,{\rm up},u_1,u_2}\|_{\rm bubble} &\le \mathrm{const}\,\mathfrak{l}_\ell,\\ \|\mathcal{D}^{(\ell)}_{\nu,{\rm dn},u_1,u_2}\|_{\rm bubble} &\le \mathrm{const}\,\mathfrak{l}_\ell.\end{aligned}$$

Proof. By the triangle inequality, Lemma III.2 and Lemma A.2,

$$\begin{aligned}\|\mathcal{D}^{(\ell)}_{\nu,{\rm up},u_1,u_2}\|_{\rm bubble} &\le \tfrac{1}{M^{|\nu|\ell}}\sum_{m\ge\ell}\|x^\nu c^{(\ell)}_{u_1}\|_{L^1}\|c^{(m)}_{u_2}\|_{L^\infty}\le\sum_{m\ge\ell}\mathrm{const}\,\tfrac{1}{M^{|\nu|\ell}}M^{(1+|\nu|)\ell}\tfrac{\mathfrak{l}_\ell}{M^m}\\ &\le \mathrm{const}\,\mathfrak{l}_\ell.\end{aligned}$$

The bound on $\|\mathcal{D}^{(\ell)}_{\nu,{\rm dn},u_1,u_2}\|_{\rm bubble}$ is proven similarly. □

Corollary IV.3. *Let $1\le\ell\le i\le j$ and $r\ge1$. Let $\kappa_1,\kappa_2\in\mathfrak{K}_r$ and g_1, g_2 and h be sectorized, translation invariant functions on $\mathfrak{Y}_{\ell,\ell}$, $\mathfrak{Y}_{\ell,\ell}$ and $\mathfrak{Y}_{i,r}$ respectively. Let W be a particle–hole bubble propagator of the form*

$$W(p_1,k_1,p_2,k_2)=\sum_{m=i}^{j}\sum_{s_1,s_2\in\Sigma_m}W^{(m)}_{s_1,s_2}(p_1,k_1,p_2,k_2)\qquad \text{if } p_2-k_2\in\kappa_1-\kappa_2$$

with $W^{(m)}_{s_1,s_2}(p_1,k_1,p_2,k_2)$ vanishing unless $\boldsymbol{\pi}(p_1),\boldsymbol{\pi}(p_2)\in\boldsymbol{\pi}(s_1)$ and $\boldsymbol{\pi}(k_1),\boldsymbol{\pi}(k_2)\in\boldsymbol{\pi}(s_2)$. Let $\mathcal{D}$ be either $\mathcal{D}^{(\ell)}_{\nu,\rm up}$ or $\mathcal{D}^{(\ell)}_{\nu,\rm dn}$, with $\nu\in\Delta$. Then

$$\begin{aligned}&\big\|(g_1\bullet\mathcal{D}\bullet g_2)^f\bullet W\bullet h\big\|_{\kappa_1,\kappa_2}\\ &\quad\le \mathrm{const}\,\big|g_1\big|_{\ell,\ell}\big|g_2\big|_{\ell,\ell}\big|h\big|_{i,r}\,\mathfrak{l}_\ell\sup_{\kappa'\in\mathfrak{K}_\ell}\sum_{m=i}^{j}\sum_{\substack{s_1,s_2\in\Sigma_m\\ \boldsymbol{\pi}(s_1-s_2)\cap\boldsymbol{\pi}(\kappa_1-\kappa_2)\ne\emptyset}}\inf_{R\in\mathcal{R}(\kappa_1-\kappa_2)}\big\|W^{(m)}_{s_1,s_2,R}\big\|_{\rm bubble}\\ &\qquad\times\#\big\{\,(u_1,u_2)\in\Sigma_\ell\times\Sigma_\ell\;\big|\;\boldsymbol{\pi}(u_1-u_2)\cap\boldsymbol{\pi}(\kappa'-s_1)\ne\emptyset\,\big\}\end{aligned}$$

$$\le \text{const}\, |g_1|_{\ell,\ell} |g_2|_{\ell,\ell} |h|_{i,r}\, \mathfrak{l}_\ell \sum_{m=i}^{j} \sup_{s_1,s_2\in\Sigma_m} \|W^{(m)}_{s_1,s_2}\|_{\text{bubble}}$$
$$\times \sup_{\kappa'\in\mathfrak{K}_\ell} \#\left\{ (u_1,u_2,s_1,s_2)\in\Sigma_\ell^2\times\Sigma_m^2 \;\middle|\; \begin{array}{l} \boldsymbol{\pi}(u_1-u_2)\cap\boldsymbol{\pi}(\kappa'-s_1)\ne\emptyset \\ \boldsymbol{\pi}(s_1-s_2)\cap\boldsymbol{\pi}(\kappa_1-\kappa_2)\ne\emptyset \end{array}\right\}.$$

Proof. The first inequality follows directly from Lemmas IV.1 and IV.2. The second inequality follows by choosing an R which is one on a large ball. □

Theorem IV.4. *Let $1\le\ell\le i\le j\le r$, $\kappa_1,\kappa_2\in\mathfrak{K}_r$ and let g_1, g_2 and h be sectorized, translation invariant functions on $\mathfrak{Y}_{\ell,\ell}$, $\mathfrak{Y}_{\ell,\ell}$ and $\mathfrak{Y}_{i,r}$ respectively. Let $\nu\in\Delta$ and $\mathcal{D}$ be either $\mathcal{D}^{(\ell)}_{\nu,\text{up}}$ or $\mathcal{D}^{(\ell)}_{\nu,\text{dn}}$.*

i) For any $\beta\in\Delta$,

$$\tfrac{1}{M^{|\beta|j}} \big\|(g_1\bullet\mathcal{D}\bullet g_2)^f\bullet \mathrm{D}^{\beta}_{1;3}\mathcal{C}^{[i,j]}_{\text{top}}\bullet h\big\|_{\kappa_1,\kappa_2} \le \text{const}\sqrt{\mathfrak{l}_\ell}\, |g_1|_{\ell,\ell}|g_2|_{\ell,\ell}|h|_{i,r},$$
$$\tfrac{1}{M^{|\beta|j}} \big\|(g_1\bullet\mathcal{D}\bullet g_2)^f\bullet \mathrm{D}^{\beta}_{2;4}\mathcal{C}^{[i,j]}_{\text{bot}}\bullet h\big\|_{\kappa_1,\kappa_2} \le \text{const}\sqrt{\mathfrak{l}_\ell}\, |g_1|_{\ell,\ell}|g_2|_{\ell,\ell}|h|_{i,r}.$$

ii)

$$\big\|(g_1\bullet\mathcal{D}\bullet g_2)^f\bullet \mathcal{C}^{[i,j]}_{\text{mid}}\bullet h\big\|_{\kappa_1,\kappa_2} \le \text{const}\,|j-i+1|\,\sqrt{\mathfrak{l}_\ell}\, |g_1|_{\ell,\ell}|g_2|_{\ell,\ell}|h|_{i,r},$$

and for any $\beta\in\Delta$ with $|\beta|\ge 1$ and $(\mu,\mu')=(1,3),(2,4)$,

$$\tfrac{1}{M^{|\beta|j}} \big\|(g_1\bullet\mathcal{D}\bullet g_2)^f\bullet \mathrm{D}^{\beta}_{\mu;\mu'}\mathcal{C}^{[i,j]}_{\text{mid}}\bullet h\big\|_{\kappa_1,\kappa_2} \le \text{const}\sqrt{\mathfrak{l}_\ell}\, |g_1|_{\ell,\ell}|g_2|_{\ell,\ell}|h|_{i,r}.$$

Proof. i) We treat $\mathcal{C}^{[i,j]}_{\text{top}}$. The proof for $\mathcal{C}^{[i,j]}_{\text{bot}}$ is similar. By Corollary IV.3, followed by Lemma III.14 and Lemma C.3,

$$\tfrac{1}{M^{|\beta|j}} \big\|(g_1\bullet\mathcal{D}\bullet g_2)^f\bullet \mathrm{D}^{\beta}_{1;3}\mathcal{C}^{[i,j]}_{\text{top}}\bullet h\big\|_{\kappa_1,\kappa_2}$$
$$\le \text{const}\, |g_1|_{\ell,\ell}|g_2|_{\ell,\ell}|h|_{i,r}\,\mathfrak{l}_\ell \sum_{m=i}^{j}\sup_{s_1,s_2\in\Sigma_m} \tfrac{1}{M^{|\beta|j}}\big\|\mathrm{D}^{\beta}_{1;3}\mathcal{C}^{(m)}_{\text{top},j,s_1,s_2}\big\|_{\text{bubble}}$$
$$\times \sup_{\kappa'\in\mathfrak{K}_\ell} \#\left\{ (u_1,u_2,s_1,s_2)\in\Sigma_\ell^2\times\Sigma_m^2 \;\middle|\; \begin{array}{l} \boldsymbol{\pi}(u_1-u_2)\cap\boldsymbol{\pi}(\kappa'-s_1)\ne\emptyset \\ \boldsymbol{\pi}(s_1-s_2)\cap\boldsymbol{\pi}(\kappa_1-\kappa_2)\ne\emptyset \end{array}\right\}$$
$$\le \text{const}\, |g_1|_{\ell,\ell}|g_2|_{\ell,\ell}|h|_{i,r}\,\mathfrak{l}_\ell \sum_{m=i}^{j} \tfrac{1}{M^{|\beta|j}}\,\mathfrak{l}_m \tfrac{M^m}{M^j} M^{|\beta|m}\tfrac{1}{\mathfrak{l}_m\sqrt{\mathfrak{l}_\ell}}$$
$$\le \text{const}\, |g_1|_{\ell,\ell}|g_2|_{\ell,\ell}|h|_{i,r}\sqrt{\mathfrak{l}_\ell}.$$

ii) By Corollary IV.3, followed by Lemma III.14 and Lemma C.3,

$$\tfrac{1}{M^{|\beta|j}} \big\|(g_1\bullet\mathcal{D}\bullet g_2)^f\bullet \mathrm{D}^{\beta}_{\mu;\mu'}\mathcal{C}^{[i,j]}_{\text{mid}}\bullet h\big\|_{\kappa_1,\kappa_2}$$
$$\le \text{const}\, |g_1|_{\ell,\ell}|g_2|_{\ell,\ell}|h|_{i,r}\,\mathfrak{l}_\ell \sum_{m=i}^{j}\sup_{s_1,s_2\in\Sigma_m} \tfrac{1}{M^{|\beta|j}}\big\|\mathrm{D}^{\beta}_{\mu;\mu'}\mathcal{C}^{(m)}_{\text{mid},j,s_1,s_2}\big\|_{\text{bubble}}$$

$$\times \sup_{\kappa' \in \mathfrak{K}_\ell} \#\left\{ (u_1, u_2, s_1, s_2) \in \Sigma_\ell^2 \times \Sigma_m^2 \;\middle|\; \begin{array}{l} \boldsymbol{\pi}(u_1 - u_2) \cap \boldsymbol{\pi}(\kappa' - s_1) \neq \emptyset \\ \boldsymbol{\pi}(s_1 - s_2) \cap \boldsymbol{\pi}(\kappa_1 - \kappa_2) \neq \emptyset \end{array} \right\}$$

$$\le \mathrm{const}\, \big|g_1\big|_{\ell,\ell} \big|g_2\big|_{\ell,\ell} \big|h\big|_{i,r}\, \mathfrak{l}_\ell \sum_{m=i}^{j} \tfrac{1}{\mathfrak{l}_m \sqrt{\mathfrak{l}_\ell}} \begin{cases} \mathfrak{l}_m & \beta = 0 \\ \mathfrak{l}_m \frac{M^m}{M^j}(j - m + 1) & |\beta| = 1 \\ \mathfrak{l}_m \frac{M^m}{M^j} & |\beta| \ge 2 \end{cases}.$$

For $\beta = 0$, $\sum_{m=i}^{j} \frac{1}{\mathfrak{l}_m} \mathfrak{l}_m = |j - i + 1|$ as desired. For $\beta \neq 0$,

$$\sum_{m=i}^{j} \tfrac{1}{\mathfrak{l}_m} \mathfrak{l}_m \tfrac{M^m}{M^j}(j - m + 1) = \sum_{m=i}^{j} M^{-(j-m)}(j - m + 1) \le \mathrm{const}$$

again, as desired. □

Proof of Theorem II.20b,c. Replacing h by $\frac{1}{M^{j|\delta_r|}} \mathrm{D}_{3,4}^{\delta_r} h$ reduces consideration to $\delta_r = 0$. Suppose that $\mathcal{D} = \mathcal{D}_{\nu,\mathrm{up}}^{(\ell)}$. Observe that, by Leibniz (Lemma II.21)

$$\begin{aligned} \mathrm{D}_{1,2}^{\delta_l}(g_1 \bullet \mathcal{D} \bullet g_2)^f &= \big(\mathrm{D}_{1;3}^{\delta_l} g_1 \bullet \mathcal{D}_{\nu,\mathrm{up}}^{(\ell)} \bullet g_2\big)^f \\ &= \sum_{\substack{\beta_1, \beta_2 \beta_3 \in \mathbb{N}_0^3 \\ \beta_1 + \beta_2 + \beta_3 = \delta_l}} \binom{\delta_l}{\beta_1, \beta_2, \beta_3} \big(\mathrm{D}_{1;3}^{\beta_1} g_1 \bullet \mathrm{D}_{1;3}^{\beta_2} \mathcal{D}_{\nu,\mathrm{up}}^{(\ell)} \bullet \mathrm{D}_{1;3}^{\beta_3} g_2\big)^f \\ &= \sum_{\substack{\beta_1, \beta_2 \beta_3 \in \mathbb{N}_0^3 \\ \beta_1 + \beta_2 + \beta_3 = \delta_l}} \binom{\delta_l}{\beta_1, \beta_2, \beta_3} M^{|\beta_2 \ell|} \big(\mathrm{D}_{1;3}^{\beta_1} g_1 \bullet \mathcal{D}_{\nu+\beta_2,\mathrm{up}}^{(\ell)} \bullet \mathrm{D}_{1;3}^{\beta_3} g_2\big)^f. \end{aligned}$$

Replacing $\frac{1}{M^{|\beta_1|\ell}} \mathrm{D}_{1;3}^{\beta_1} g_1$ by g_1, $\nu + \beta_2$ by ν and $\frac{1}{M^{|\beta_1|\ell}} \mathrm{D}_{1;3}^{\beta_3} g_2$ by g_2, Theorem IV.4, with $r = j$, gives bounds on

$$\tfrac{1}{M^{|\beta|j}} \big\| \tfrac{1}{M^{|\delta_l|\ell}} M^{|\beta_2 \ell|} \big(\mathrm{D}_{1;3}^{\beta_1} g_1 \bullet \mathcal{D}_{\nu+\beta_2,\mathrm{up}}^{(\ell)} \bullet \mathrm{D}_{1;3}^{\beta_3} g_2\big)^f \bullet \mathrm{D}_{\mu;\mu'}^{\beta} \mathcal{C}_{\mathrm{loc}}^{[i,j]} \bullet h \big\|_{\kappa_1,\kappa_2}$$

for each of loc = top, mid, bot. Theorem II.20b,c now follows by Remark III.8. □

Theorem IV.5. *Let $1 \le \ell \le i \le j \le r$ and $\kappa_1, \kappa_2 \in \mathfrak{K}_r$. Set $d = \kappa_1 - \kappa_2$ and let $\mathbf{d}$, the projection of d onto $\{0\} \times \mathbb{R}^2$ identified with $\mathbb{R}^2$, be contained in a disc of radius $2\mathfrak{l}_r$ and centre $\boldsymbol{\tau}$. Furthermore, set $\tau_0 = \inf\big\{ |t_0| \;\big|\; (t_0, \mathbf{t}) \in d \text{ for some } \mathbf{t} \in \mathbb{R}^2 \big\}$. Assume that*

$$\tau_0 \le \tfrac{1}{M^{j-1}}, \qquad |\boldsymbol{\tau}| \le \max\big\{\tfrac{1}{M^j}, r^3 \mathfrak{l}_r\big\}, \qquad M^i \le \mathfrak{l}_j M^j.$$

Also assume that $p^{(i)}$ vanishes for all $i > j + 1$. Let $\nu \in \mathbb{N}_0 \times \mathbb{N}_0^2$, with $\nu + \alpha \in \Delta$ for all $|\alpha| \le 3$ and let $\mathcal{D}$ be either $\mathcal{D}_{\nu,\mathrm{up}}^{(\ell)}$ or $\mathcal{D}_{\nu,\mathrm{dn}}^{(\ell)}$. For any sectorized, translation invariant functions g_1, g_2 and h on $\mathfrak{Y}_{\ell,\ell}$, $\mathfrak{Y}_{\ell,\ell}$ and $\mathfrak{Y}_{i,r}$ respectively,

$$\big\|(g_1 \bullet \mathcal{D} \bullet g_2)^f \bullet \mathcal{C}^{[i,j]} \bullet h\big\|_{\kappa_1,\kappa_2} \le \mathrm{const}\, \sqrt{\mathfrak{l}_\ell} \max_{\substack{\alpha_{\mathrm{up}}, \alpha_{\mathrm{dn}}, \alpha_l \in \mathbb{N}_0 \times \mathbb{N}_0^2 \\ |\alpha_{\mathrm{up}}| + |\alpha_{\mathrm{dn}}| + |\alpha_l| \le 3}} \big|g_1\big|_\ell^{[\![\alpha_{\mathrm{up}}]\!]} \big|g_2\big|_\ell^{[\![\alpha_{\mathrm{dn}}]\!]} \big|h\big|_{i,r}^{[\alpha_l,0,0]}.$$

Theorem IV.5 is proven at the end of this section.

Proof of Theorem II.20a (assuming Theorem IV.5). As in the proof of Theorem II.20b,c, we may assume without loss of generality that $\delta_{\rm l} = \delta_{\rm r} = 0$. Fix $1 \le \ell \le i \le j$, $\nu \in \mathbb{N}_0 \times \mathbb{N}_0^2$, $\mathcal{D}$ and sectorized, translation invariant functions g_1, g_2 and h on $\mathfrak{Y}_{\ell,\ell}$, $\mathfrak{Y}_{\ell,\ell}$ and $\mathfrak{Y}_{i,j}$ as in Theorem II.20. By Remark III.8, it suffices to prove that

$$\begin{aligned}&\big\|(g_1 \bullet \mathcal{D} \bullet g_2)^f \bullet \mathcal{C}^{[i,j]} \bullet h\big\|_{\kappa_1,\kappa_2} \\ &\quad\le \text{const } i\,\sqrt{\mathfrak{l}_\ell} \max_{\substack{\alpha_{\rm up},\alpha_{\rm dn},\alpha_{\rm l}\in\mathbb{N}_0\times\mathbb{N}_0^2 \\ |\alpha_{\rm up}|+|\alpha_{\rm dn}|+|\alpha_{\rm l}|\le 3}} \big|g_1\big|_\ell^{[\![\alpha_{\rm up}]\!]}\big|g_2\big|_\ell^{[\![\alpha_{\rm dn}]\!]}\big|h\big|_{i,j}^{[\alpha_{\rm l},0,0]}\end{aligned} \tag{IV.4}$$

for all $\kappa_1, \kappa_2 \in \mathfrak{K}_j$. Fix $\kappa_1, \kappa_2 \in \mathfrak{K}_j$. Set $d = \kappa_1 - \kappa_2$ and denote by $\mathbf{d}$ the projection of d onto $\{0\} \times \mathbb{R}^2$ identified with $\mathbb{R}^2$. By Remark III.12, the set $\mathbf{d}$ is contained in a disc of radius $2\mathfrak{l}_j$. We fix such a disk and denote by $\boldsymbol{\tau}$ its centre. Furthermore, as in the proof of Theorem II.19a, we define $\tau_0 = \inf\big\{\, |t_0| \;\big|\; (t_0, \mathbf{t}) \in d \text{ for some } \mathbf{t} \in \mathbb{R}^2 \,\big\}$ and

$$j_0 = \begin{cases} \max\big\{\, n \in \mathbb{N}_0 \;\big|\; \tau_0 \le \frac{1}{M^{n-1}} \,\big\} & \text{if } 0 < \tau_0 \le M \\ 0 & \text{if } \tau_0 \ge M \\ \infty & \text{if } \tau_0 = 0 \end{cases},$$

$$j_1 = \begin{cases} \max\big\{\, n \in \mathbb{N}_0 \;\big|\; |\boldsymbol{\tau}| \le \frac{1}{M^{n}} \,\big\} & \text{if } j^3\mathfrak{l}_j < |\boldsymbol{\tau}| \le 1 \\ 0 & \text{if } |\boldsymbol{\tau}| \ge 1 \\ \infty & \text{if } |\boldsymbol{\tau}| \le j^3\mathfrak{l}_j \end{cases},$$

$$\bar{j} = \max\Big\{i - 1, \min\{j, j_0, j_1\}\Big\}.$$

The analog of Prop. III.16 in the current double bubble setting is □

Proposition IV.6 (Large transfer momentum).

$$\big\|(g_1 \bullet \mathcal{D} \bullet g_2)^f \bullet \mathcal{C}^{[\bar{j}+1,j]} \bullet h\big\|_{\kappa_1,\kappa_2} \le \text{const}\sqrt{\mathfrak{l}_\ell}\,\big|g_1\big|_{\ell,\ell}\big|g_2\big|_{\ell,\ell}\big|h\big|_{i,j}.$$

Proof. If $\min\{j, j_0, j_1\} = j$, then $\bar{j} = j$ and $\mathcal{C}^{[\bar{j}+1,j]} = 0$ so that there is nothing to prove. So we may assume that $\min\{j_0, j_1\} < j$.

Case 1. $j_0 \le j_1$. In this case, $\big\|(g_1 \bullet \mathcal{D} \bullet g_2)^f \bullet \mathcal{C}^{[\bar{j}+1,j]} \bullet h\big\|_{\kappa_1,\kappa_2} = 0$, because $\mathcal{C}^{[\bar{j}+1,j]}(p,k)$ vanishes unless $|p_0|, |k_0| \le \frac{\sqrt{2M}}{M^{\bar{j}+1}}$, and hence unless $|p_0 - k_0| \le \frac{2\sqrt{2M}}{M^{\bar{j}+1}} < \frac{1}{M^{\bar{j}}} < \tau_0$, while $|t_0| \ge \tau_0$ for all $t \in d$.

Case 2. $j_1 < j_0$. In this case $|\boldsymbol{\tau}| \ge j^3\mathfrak{l}_j$. By Cor. IV.3, Lemma III.14 and Lemma C.4,

$$\begin{aligned}&\big\|(g_1 \bullet \mathcal{D} \bullet g_2)^f \bullet \mathcal{C}^{[\bar{j}+1,j]} \bullet h\big\|_{\kappa_1,\kappa_2} \\ &\quad\le \text{const}\,\big|g_1\big|_{\ell,\ell}\big|g_2\big|_{\ell,\ell}\big|h\big|_{i,j}\,\mathfrak{l}_\ell \sum_{m=\bar{j}+1}^{j}\ \sup_{s_1,s_2\in\Sigma_m} \big\|\mathcal{C}^{(m)}_{s_1,s_2}\big\|_{\rm bubble} \\ &\qquad\times \sup_{\kappa'\in\mathfrak{K}_\ell} \#\left\{(u_1,u_2,s_1,s_2) \in \Sigma_\ell^2 \times \Sigma_m^2 \;\middle|\; \begin{array}{l}\boldsymbol{\pi}(u_1-u_2)\cap\boldsymbol{\pi}(\kappa'-s_1)\ne\emptyset \\ \boldsymbol{\pi}(s_1-s_2)\cap\boldsymbol{\pi}(\kappa_1-\kappa_2)\ne\emptyset\end{array}\right\} \\ &\quad\le \text{const}\,\big|g_1\big|_{\ell,\ell}\big|g_2\big|_{\ell,\ell}\big|h\big|_{i,j} \sum_{m=\bar{j}+1}^{j}\Big[\min\big(\ell\mathfrak{l}_\ell, \tfrac{1+M^m\mathfrak{l}_j}{M^m|\boldsymbol{\tau}|}\big) + \tfrac{\sqrt{\mathfrak{l}_\ell}}{|\boldsymbol{\tau}|}\big(\tfrac{1}{M^m} + \mathfrak{l}_j\big) + \sqrt{\mathfrak{l}_m}\Big].\end{aligned}$$

The last sum

$$\sum_{m=\bar{j}+1}^{j} \sqrt{\mathfrak{l}_m} \le \sqrt{\mathfrak{l}_{\bar{j}}} \le \sqrt{\mathfrak{l}_{i-1}} \le \text{const}\sqrt{\mathfrak{l}_\ell}.$$

As, by the definition of j_1,

$$\tfrac{1}{|\boldsymbol{\tau}|}\sum_{m=j_1+1}^{j} \big(\tfrac{1}{M^m} + \mathfrak{l}_j\big) \le \tfrac{1}{|\boldsymbol{\tau}|}\big(\tfrac{1}{M^{j_1}} + j\,\mathfrak{l}_j\big) \le \text{const}\,,$$

the middle sum

$$\sum_{m=\bar{j}+1}^{j} \tfrac{\sqrt{\mathfrak{l}_\ell}}{|\boldsymbol{\tau}|}\big(\tfrac{1}{M^m} + \mathfrak{l}_j\big) \le \text{const}\sqrt{\mathfrak{l}_\ell}.$$

As $\min\big(\ell\mathfrak{l}_\ell,\, \frac{1+M^m\mathfrak{l}_j}{M^m|\boldsymbol{\tau}|}\big) \le \min\big(\ell\mathfrak{l}_\ell,\, \frac{1}{M^m|\boldsymbol{\tau}|}\big) + \min\big(\ell\mathfrak{l}_\ell,\, \frac{\mathfrak{l}_j}{|\boldsymbol{\tau}|}\big)$, the first sum

$$\begin{aligned}
\sum_{m=\bar{j}+1}^{j} \min\big(\ell\mathfrak{l}_\ell,\, \tfrac{1+M^m\mathfrak{l}_j}{M^m|\boldsymbol{\tau}|}\big) &\le j\min\big(\ell\mathfrak{l}_\ell,\, \tfrac{\mathfrak{l}_j}{|\boldsymbol{\tau}|}\big) + \sum_{m=\bar{j}+1}^{j} \min\big(\ell\mathfrak{l}_\ell,\, \tfrac{1}{M^m|\boldsymbol{\tau}|}\big) \\
&\le j\min\big(\ell\mathfrak{l}_\ell,\, \tfrac{\mathfrak{l}_j}{|\boldsymbol{\tau}|}\big) + \sum_{m=\bar{j}+1}^{j} (\ell\mathfrak{l}_\ell)^{2/3}\tfrac{1}{(M^m|\boldsymbol{\tau}|)^{1/3}} \\
&\le j\min\big(\ell\mathfrak{l}_\ell,\, \tfrac{\mathfrak{l}_j}{|\boldsymbol{\tau}|}\big) + (\ell\mathfrak{l}_\ell)^{2/3}\tfrac{1}{(M^{\bar{j}}|\boldsymbol{\tau}|)^{1/3}} \\
&\le j\min\big(\ell\mathfrak{l}_\ell,\, \tfrac{\mathfrak{l}_j}{|\boldsymbol{\tau}|}\big) + \text{const}\,(\ell\mathfrak{l}_\ell)^{2/3}.
\end{aligned}$$

If $j \le \frac{1}{\mathfrak{l}_\ell^{1/3}}$, then

$$j\,\ell\mathfrak{l}_\ell \le \ell\mathfrak{l}_\ell^{2/3} \le \text{const}\sqrt{\mathfrak{l}_\ell}$$

while, if $j \ge \frac{1}{\mathfrak{l}_\ell^{1/3}}$, then

$$\tfrac{j\,\mathfrak{l}_j}{|\boldsymbol{\tau}|} \le \tfrac{1}{j^2} \le \mathfrak{l}_\ell^{2/3}. \quad \square$$

Continuation of the proof of Theorem II.20a (assuming Theorem IV.5). When $M^i \ge \mathfrak{l}_{\bar{j}}M^{\bar{j}} = M^{(1-\aleph)\bar{j}}$, we have $|\bar{j} - i + 1| \le \text{const}\, i$. In this case Theorem IV.4, with $r = j$ and $j = \bar{j}$, gives

$$\big\|(g_1 \bullet \mathcal{D} \bullet g_2)^f \bullet \mathcal{C}^{[i,\bar{j}]} \bullet h\big\|_{\kappa_1,\kappa_2} \le \text{const}\, i\, \sqrt{\mathfrak{l}_\ell}\, |g_1|_{\ell,\ell}|g_2|_{\ell,\ell}|h|_{i,j}.$$

This together with Prop. IV.6 yields (IV.4). Therefore, we may assume that

$$M^i \le \mathfrak{l}_{\bar{j}}M^{\bar{j}}. \tag{IV.5}$$

Furthermore, if $\bar{j} = i - 1$, $\mathcal{C}^{[i,\bar{j}]} = 0$ and there is nothing more to prove. So we may also assume that $j_0, j_1 \ge i$ and $\bar{j} \le j, j_0, j_1$.

Set $v' = \sum_{i=2}^{\bar{j}+1} p^{(i)}$ and $\mathcal{C}'^{[i,\bar{j}]} = \mathcal{C}'^{[i,\bar{j}]}_{\rm top} + \mathcal{C}'^{[i,\bar{j}]}_{\rm mid} + \mathcal{C}'^{[i,\bar{j}]}_{\rm bot}$ with

$$\mathcal{C}'^{[i,\bar{j}]}_{\rm top} = \sum_{\substack{i\le i_t\le \bar{j}\\ i_b>\bar{j}}} C^{(i_t)}_{v'} \otimes C^{(i_b)\,t}_{v'}, \qquad \mathcal{C}'^{[i,\bar{j}]}_{\rm mid} = \sum_{\substack{i\le i_t\le \bar{j}\\ i\le i_b\le \bar{j}}} C^{(i_t)}_{v'} \otimes C^{(i_b)\,t}_{v'},$$
$$\mathcal{C}'^{[i,\bar{j}]}_{\rm bot} = \sum_{\substack{i_t>\bar{j}\\ i\le i_b\le \bar{j}}} C^{(i_t)}_{v'} \otimes C^{(i_b)\,t}_{v'}$$

as in §III. Again, $v - v'$ is supported on the $(\bar{j}+2)^{\rm nd}$ extended neighbourhood and $\mathcal{C}^{[i,\bar{j}]}_{\rm mid} = \mathcal{C}'^{[i,\bar{j}]}_{\rm mid}$. Hence, by Theorem IV.4.i, with $\beta = 0$, $r = j$ and $j = \bar{j}$,

$$\big\|(g_1 \bullet \mathcal{D} \bullet g_2)^f \bullet \big[\mathcal{C}^{[i,\bar{j}]} - \mathcal{C}'^{[i,\bar{j}]}\big] \bullet h\big\|_{\kappa_1,\kappa_2} \le {\rm const}\, \sqrt{\mathfrak{l}_\ell}\, \big|g_1\big|_{\ell,\ell} \big|g_2\big|_{\ell,\ell} \big|h\big|_{i,j}. \quad \text{(IV.6)}$$

By (IV.5) and the definitions of $\bar{j}$ and $\mathcal{C}'^{[i,\bar{j}]}$, the hypotheses of Theorem IV.5, with $r = j$ and $j = \bar{j}$, apply to $(g_1 \bullet \mathcal{D} \bullet g_2)^f \bullet \mathcal{C}'^{[i,\bar{j}]} \bullet h$. Hence

$$\big\|(g_1 \bullet \mathcal{D} \bullet g_2)^f \bullet \mathcal{C}'^{[i,\bar{j}]} \bullet h\big\|_{\kappa_1,\kappa_2} \le {\rm const}\, \sqrt{\mathfrak{l}_\ell} \max_{\substack{\alpha_{\rm up},\alpha_{\rm dn},\alpha_{\rm l}\in\mathbb{N}_0\times\mathbb{N}_0^2\\ |\alpha_{\rm up}|+|\alpha_{\rm dn}|+|\alpha_{\rm l}|\le 3}} \big|g_1\big|_\ell^{[\![\alpha_{\rm up}]\!]} \big|g_2\big|_\ell^{[\![\alpha_{\rm dn}]\!]} \big|h\big|_{i,j}^{[\alpha_{\rm l},0,0]}.$$

This together with (IV.6) and Prop. IV.6 yields (IV.4). This completes the proof that Theorem IV.5 implies Theorem II.20.a. □

The rest of this section is devoted to the proof of Theorem IV.5. So we fix $\nu \in \Delta$, $1 \le \ell \le i \le j \le r$, $\mathcal{D} = \mathcal{D}^{(\ell)}_{\nu,{\rm up}}$ or $\mathcal{D}^{(\ell)}_{\nu,{\rm dn}}$ and sectorized, translation invariant functions, g_1, g_2 and h, on $\mathfrak{Y}_{\ell,\ell}$, $\mathfrak{Y}_{\ell,\ell}$ and $\mathfrak{Y}_{i,r}$ respectively. We also fix $\kappa_1, \kappa_2 \in \mathfrak{K}_r$ and assume that

$$\tau_0 \le \tfrac{1}{M^{j-1}}, \qquad |\boldsymbol{\tau}| \le \max\big\{\tfrac{1}{M^j}, r^3\mathfrak{l}_r\big\}, \qquad M^i \le \mathfrak{l}_j M^j, \qquad \text{(IV.7)}$$

and that $p^{(i)}$ vanishes for all $i > j+1$. As in §III, we reduce the particle–hole bubble propagator $\mathcal{C}^{[i,j]}$ to the model bubble propagator of (III.22). This is done in the following two lemmata.

Lemma IV.7. *Let $\mathcal{Z}$ be the operator defined in (III.18). Then*

$$\big\|(g_1 \bullet \mathcal{D} \bullet g_2)^f \bullet (\mathcal{Z} \bullet \mathcal{C}^{[i,j]} \bullet \mathcal{Z}^t - \mathcal{M}) \bullet h\big\|_{\kappa_1,\kappa_2} \le {\rm const}\, \sqrt{\mathfrak{l}_\ell}\, \big|g_1\big|_{\ell,\ell} \big|g_2\big|_{\ell,\ell} \big|h\big|_{i,r}.$$

Proof. Expand

$$\mathcal{Z} \bullet \mathcal{C}^{[i,j]} \bullet \mathcal{Z}^t - \mathcal{M} = (\widetilde{\mathcal{C}}^{[i,j]} - \mathcal{M}) + (\mathcal{Z} \bullet \mathcal{C}^{[i,j]} \bullet \mathcal{Z}^t - \widetilde{\mathcal{C}}^{[i,j]}),$$

where $\widetilde{\mathcal{C}}^{[i,j]}$ was defined in (III.20). Then, for $|p_0| \le \frac{M+4\sqrt{2M}}{M^j}$,

$$\begin{aligned}&(\widetilde{\mathcal{C}}^{[i,j]} - \mathcal{M}) + (\mathcal{Z} \bullet \mathcal{C}^{[i,j]} \bullet \mathcal{Z}^t - \widetilde{\mathcal{C}}^{[i,j]})\\ &\quad = \sum_{m=i}^{j} \sum_{s_1,s_2\in\Sigma_m} \mathcal{D}^{(m)}_{s_1,s_2} + \sum_{m=i}^{j} \sum_{s_1,s_2\in\Sigma_m} (\mathcal{Z} \bullet \mathcal{C}^{(m)}_{s_1,s_2} \bullet \mathcal{Z}^t - \widetilde{\mathcal{C}}^{(m)}_{s_1,s_2}),\end{aligned}$$

where $\mathcal{D}^{(m)}_{s_1,s_2}$ was defined in (III.23). So, by Cor. IV.3,

$$\begin{aligned}
&\big\|(g_1\bullet\mathcal{D}\bullet g_2)^f\bullet\big(\widetilde{\mathcal{C}}^{[i,j]}-\mathcal{M}+\mathcal{Z}\bullet\mathcal{C}^{[i,j]}\bullet\mathcal{Z}^t-\widetilde{\mathcal{C}}^{[i,j]}\big)\bullet h\big\|_{\kappa_1,\kappa_2}\\
&\quad\le \text{const}\,\big|g_1\big|_{\ell,\ell}\big|g_2\big|_{\ell,\ell}\big|h\big|_{i,r}\,\mathfrak{l}_\ell\sum_{m=i}^{j}\sup_{\kappa'\in\mathfrak{K}_\ell}\#\{(s_1,s_2,u_1,u_2)\\
&\qquad\times\boldsymbol{\pi}(s_1-s_2)\cap\boldsymbol{\pi}(\kappa_1-\kappa_2)\neq\emptyset,\ \boldsymbol{\pi}(u_1-u_2)\cap\boldsymbol{\pi}(\kappa'-s_1)\neq\emptyset\big\}\\
&\qquad\times\max_{s_1,s_2\in\Sigma_m}\Big[\big\|\mathcal{D}^{(m)}_{s_1,s_2}\big\|_{\text{bubble}}+\big\|\mathcal{Z}\bullet\mathcal{C}^{(m)}_{s_1,s_2,\phi}\bullet\mathcal{Z}^t-\big(\widetilde{\mathcal{C}}^{(m)}_{s_1,s_2}\big)_\phi\big\|_{\text{bubble}}\Big],
\end{aligned}$$

where ϕ was defined at the beginning of the proof of Prop. III.22. Then by Lemma C.3, Lemma III.25 and Lemma III.23,

$$\begin{aligned}
&\big\|(g_1\bullet\mathcal{D}\bullet g_2)^f\bullet\big(\widetilde{\mathcal{C}}^{[i,j]}-\mathcal{M}+\mathcal{Z}\bullet\mathcal{C}^{[i,j]}\bullet\mathcal{Z}^t-\widetilde{\mathcal{C}}^{[i,j]}\big)\bullet h\big\|_{\kappa_1,\kappa_2}\\
&\quad\le \text{const}\,\big|g_1\big|_{\ell,\ell}\big|g_2\big|_{\ell,\ell}\big|h\big|_{i,r}\,\mathfrak{l}_\ell\Big[\sum_{m=i}^{i+1}\tfrac{1}{\mathfrak{l}_m\sqrt{\mathfrak{l}_\ell}}\mathfrak{l}_m\\
&\qquad+\sum_{m=i+2}^{j}\tfrac{j-m+1}{\mathfrak{l}_m\sqrt{\mathfrak{l}_\ell}}\tfrac{M^m}{M^j}\mathfrak{l}_m+\sum_{m=i}^{j}\tfrac{1}{\mathfrak{l}_m\sqrt{\mathfrak{l}_\ell}}\mathfrak{l}_m\tfrac{M^m}{M^j}\Big]\\
&\quad\le \text{const}\sqrt{\mathfrak{l}_\ell}\,\big|g_1\big|_{\ell,\ell}\big|g_2\big|_{\ell,\ell}\big|h\big|_{i,r}.\qquad\square
\end{aligned}$$

Lemma IV.8.

$$\begin{aligned}
&\big\|(g_1\bullet\mathcal{D}\bullet g_2)^f\bullet(\mathcal{C}^{[i,j]}-\mathcal{Z}\bullet\mathcal{C}^{[i,j]}\bullet\mathcal{Z}^t)\bullet h\big\|_{\kappa_1,\kappa_2}\\
&\quad\le\text{const}\sqrt{\mathfrak{l}_\ell}\max_{|\alpha_{\text{up}}|+|\alpha_{\text{dn}}|\le1}\big|g_1\big|_\ell^{[\![\alpha_{\text{up}}]\!]}\big|g_2\big|_\ell^{[\![\alpha_{\text{dn}}]\!]}\big|h\big|_{i,r}^{[1,0,0]}.
\end{aligned}$$

Proof. By Lemma III.20

$$\begin{aligned}
&(g_1\bullet\mathcal{D}\bullet g_2)^f\bullet(\mathcal{C}^{[i,j]}-\mathcal{Z}\bullet\mathcal{C}^{[i,j]}\bullet\mathcal{Z}^t)\bullet h\\
&\quad=(g_1\bullet\mathcal{D}\bullet g_2)^f_{\text{r}}\bullet D_{\text{l}}\mathcal{C}^{[i,j]}\bullet h+(g_1\bullet\mathcal{D}\bullet g_2)^f\bullet\mathcal{Z}\bullet D_{\text{r}}\mathcal{C}^{[i,j]}\bullet h_1.
\end{aligned}$$

Both terms are bounded as in the previous lemma, using Lemma C.3 to bound the number of allowed 4–tuples (s_1,s_2,u_1,u_2) by $\frac{1}{\mathfrak{l}_m\sqrt{\mathfrak{l}_\ell}}$. Lemma III.21, (and, for the second term, Lemma III.18.iii) are used to bound $\|D_{\text{l}}\mathcal{C}^{(m)}_{s_1,s_2}\|_{\text{bubble}}\le\text{const}\,\frac{\mathfrak{l}_m}{M^m}$ and $\|\mathcal{Z}\bullet D_{\text{r}}\mathcal{C}^{(m)}_{s_1,s_2}\|_{\text{bubble}}\le\|D_{\text{r}}\mathcal{C}^{(m)}_{s_1,s_2}\|_{\text{bubble}}\le\text{const}\,\frac{\mathfrak{l}_m}{M^m}$. The right derivative in $(g_1\bullet\mathcal{D}\bullet g_2)^f_{\text{r}}$ acts as a central derivative on $g_1\bullet\mathcal{D}\bullet g_2$ and may be written, using Leibniz's rule, as a sum of three terms with the first containing a central derivative acting on g_1, the second a central derivative acting on g_2 and the third having one component of $\mathcal{D}$'s index ν increased by one. Lemma IV.2 is used to bound the bubble norms of the sectorized contributions to $\mathcal{D}$. All together,

$$\begin{aligned}
&\big\|(g_1\bullet\mathcal{D}\bullet g_2)^f\bullet\big(\mathcal{C}^{[i,j]}-\mathcal{Z}\bullet\mathcal{C}^{[i,j]}\bullet\mathcal{Z}^t\big)\bullet h\big\|_{\kappa_1,\kappa_2}\\
&\quad\le\big\|(g_1\bullet\mathcal{D}\bullet g_2)^f_{\text{r}}\bullet D_{\text{l}}\mathcal{C}^{[i,j]}\bullet h\big\|_{\kappa_1,\kappa_2}+\big\|(g_1\bullet\mathcal{D}\bullet g_2)^f\bullet\mathcal{Z}\bullet D_{\text{r}}\mathcal{C}^{[i,j]}\bullet h_1\|_{\kappa_1,\kappa_2}\\
&\quad\le\text{const}\Big[M^\ell\big|g_1\big|_\ell^{[\![(1,0,0)]\!]}\,\mathfrak{l}_\ell\,\big|g_2\big|_{\ell,\ell}+\big|g_1\big|_{\ell,\ell}\,M^\ell\mathfrak{l}_\ell\,\big|g_2\big|_{\ell,\ell}\\
&\qquad+\big|g_1\big|_{\ell,\ell}\,\mathfrak{l}_\ell\,M^\ell\big|g_2\big|_\ell^{[\![(1,0,0)]\!]}\Big]\big|h\big|_{i,r}\sum_{m=i}^{j}\tfrac{1}{\mathfrak{l}_m\sqrt{\mathfrak{l}_\ell}}\tfrac{\mathfrak{l}_m}{M^m}
\end{aligned}$$

$$
\begin{aligned}
&+\mathrm{const}\,\big|g_1\big|_{\ell,\ell}\ \mathfrak{l}_\ell\ \big|g_2\big|_{\ell,\ell}|h_1|_{i,r}\sum_{m=i}^{j}\tfrac{1}{\mathfrak{l}_m\sqrt{\mathfrak{l}_\ell}}\tfrac{\mathfrak{l}_m}{M^m}\\
&\le \mathrm{const}\sqrt{\mathfrak{l}_\ell}\max_{|\alpha_{\rm up}|+|\alpha_{\rm dn}|\le 1}\big|g_1\big|_\ell^{[\![\alpha_{\rm up}]\!]}\big|g_2\big|_\ell^{[\![\alpha_{\rm dn}]\!]}\big|h\big|_{i,r}^{[1,0,0]}\sum_{m=i}^{j}\tfrac{M^i}{M^m}\\
&\le \mathrm{const}\sqrt{\mathfrak{l}_\ell}\max_{|\alpha_{\rm up}|+|\alpha_{\rm dn}|\le 1}\big|g_1\big|_\ell^{[\![\alpha_{\rm up}]\!]}\big|g_2\big|_\ell^{[\![\alpha_{\rm dn}]\!]}\big|h\big|_{i,r}^{[1,0,0]}. \quad\square
\end{aligned}
$$

Proposition IV.9.

$$
\big\|(g_1\bullet\mathcal{D}\bullet g_2)^f\bullet\mathcal{M}\bullet h\big\|_{\kappa_1,\kappa_2}\le \mathrm{const}\sqrt{\mathfrak{l}_\ell}\max_{\substack{\alpha_{\rm up},\alpha_{\rm dn},\alpha_1\in\mathbb{N}_0\times\mathbb{N}_0^2\\ |\alpha_{\rm up}|+|\alpha_{\rm dn}|+|\alpha_1|\le 3}}\big|g_1\big|_\ell^{[\![\alpha_{\rm up}]\!]}\big|g_2\big|_\ell^{[\![\alpha_{\rm dn}]\!]}\big|h\big|_{i,r}^{[\alpha_1,0,0]}.
$$

Proof. Write

$$
\mathcal{M}(p,k)=\sum_{s_1,s_2\in\Sigma_i}\mathcal{M}_{s_1,s_2}(p,k),
$$

where

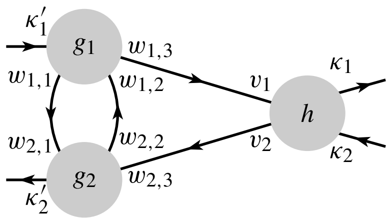
$$
\mathcal{M}_{s_1,s_2}(p,k)=\mathcal{M}(p,k)\rho(\mathbf{p}-\mathbf{k})\chi_{s_1}(\mathbf{p})\chi_{s_2}(\mathbf{k})
$$

and ρ was defined just before (III.24). Then

$$
\begin{aligned}
&(g_1\bullet\mathcal{D}\bullet g_2)^f\bullet\mathcal{M}\bullet h\\
&\quad=\sum_{\substack{u_1,u_2\in\Sigma_\ell\\ w_{1,1},w_{1,2}\in\Sigma_\ell\\ w_{2,1},w_{2,2}\in\Sigma_\ell}}\ \sum_{\substack{w_{1,3},w_{2,3}\in\Sigma_\ell\\ s_1,s_2\in\Sigma_i\\ v_1,v_2\in\Sigma_i}}\\
&\quad\times\Big(g_1\big(y_1,(\,\cdot\,,w_{1,3}),(\,\cdot\,,w_{1,1}),(\,\cdot\,,w_{1,2})\big)\circ\mathcal{D}_{u_1,u_2}\circ g_2\big((\,\cdot\,,w_{2,1}),(\,\cdot\,,w_{2,2}),y_2,(\,\cdot\,,w_{2,3})\big)\Big)^f,\\
&\quad\times\ \circ\,\mathcal{M}_{s_1,s_2}\circ h\big((\,\cdot\,,v_1),(\,\cdot\,,v_2),y_3,y_4\big)
\end{aligned}
$$

where, for $\nu=1,2$,

$$
y_\nu=\begin{cases}\kappa'_\nu & \text{if } \kappa'_\nu\in\mathbb{M}\\ (x_\nu,\kappa'_\nu) & \text{if } \kappa'_\nu\in\Sigma_\ell\end{cases}\qquad y_{\nu+2}=\begin{cases}\kappa_\nu & \text{if } \kappa_\nu\in\mathbb{M}\\ (x_{\nu+2},\kappa_\nu) & \text{if } \kappa_\nu\in\Sigma_r\end{cases}
$$



The multiple convolution vanishes unless

$$
\boldsymbol{\pi}(s_1-s_2)\cap\boldsymbol{\pi}(\kappa_1-\kappa_2)\neq\emptyset,\qquad \boldsymbol{\pi}(u_1-u_2)\cap\boldsymbol{\pi}(\kappa'_1-s_1)\neq\emptyset,
$$

and

$$
w_{1,1}\cap u_1\neq\emptyset\quad w_{2,1}\cap u_1\neq\emptyset,\quad w_{1,2}\cap u_2\neq\emptyset\quad w_{2,2}\cap u_2\neq\emptyset,
$$

$$\boldsymbol{\pi}(w_{1,3})\cap\boldsymbol{\pi}(s_1)\neq\emptyset,\quad \boldsymbol{\pi}(v_1)\cap\boldsymbol{\pi}(s_1)\neq\emptyset,\quad \boldsymbol{\pi}(w_{2,3})\cap\boldsymbol{\pi}(s_2)\neq\emptyset,$$
$$\boldsymbol{\pi}(v_2)\cap\boldsymbol{\pi}(s_2)\neq\emptyset. \tag{IV.8}$$

Fix the external sectors/momenta $(\kappa_1,\kappa_2,\kappa_1',\kappa_2')$. Then, for each fixed (s_1,s_2,u_1,u_2), there are at most 3^8 8–tuples $(w_{1,1},w_{2,1},w_{1,2},w_{2,2},w_{1,3},v_1,w_{2,3},v_2)$ satisfying (IV.8). By Lemma C.3, the number of allowed 4–tuples (s_1,s_2,u_1,u_2) is bounded by $\frac{1}{\mathfrak{l}_i\sqrt{\mathfrak{l}_\ell}}$. Set, for each $\varsigma=(\kappa_1',\kappa_2',w_{1,1},w_{1,2},w_{1,3},w_{2,1},w_{2,2},w_{2,3},u_1,u_2)\in\mathfrak{K}_\ell^2\times\Sigma_\ell^8$,

$$g'_\varsigma = g_1(y_1,(\,\cdot\,,w_{1,3}),(\,\cdot\,,w_{1,1}),(\,\cdot\,,w_{1,2}))\circ\mathcal{D}_{u_1,u_2}\circ g_2((\,\cdot\,,w_{2,1}),(\,\cdot\,,w_{2,2}),y_2,(\,\cdot\,,w_{2,3})),$$

for each $\tau=(w_{1,1},w_{1,2},w_{2,1},w_{2,2},,u_1,u_2)\in\Sigma_\ell^6$,

$$g_\tau = g_1(\,\cdot\,,\cdot\,,(\,\cdot\,,w_{1,1}),(\,\cdot\,,w_{1,2}))\circ\mathcal{D}_{u_1,u_2}\circ g_2((\,\cdot\,,w_{2,1}),(\,\cdot\,,w_{2,2}),\cdot\,,\cdot\,),$$

and, for each $v_1,v_2\in\Sigma_i$,

$$h_{v_1,v_2}(x_1,x_2,y_3,y_4)=h\big((x_1,v_1),(x_2,v_2),y_3,y_4\big).$$

Then

$$\big\|(g_1\bullet\mathcal{D}\bullet g_2)^f\bullet\mathcal{M}\bullet h\big\|_{\kappa_1,\kappa_2}\le \text{const}\,\frac{1}{\mathfrak{l}_i\sqrt{\mathfrak{l}_\ell}}\sup_{\varsigma\in\mathfrak{K}_\ell^2\times\Sigma_\ell^8}\max_{\substack{s_1,s_2\in\Sigma_i\\ v_1,v_2\in\Sigma_i}}\big|\!\big|\!\big|{g'_\varsigma}^f\circ\mathcal{M}_{s_1,s_2}\circ h_{v_1,v_2}\big|\!\big|\!\big|_{1,\infty}.$$

By (III.28)

$$\big|\!\big|\!\big|{g'_\varsigma}^f\circ\mathcal{M}_{s_1,s_2}\circ h_{v_1,v_2}\big|\!\big|\!\big|_{1,\infty}\le\text{const}\,\mathfrak{l}_i\max_{\tau\in\Sigma_\ell^6}\max_{\substack{\alpha_{\rm r},\alpha_{\rm l}\in\mathbb{N}_0\times\mathbb{N}_0^2\\ |\alpha_{\rm r}|+|\alpha_{\rm l}|\le3}}\big|g_\tau^f\big|_{\ell,i}^{[0,0,\alpha_{\rm r}]}\big|h\big|_{i,r}^{[\alpha_{\rm l},0,0]}.$$

Bounding

$$\begin{aligned}\big|g_\tau^f\big|_{\ell,i}^{[0,0,\alpha_{\rm r}]}&\le|g_\tau|_{\ell,\ell}^{[0,\alpha_{\rm r},0]}\\&\le\text{const}\,\mathfrak{l}_\ell\max_{\substack{\alpha_{\rm up},\alpha_{\rm dn}\in\mathbb{N}_0\times\mathbb{N}_0^2\\ \alpha_{\rm up}+\alpha_{\rm dn}\le\alpha_{\rm r}}}|g_1|_\ell^{[\![\alpha_{\rm up}]\!]}|g_2|_\ell^{[\![\alpha_{\rm dn}]\!]}\end{aligned}$$

by Leibniz and Lemma IV.2, yields

$$\begin{aligned}&\big\|(g_1\bullet\mathcal{D}\bullet g_2)^f\bullet\mathcal{M}\bullet h\big\|_{\kappa_1,\kappa_2}\\&\quad\le\text{const}\,\frac{1}{\mathfrak{l}_i\sqrt{\mathfrak{l}_\ell}}\,\mathfrak{l}_i\mathfrak{l}_\ell\max_{\substack{\alpha_{\rm up},\alpha_{\rm dn},\alpha_{\rm l}\in\mathbb{N}_0\times\mathbb{N}_0^2\\ |\alpha_{\rm up}|+|\alpha_{\rm dn}|+|\alpha_{\rm l}|\le3}}\big|g_1\big|_\ell^{[\![\alpha_{\rm up}]\!]}\big|g_2\big|_\ell^{[\![\alpha_{\rm dn}]\!]}\big|h\big|_{i,r}^{[\alpha_{\rm l},0,0]}\\&\quad\le\text{const}\,\sqrt{\mathfrak{l}_\ell}\max_{\substack{\alpha_{\rm up},\alpha_{\rm dn},\alpha_{\rm l}\in\mathbb{N}_0\times\mathbb{N}_0^2\\ |\alpha_{\rm up}|+|\alpha_{\rm dn}|+|\alpha_{\rm l}|\le3}}\big|g_1\big|_\ell^{[\![\alpha_{\rm up}]\!]}\big|g_2\big|_\ell^{[\![\alpha_{\rm dn}]\!]}\big|h\big|_{i,r}^{[\alpha_{\rm l},0,0]}.\qquad\square\end{aligned}$$

Proof of Theorem IV.5. By Lemmas IV.7, IV.8 and Prop. IV.9,

$$\begin{aligned}&\big\|(g_1\bullet\mathcal{D}\bullet g_2)^f\bullet\mathcal{C}^{[i,j]}\bullet h\big\|_{\kappa_1,\kappa_2}\\&\quad\le\big\|(g_1\bullet\mathcal{D}\bullet g_2)^f\bullet(\mathcal{Z}\bullet\mathcal{C}^{[i,j]}\bullet\mathcal{Z}^t-\mathcal{M})\bullet h\big\|_{\kappa_1,\kappa_2}\\&\qquad+\big\|(g_1\bullet\mathcal{D}\bullet g_2)^f\bullet(\mathcal{C}^{[i,j]}-\mathcal{Z}\bullet\mathcal{C}^{[i,j]}\bullet\mathcal{Z}^t)\bullet h\big\|_{\kappa_1,\kappa_2}\\&\qquad+\big\|(g_1\bullet\mathcal{D}\bullet g_2)^f\bullet\mathcal{M}\bullet h\big\|_{\kappa_1,\kappa_2}\\&\quad\le\text{const}\,\sqrt{\mathfrak{l}_\ell}\max_{\substack{\alpha_{\rm up},\alpha_{\rm dn},\alpha_{\rm l}\in\mathbb{N}_0\times\mathbb{N}_0^2\\ |\alpha_{\rm up}|+|\alpha_{\rm dn}|+|\alpha_{\rm l}|\le3}}\big|g_1\big|_\ell^{[\![\alpha_{\rm up}]\!]}\big|g_2\big|_\ell^{[\![\alpha_{\rm dn}]\!]}\big|h\big|_{i,r}^{[\alpha_{\rm l},0,0]}\end{aligned}$$

as desired. $\square$

Appendix A. Bounds on Propagators

Fix, as in Theorem I.20, a sequence, $p^{(2)}, p^{(3)}, \cdots$, of sectorized, translation invariant functions $p^{(i)}$ on $\Big((\mathbb{R}\times\mathbb{R}^2)\times\Sigma_i\Big)^2$ obeying

$$|p^{(i)}|_{1,\Sigma_i} \le \tfrac{\rho\,\mathfrak{l}_i}{M^i}\mathfrak{c}_i, \qquad \check p^{(i)}(0,\mathbf{k}) = 0,$$

and set, for all $j \ge 1$,

$$v(k) = \sum_{i=2}^{\infty} \check p^{(i)}(k), \qquad e'(k) = e(\mathbf{k}) - v(k),$$
$$v_j(k) = \sum_{i=2}^{j} \check p^{(i)}(k), \qquad e'_j(k) = e(\mathbf{k}) - v_j(k).$$

Lemma A.1. *There is a $\tilde\rho_0 > 0$ such that for all $\rho \le \tilde\rho_0$ and $(k_0,\mathbf{k})$ in the first neighbourhood*

i)

$$\nabla_{\mathbf{k}} e'(k_0,\mathbf{k}) \ne 0, \qquad \nabla_{\mathbf{k}} e'_j(k_0,\mathbf{k}) \ne 0,$$
$$\big|\partial_{k_0} e'(k_0,\mathbf{k})\big| \le \rho < \tfrac12, \qquad \big|\partial_{k_0} e'_j(k_0,\mathbf{k})\big| \le \rho < \tfrac12,$$
$$\big|\imath k_0 - e'(k)\big| \ge \tfrac12\big|\imath k_0 - e(\mathbf{k})\big|, \qquad \big|\imath k_0 - e'_j(k)\big| \ge \tfrac12\big|\imath k_0 - e(\mathbf{k})\big|.$$

ii) $\big|\partial_k^\alpha e'_j(k)\big| \le const \begin{cases} 1 & \text{if } |\alpha|\le 1 \\ \mathfrak{l}_j M^{(|\alpha|-1)j} & \text{if } \alpha\in\Delta,\ |\alpha|\ge 2 \end{cases}.$

iii) If $|\gamma| = 1$, then

$$\big|\partial_k^\gamma e'_j(p_0,\mathbf{k}) - \partial_k^\gamma e'_j(k_0,\mathbf{k})\big|,\ \big|\partial_k^\gamma e'(p_0,\mathbf{k}) - \partial_k^\gamma e'(k_0,\mathbf{k})\big| \le const\,\rho\,|p_0-k_0|^{\aleph},$$
$$\big|\partial_{k_0} e'_j(k) - \partial_{k_0} e'_j(k')\big|,\ \big|\partial_{k_0} e'(k) - \partial_{k_0} e'(k')\big| \le const\,\rho\,|k-k'|^{\aleph},$$
$$\big|\partial_k^\gamma e'_j(k) - \partial_k^\gamma e'_j(k')\big|,\ \big|\partial_k^\gamma e'(k) - \partial_k^\gamma e'(k')\big| \le const\,|k-k'|^{\aleph}.$$

Proof. i) By setting $p^{(i)} = 0$ for all $i > j$, it suffices to prove the statements regarding e'. All statements follow from

$$\sup_k \big|\partial_k v(k)\big| \le \sum_{i=2}^{\infty} \sup_k \big|\partial_k \check p^{(i)}(k)\big| \le \sum_{i=2}^{\infty} 2\rho\mathfrak{l}_i \le \rho.$$

For the second inequality, we used Lemma XII.12 of [FKTo3].

ii) Again by Lemma XII.12 of [FKTo3],

$$\big|\partial_k^\alpha e'_j(k)\big| \le \big|\partial_k^\alpha e(\mathbf{k})\big| + \sum_{i=2}^{j}\big|\partial_k^\alpha \check p^{(i)}(k)\big| \le \big|\partial_k^\alpha e(\mathbf{k})\big| + \sum_{i=2}^{j} 2\alpha!\rho\mathfrak{l}_i M^{(|\alpha|-1)i}$$
$$\le const \begin{cases} 1 & \text{if } |\alpha|\le 1 \\ \mathfrak{l}_j M^{(|\alpha|-1)j} & \text{if } \alpha\in\Delta,\ |\alpha|\ge 2 \end{cases}.$$

iii) Apply Lemma C.1 of [FKTf3] with $C_0 = C_1 = \mathit{const}\,\rho, \alpha = \aleph$ and $\beta = 1-\aleph$. When dealing with e', use $f_i(t) = \partial_k^\gamma \check{p}_i(t, \mathbf{k})$ for the first bound, $f_i(t) = \partial_{k_0} \check{p}_i\big(k+t\frac{k'-k}{|k'-k|}\big)$ for the second bound and $f_i(t) = \partial_k^\gamma \check{p}_i\big(k + t\frac{k'-k}{|k'-k|}\big)$ for the third bound. When dealing with e'_j use the above f_i's for $2 \le i \le j$ and zero otherwise. The contribution from $e(\mathbf{k})$ vanishes in the first two bounds and is bounded by $\mathit{const}\,|k - k'|$ in the third. □

Recall that, for $j \ge 1$, $C_v^{(j)}(k) = \frac{\nu^{(j)}(k)}{ik_0 - e'(k)}$. Set, for $m \ge 1$ and $s \in \Sigma_m$,

$$c_s^{(j)}(k) = C_v^{(j)}(k)\chi_s(k),$$

and denote by $c_s^{(j)}(x)$ its Fourier transform.

Lemma A.2. *There are M–dependent constants* const *and* $\tilde\rho_0$ *such that the following holds for all* $\rho \le \tilde\rho_0$, $\beta \in \Delta$ *and* $j \ge 1$*:*

i) For $s \in \Sigma_j$,

$$\big\|x^\beta c_s^{(j)}(x)\big\|_{L^1} \le \mathrm{const}\, M^{(1+|\beta|)j}.$$

ii) For $1 \le m \le j$ *and* $s \in \Sigma_m$,

$$\big\|x^\beta c_s^{(j)}(x)\big\|_{L^\infty} \le \mathrm{const}\, \mathfrak{l}_m M^{(|\beta|-1)j}.$$

iii) For $m \ge 1$ *and* $s \in \Sigma_m$,

$$\big\|c_s^{(j)}(x)\big\|_{L^\infty} \le \mathrm{const}\,\tfrac{\mathfrak{l}_m}{M^j} \qquad \Big\|\tfrac{\partial}{\partial x_0} c_s^{(j)}(x)\Big\|_{L^\infty} \le \mathrm{const}\,\tfrac{\mathfrak{l}_m}{M^{2j}}.$$

iv) For $s \in \Sigma_j$,

$$\Big\|\tfrac{\partial}{\partial x_0} c_s^{(j)}(x)\Big\|_{L^1} \le \mathrm{const}\,.$$

Proof. For any sector s' of any scale, $c_{s'}^{(j)}(k)$ is supported on the j^{th} shell and $\check{p}^{(i)}(k)$ is supported on the i^{th} extended neighbourhood. If $i > j+1$, the j shell and i^{th} extended neighbourhood do not intersect, so we may assume that $p^{(i)} = 0$ for all $i > j+1$. Therefore, by Cor. XIX.13 and Prop. XIX.4.iii of [FKTo4],

$$|\!|\!|\nu|\!|\!|_{1,\Sigma_j} \le \sum_{i=2}^{j+1} |\!|\!|p^{(i)}|\!|\!|_{1,\Sigma_j} \le \mathrm{const} \sum_{i=2}^{j+1} \tfrac{\rho\,\mathfrak{l}_i}{M^j}\mathfrak{c}_j \le \mathrm{const}\,\tfrac{\rho}{M^j}\mathfrak{c}_j.$$

Hence the hypotheses of Prop. XIII.5 of [FKTo3] are fulfilled. To apply this proposition, we let, for $s', s'' \in \Sigma_j$, $\tilde c\big((\xi, s'), (\xi', s'')\big)$ be the Fourier transform, as in Def. IX.3 of [FKTo2], of $\chi_{s'}(k)\, C_v^{(j)}(k)\, \chi_{s''}(k)$. Comparing this Fourier transform with that specified before Def. I.17, we see that

$$c_{s'}^{(j)}(x) = \sum_{s'' \in \Sigma_j} \tilde c\big((0, \uparrow, 0, s'), (x, \uparrow, a, s'')\big).$$

By conservation of momentum, only three s'''s give nonzero contributions to the right hand side for each s'. Hence, by parts (ii) and (iii) of Prop. XIII.5 of [FKTo3] and Cor. A.5.i of [FKTo1],

$$\sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \tfrac{1}{\delta!}\left\|x^\delta c^{(j)}_{s'}(x)\right\|_{L^1} t^\delta \le \mathit{const}\,\left|\tilde c\right|_{1,\Sigma_j} \le \mathrm{const}\,\frac{M^j\mathfrak{c}_j}{1-\mathrm{const}\,\rho\,\mathfrak{c}_j} \le \mathrm{const}\,M^j\mathfrak{c}_j, \quad \text{(A.1)}$$

$$\sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \tfrac{1}{\delta!}\left\|x^\delta c^{(j)}_{s'}(x)\right\|_{L^\infty} t^\delta \le \mathrm{const}\,\frac{\mathfrak{l}_j}{M^j}\frac{\mathfrak{c}_j}{1-\mathrm{const}\,\rho\,\mathfrak{c}_j} \le \mathrm{const}\,\frac{\mathfrak{l}_j}{M^j}\mathfrak{c}_j. \quad \text{(A.2)}$$

i) follows from (A.1) by choosing $s' = s$.

ii) By (A.2),

$$\left\|x^\delta c^{(j)}_{s'}(x)\right\|_{L^\infty} \le \mathrm{const}\,\frac{\mathfrak{l}_j}{M^j} M^{|\delta| j}$$

for all $s' \in \Sigma_j$ and $\delta \in \Delta$. Write $c^{(j)}_s(k) = \sum_{\substack{s'\in\Sigma_j\\ s\cap s'\neq\emptyset}} c^{(j)}_{s'}(k)\chi_s(k)$. By Lemma XII.3.iii of [FKTo3],

$$\begin{aligned}\left\|x^\beta c^{(j)}_s(x)\right\|_{L^\infty} &\le \mathit{const} \sum_{\substack{s'\in\Sigma_j\\ s\cap s'\neq\emptyset}} \sum_{\substack{\delta,\delta'\in\mathbb{N}_0\times\mathbb{N}_0^2\\ \delta+\delta'=\beta}} \left\|x^\delta c^{(j)}_{s'}(x)\right\|_{L^\infty}\left\|x^{\delta'}\hat\chi_s(x)\right\|_{L^1}\\ &\le \mathrm{const} \sum_{\substack{s'\in\Sigma_j\\ s\cap s'\neq\emptyset}} \sum_{\substack{\delta,\delta'\in\mathbb{N}_0\times\mathbb{N}_0^2\\ \delta+\delta'=\beta}} \frac{\mathfrak{l}_j}{M^j} M^{|\delta| j} M^{|\delta'| j} \le \mathrm{const}\,\frac{\mathfrak{l}_m}{\mathfrak{l}_j}\frac{\mathfrak{l}_j}{M^j} M^{|\beta| j}.\end{aligned}$$

iii) follows from the observations that

$$\sup_k \left|c^{(j)}_s(k)\right| \le \mathrm{const}\,M^j, \qquad \sup_k \left|k_0 c^{(j)}_s(k)\right| \le \mathrm{const}\,,$$

and $c^{(j)}_s(k)$ is supported in a region of volume const $\frac{\mathfrak{l}_m}{M^{2j}}$.

iv) follows from part (iv) of Prop. XIII.5 of [FKTo3]. □

Recall from Lemma III.25 that

$$\Delta^{m_1,m_2}_{s_1,s_2}(\mathbf{p},\mathbf{k}) = \int_{-b(m_1,m_2)}^{b(m_1,m_2)} d\omega\, \frac{\mu^{(m_1,m_2)}(\omega,\mathbf{k},\mathbf{p})\chi_{s_1}(\mathbf{p})\chi_{s_2}(\mathbf{k})}{[i\omega - e'(\omega,\mathbf{p})][i\omega - e'(\omega,\mathbf{k})]},$$

where

$$\mu^{(m_1,m_2)}(\omega,\mathbf{k},\mathbf{p}) = \Big[1-\nu_0(\omega)\nu_1(\mathbf{p},\mathbf{k})\Big]\nu^{(m_1)}((\omega,\mathbf{p}))\nu^{(m_2)}((\omega,\mathbf{k}))$$

and

$$b(m_1,m_2) = \begin{cases} \frac{\mathrm{const}}{M^{\max\{m_1,m_2\}}} & \text{if } m = i, i+1\\ \min\left\{\frac{1}{M^{j-3/4}}, \frac{\mathrm{const}}{M^{\max\{m_1,m_2\}}}\right\} & \text{if } m \ge i+2\end{cases}.$$

The functions ν_0 and ν_1 were defined just before (III.22).

Lemma A.3. *Let $i \le m \le j$, $\min\{m_1,m_2\} = m$ and $s_1, s_2 \in \Sigma_m$. Then*

$$\int d\mathbf{z}_1 \sup_{\mathbf{z}_2}\left|\hat\Delta^{m_1,m_2}_{s_1,s_2}(\mathbf{z}_1,\mathbf{z}_2)\right| \le \mathrm{const}\, b(m_1,m_2)\,\mathfrak{l}_m^2\frac{M^{m_1}}{\mathfrak{l}_{m_1}},$$

$$\int d\mathbf{z}_2 \sup_{\mathbf{z}_1}\left|\hat\Delta^{m_1,m_2}_{s_1,s_2}(\mathbf{z}_1,\mathbf{z}_2)\right| \le \mathrm{const}\, b(m_1,m_2)\,\mathfrak{l}_m^2\frac{M^{m_2}}{\mathfrak{l}_{m_2}}.$$

Proof. We may write $\Delta^{m_1,m_2}_{s_1,s_2}(\mathbf{p},\mathbf{k})$ as a sum of two pieces, each of the form

$$\sum_{\substack{u_1,v_1\in\Sigma_{m_1}\\ u_1\cap s_1\neq\emptyset\\ u_1\cap v_1\neq\emptyset}} \sum_{\substack{u_2,v_2\in\Sigma_{m_2}\\ u_2\cap s_2\neq\emptyset\\ u_2\cap v_2\neq\emptyset}} \Delta_{u_1,v_1,u_2,v_2}(\mathbf{p},\mathbf{k})$$

with

$$\Delta_{u_1,v_1,u_2,v_2}(\mathbf{p},\mathbf{k}) = \pm\int d\omega\, \zeta_{m_1,m_2}(\omega) c^{(m_1)}_{u_1}(\omega,\mathbf{p}) c^{(m_2)}_{u_2}(\omega,\mathbf{k}) \chi_{1,v_1}(\omega,\mathbf{p}) \chi_{2,v_2}(\omega,\mathbf{k}),$$

where

$$\begin{aligned}
\chi_{1,v_1}(\omega,\mathbf{p}) &= \left\{\begin{matrix} 1 & \text{or} \\ \sum\limits_{\ell=i+1}^{\infty} \nu\big(M^{2\ell} e(\mathbf{p})^2\big) & \end{matrix}\right\} \chi_{v_1}(\omega,\mathbf{p})\chi_{s_1}(0,\mathbf{p}),\\
\chi_{2,v_2}(\omega,\mathbf{k}) &= \left\{\begin{matrix} 1 & \text{or} \\ \sum\limits_{\ell=i+1}^{\infty} \nu\big(M^{2\ell} e(\mathbf{k})^2\big) & \end{matrix}\right\} \chi_{v_2}(\omega,\mathbf{k})\chi_{s_2}(0,\mathbf{k}),\\
\zeta_{m_1,m_2}(\omega) &= \left\{\begin{matrix} 1 & \text{or} \\ \nu_0(\omega) & \end{matrix}\right\} \chi_{m_1,m_2}(\omega),
\end{aligned}$$

and $\chi_{m_1,m_2}(\omega)$ is the characteristic function of the interval $\big[-b(m_1,m_2), b(m_1,m_2)\big]$. The Fourier transform of $\Delta_{u_1,v_1,u_2,v_2}(\mathbf{p},\mathbf{k})$ is then

$$\begin{aligned}
\hat{\Delta}_{u_1,v_1,u_2,v_2}(\mathbf{z}_1,\mathbf{z}_2) = \int d\mathbf{x}_1\, d\mathbf{x}_2 \prod_{\ell=1}^{4} dt_\ell\; & c^{(m_1)}_{u_1}(t_1,\mathbf{x}_1)\hat{\chi}_{1,v_1}(t_2-t_1,\mathbf{z}_1-\mathbf{x}_1)\\
&\times c^{(m_2)}_{u_2}(t_3-t_2,\mathbf{x}_2)\hat{\chi}_{2,v_2}(t_4-t_3,\mathbf{z}_2-\mathbf{x}_2)\hat{\zeta}_{m_1,m_2}(-t_4).
\end{aligned}$$

This is bounded using

$$\begin{aligned}
\sup_{t_4} \big|\hat{\zeta}_{m_1,m_2}(-t_4)\big| &\le 2b(m_1,m_2),\\
\sup_{t_3,\mathbf{x}_2,\mathbf{z}_2} \int dt_4\, \big|\hat{\chi}_{2,v_2}(t_4-t_3,\mathbf{z}_2-\mathbf{x}_2)\big| &\le \text{const}\, \tfrac{\mathfrak{l}_{m_2}}{M^{m_2}},\\
\sup_{t_2} \int dt_3\, d\mathbf{x}_2\, \big|c^{(m_2)}_{u_2}(t_3-t_2,\mathbf{x}_2)\big| &\le \text{const}\, M^{m_2}, \qquad\qquad (A.3)\\
\sup_{t_1,\mathbf{x}_1} \int dt_2\, d\mathbf{z}_1\, \big|\hat{\chi}_{1,v_1}(t_2-t_1,\mathbf{z}_1-\mathbf{x}_1)\big| &\le \text{const}\,,\\
\int dt_1\, d\mathbf{x}_1\, \big|c^{(m_1)}_{u_1}(t_1,\mathbf{x}_1)\big| &\le \text{const}\, M^{m_1}.
\end{aligned}$$

The supremum of $|\hat{\zeta}_{m_1,m_2}(-t_4)|$ was bounded by the L^1 norm of $\zeta_{m_1,m_2}(\omega)$. The bounds on $c^{(m_\ell)}_{u_\ell}$, $\ell=1,2$ are immediate consequences of Lemma A.2.i with $\beta=0$. The bounds on $\hat{\chi}_{\ell,v_\ell}$, $\ell=1,2$ are proven much as Lemma XIII.3 of [FKTo3]. Indeed (XIII.3) of [FKTo3] applies with $j=m_\ell$ and $\mathfrak{l}=\mathfrak{l}_{m_\ell}$. Denoting by $\mathbf{x}_\perp$ and $\mathbf{x}_\parallel$ the components of

$\mathbf{x}$ perpendicular and parallel, respectively, to the Fermi curve at the centre of v_ℓ, Prop. XIII.1.i gives

$$\big|\hat{\chi}_{\ell,v_\ell}(t,\mathbf{x})\big| \le \text{const}\,\tfrac{\mathfrak{l}_{m_\ell}}{M^{2m_\ell}}\,\tfrac{1}{\left[1+M^{-m_\ell}|t|\right]^2\,\left[1+M^{-m_\ell}|\mathbf{x}_\perp|+\mathfrak{l}_{m_\ell}|\mathbf{x}_\parallel|\right]^3}$$

from which the desired bounds follow. Applying, in order, the bounds of (A.3) yields

$$\int d\mathbf{z}_1 \sup_{\mathbf{z}_2}\big|\hat{\Delta}_{u_1,v_1,u_2,v_2}(\mathbf{z}_1,\mathbf{z}_2)\big| \le \text{const}\,b(m_1,m_2)\;\mathfrak{l}_{m_2}M^{m_1}.$$

Similarly,

$$\int d\mathbf{z}_2 \sup_{\mathbf{z}_1}\big|\hat{\Delta}_{u_1,v_1,u_2,v_2}(\mathbf{z}_1,\mathbf{z}_2)\big| \le \text{const}\,b(m_1,m_2)\;\mathfrak{l}_{m_1}M^{m_2}.$$

For each fixed $s_1 \in \Sigma_m$, there are at most const $\frac{\mathfrak{l}_m}{\mathfrak{l}_{m_1}}$ pairs $(u_1,v_1) \in \Sigma_{m_1}^2$ obeying $u_1 \cap s_1 \ne \emptyset$, $u_1 \cap v_1 \ne \emptyset$ and for each fixed $s_2 \in \Sigma_m$, there are at most const $\frac{\mathfrak{l}_m}{\mathfrak{l}_{m_2}}$ pairs $(u_2,v_2) \in \Sigma_{m_2}^2$ obeying $u_2 \cap s_2 \ne \emptyset$, $u_2 \cap v_2 \ne \emptyset$. Hence

$$\begin{aligned}
\int d\mathbf{z}_1 \sup_{\mathbf{z}_2}\big|\hat{\Delta}^{m_1,m_2}_{s_1,s_2}(\mathbf{z}_1,\mathbf{z}_2)\big| &\le \text{const}\,b(m_1,m_2)\;\tfrac{\mathfrak{l}_m}{\mathfrak{l}_{m_1}}\,\tfrac{\mathfrak{l}_m}{\mathfrak{l}_{m_2}}\;\mathfrak{l}_{m_2}M^{m_1},\\
&\le \text{const}\,b(m_1,m_2)\;\mathfrak{l}_m^2\,\tfrac{M^{m_1}}{\mathfrak{l}_{m_1}},\\
\int d\mathbf{z}_2 \sup_{\mathbf{z}_1}\big|\hat{\Delta}^{m_1,m_2}_{s_1,s_2}(\mathbf{z}_1,\mathbf{z}_2)\big| &\le \text{const}\,b(m_1,m_2)\;\tfrac{\mathfrak{l}_m}{\mathfrak{l}_{m_1}}\,\tfrac{\mathfrak{l}_m}{\mathfrak{l}_{m_2}}\;\mathfrak{l}_{m_1}M^{m_2}\\
&\le \text{const}\,b(m_1,m_2)\;\mathfrak{l}_m^2\,\tfrac{M^{m_2}}{\mathfrak{l}_{m_2}}. \quad \square
\end{aligned}$$

Appendix B. Bound on the Generalized Model Bubble

Fix, as in Theorem I.20, a sequence, $p^{(2)}, p^{(3)}, \cdots$, of sectorized, translation invariant functions $p^{(i)}$ on $\left(\left(\mathbb{R}\times\mathbb{R}^2\right)\times\Sigma_i\right)^2$ obeying

$$|\!|\!|p^{(i)}|\!|\!|_{1,\Sigma_i} \le \tfrac{\rho\,\mathfrak{l}_i}{M^i}\,\mathfrak{c}_i, \qquad \check{p}^{(i)}(0,\mathbf{k}) = 0.$$

Let I be an interval of length $\mathfrak{l}$ on the Fermi surface F and $u(\mathbf{k},\mathbf{t})$ a function that vanishes unless $\pi_F(\mathbf{k}) \in I$, where π_F is projection on the Fermi curve F. Set, for $1 \le i \le j$,

$$B_{i,j}(\mathbf{t}) = \int dk\;\frac{\nu_0^{(i,j)}(k_0)\,u(\mathbf{k},\mathbf{t})}{[\imath k_0 - e'(k_0,\mathbf{k})][\imath k_0 - e'(k_0,\mathbf{k}+\mathbf{t})]},$$

where

$$\nu_0^{(i,j)}(k_0) = \sum_{\ell=i+1}^{j-1}\nu(M^{2\ell}k_0^2), \qquad e'(k) = e(\mathbf{k}) - \sum_{\ell=2}^{j+1}\check{p}^{(\ell)}(k).$$

Lemma B.1. *Let* $1 \le i \le j$ *obey* $M^i \le \mathfrak{l}_j M^j$. *Then*

$$\left|\partial_{\mathbf{t}}^{\alpha} B_{i,j}(\mathbf{t})\right| \le \mathrm{const}\, \mathfrak{l} \max\{1,\ j\, \mathfrak{l}_j M^{|\alpha| j}\} \max_{|\beta+\gamma|\le|\alpha|} \sup_{\mathbf{k},\mathbf{t}} \tfrac{1}{M^{i|\beta+\gamma|}} \left|\partial_{\mathbf{t}}^{\beta} \partial_{\mathbf{k}}^{\gamma} u(\mathbf{k},\mathbf{t})\right|$$

for all $|\alpha| \le 4$ *and all* $\mathbf{t}$ *in a neighbourhood of the origin.*

Proof. Let

$$\begin{aligned}
E'(k_0,\mathbf{k},\mathbf{t},s) &= s e'(k_0,\mathbf{k}) + (1-s) e'(k_0,\mathbf{k}+\mathbf{t}),\\
E(\mathbf{k},\mathbf{t},s) &= E'(0,\mathbf{k},\mathbf{t},s) = s e(\mathbf{k}) + (1-s) e(\mathbf{k}+\mathbf{t}),\\
w(\mathbf{k},\mathbf{t},s) &= 1 - \tfrac{1}{\imath} \tfrac{\partial E'}{\partial k_0}(0,\mathbf{k},\mathbf{t},s),\\
\tilde{E}(k_0,\mathbf{k},\mathbf{t},s) &= E'(k_0,\mathbf{k},\mathbf{t},s) - E'(0,\mathbf{k},\mathbf{t},s) - k_0 \tfrac{\partial E'}{\partial k_0}(0,\mathbf{k},\mathbf{t},s).
\end{aligned}$$

Then

$$\begin{aligned}
B_{i,j}(\mathbf{t}) &= \int dk\, \frac{\nu_0^{(i,j)}(k_0) u(\mathbf{k},\mathbf{t})}{[\imath k_0 - e'(k_0,\mathbf{k})][\imath k_0 - e'(k_0,\mathbf{k}+\mathbf{t})]}\\
&= \int dk\, \frac{\nu_0^{(i,j)}(k_0) u(\mathbf{k},\mathbf{t})}{e'(k_0,\mathbf{k}) - e'(k_0,\mathbf{k}+\mathbf{t})} \left[\frac{1}{\imath k_0 - e'(k_0,\mathbf{k})} - \frac{1}{\imath k_0 - e'(k_0,\mathbf{k}+\mathbf{t})}\right]\\
&= \int dk \int_0^1 ds\, \frac{\nu_0^{(i,j)}(k_0) u(\mathbf{k},\mathbf{t})}{[\imath k_0 - E'(k_0,\mathbf{k},\mathbf{t},s)]^2}\\
&= \int dk \int_0^1 ds\, \frac{\nu_0^{(i,j)}(k_0) u(\mathbf{k},\mathbf{t})}{[\imath w(\mathbf{k},\mathbf{t},s) k_0 - E(\mathbf{k},\mathbf{t},s) - \tilde{E}(k_0,\mathbf{k},\mathbf{t},s)]^2}.
\end{aligned} \tag{B.1}$$

Case i. $|\alpha| \ge 1$. Make, for each fixed s, the change of variables from $\mathbf{k}$ to E and an "angular" variable θ. Denote by $J(E,\mathbf{t},\theta,s)$ the Jacobian of this change of variables. Then

$$B_{i,j}(\mathbf{t}) = \int_0^1 ds \int dk_0 \int d\theta dE\, \frac{\nu_0^{(i,j)}(k_0) u\big(\mathbf{k}(E,\mathbf{t},\theta,s),\mathbf{t}\big) J(E,\mathbf{t},\theta,s)}{[\imath w(\mathbf{k}(E,\mathbf{t},\theta,s),\mathbf{t},s) k_0 - E - \tilde{E}(k_0,\mathbf{k}(E,\mathbf{t},\theta,s),\mathbf{t},s)]^2}. \tag{B.2}$$

Since $E(\mathbf{k},\mathbf{t},s) = s e(\mathbf{k}) + (1-s) e(\mathbf{k}+\mathbf{t})$ and $\mathbf{t}$ is restricted to a small neighbourhood of the origin,

$$\left|\nabla_{\mathbf{k}} E(\mathbf{k},\mathbf{t},s)\right| \ge \mathit{const} > 0, \qquad \left|\partial_k^{\alpha} \partial_{\mathbf{t}}^{\beta} E(\mathbf{k},\mathbf{t},s)\right| \le \mathit{const}' \tag{B.3}$$

for all $\alpha + \beta$ having spatial component at most r. Using

$$\partial_{\mathbf{t}_i} \mathbf{k}_\ell(E,\mathbf{t},\theta,s) = -\tfrac{\partial_{\mathbf{k}_2}\theta(\mathbf{k})\delta_{\ell,1} - \partial_{\mathbf{k}_1}\theta(\mathbf{k})\delta_{\ell,2}}{\partial_{\mathbf{k}_1} E(\mathbf{k},\mathbf{t},s)\, \partial_{\mathbf{k}_2}\theta(\mathbf{k}) - \partial_{\mathbf{k}_2} E(\mathbf{k},\mathbf{t},s)\, \partial_{\mathbf{k}_1}\theta(\mathbf{k})} \partial_{\mathbf{t}_i} E(\mathbf{k},\mathbf{t},s)\Big|_{\mathbf{k}=\mathbf{k}(E,\mathbf{t},\theta,s)},$$

one proves, by induction on $|\beta|$, that, for $|\beta| \le r$,

$$\left|\partial_{\mathbf{t}}^{\beta} \mathbf{k}(E,\mathbf{t},\theta,s)\right| \le \mathit{const}''. \tag{B.4}$$

Using this bound and

$$J(E,\mathbf{t},\theta,s) = \tfrac{1}{|\partial_{\mathbf{k}_1} E(\mathbf{k},\mathbf{t},s)\, \partial_{\mathbf{k}_2}\theta(\mathbf{k}) - \partial_{\mathbf{k}_2} E(\mathbf{k},\mathbf{t},s)\, \partial_{\mathbf{k}_1}\theta(\mathbf{k})|}\Big|_{\mathbf{k}=\mathbf{k}(E,\mathbf{t},\theta,s)},$$

one proves that, for $|\beta| \le r-1$,

$$\left|\partial_{\mathbf{t}}^{\beta} J(E,\mathbf{t},\theta,s)\right| \le const''. \tag{B.5}$$

By Lemma A.1.ii, for $\alpha+\beta+(1,0,0)\in\Delta$,

$$\left|\partial_k^{\alpha}\partial_{\mathbf{t}}^{\beta} w(\mathbf{k},\mathbf{t},s)\right| \le const \begin{cases} 1 & \text{if } |\alpha|+|\beta|=0 \\ \mathfrak{l}_j M^{(|\alpha|+|\beta|)j} & \text{if } |\alpha|+|\beta|\ge 1 \end{cases}. \tag{B.6}$$

Since

$$\begin{aligned}\tilde{E}(k_0,\mathbf{k},\mathbf{t},s) &= E'(k_0,\mathbf{k},\mathbf{t},s) - E'(0,\mathbf{k},\mathbf{t},s) - k_0\tfrac{\partial E'}{\partial k_0}(0,\mathbf{k},\mathbf{t},s)\\ &= \int_0^{k_0} d\kappa\,\left[\tfrac{\partial E'}{\partial k_0}(\kappa,\mathbf{k},\mathbf{t},s) - \tfrac{\partial E'}{\partial k_0}(0,\mathbf{k},\mathbf{t},s)\right],\end{aligned}$$

parts (ii) and (iii) of Lemma A.1 imply that, again for $\alpha+\beta+(1,0,0)\in\Delta$,

$$\left|\partial_k^{\alpha}\partial_{\mathbf{t}}^{\beta}\tilde{E}(k_0,\mathbf{k},\mathbf{t},s)\right| \le const\,\rho \begin{cases} |k_0|^{1+\aleph} & \text{if } |\alpha|+|\beta|=0 \\ |k_0|^{\aleph}+|k_0|\mathfrak{l}_j M^j & \text{if } |\alpha|+|\beta|=1 \\ \mathfrak{l}_j M^{(|\alpha|+|\beta|-1)j}+|k_0|\mathfrak{l}_j M^{(|\alpha|+|\beta|)j} & \text{if } |\alpha|+|\beta|>1 \end{cases}.$$

For k_0 in the support of $\nu_0^{(i,j)}(k_0)$, $|k_0|\ge \text{const}\,\frac{1}{M^j}$ and

$$\left|\partial_k^{\alpha}\partial_{\mathbf{t}}^{\beta}\tilde{E}(k_0,\mathbf{k},\mathbf{t},s)\right| \le \text{const}\;\rho \begin{cases} |k_0|^{1+\aleph} & \text{if } |\alpha|+|\beta|=0 \\ |k_0|\mathfrak{l}_j M^{(|\alpha|+|\beta|)j} & \text{if} |\alpha|+|\beta|\ge 1 \end{cases}. \tag{B.7}$$

Applying $\partial_{\mathbf{t}}^{\alpha}$ to (B.2) yields an integral whose integrand is a sum of terms (whose number is bounded by a universal constant) of the form a combinatorial factor (which is bounded by a universal constant) times $\nu_0^{(i,j)}(k_0)$ times

$$\begin{aligned}&\tfrac{1}{[\imath w k_0 - E - \tilde{E}]^{m+2}} \prod_{p=1}^{m}\Big[-\partial_{\mathbf{k}}^{\gamma^{(p)}}\partial_{\mathbf{t}}^{\beta^{(p)}}(\imath k_0 w-\tilde{E})\prod_{\ell=1}^{|\gamma^{(p)}|}\partial_{\mathbf{t}}^{\alpha^{(p,\ell)}}\mathbf{k}_{i_\ell}\Big]\\ &\quad\times\Big[\partial_{\mathbf{k}}^{\gamma'}\partial_{\mathbf{t}}^{\beta'}u\prod_{\ell'=1}^{|\gamma'|}\partial_{\mathbf{t}}^{\alpha'^{(\ell')}}\mathbf{k}_{i_{\ell'}}\Big]\quad\partial_{\mathbf{t}}^{\beta''}J\end{aligned}$$

with the various degrees obeying

$$\begin{aligned}&\beta''+\beta'+\sum_{\ell'=1}^{|\gamma'|}\alpha'^{(\ell')}+\sum_{p=1}^{m}\Big[\beta^{(p)}+\sum_{\ell=1}^{|\gamma^{(p)}|}\alpha^{(p,\ell)}\Big]=\alpha,\\ &|\gamma^{(p)}|+|\beta^{(p)}|\ge 1 \qquad \text{for all } 1\le p\le m,\\ &|\alpha^{(p,\ell)}|\ge 1 \qquad \text{for all } 1\le p\le m,\ 1\le\ell\le|\gamma^{(p)}|,\\ &|\alpha'^{(\ell')}|\ge 1 \qquad \text{for all } 1\le\ell'\le|\gamma'|.\end{aligned}$$

Using the fact, from Lemma A.1.i, that $|w(\mathbf{k},\mathbf{t},s)-1|\le\rho\le\frac{1}{2}$ and the bounds on the derivatives of $\mathbf{k}$, J, w and $\tilde{E}$ of (B.4–B.7), we may bound this term by

$$const'\,\tfrac{1}{|\imath k_0 - E|^{m+2}}\prod_{p=1}^{m}\Big[|k_0|\mathfrak{l}_j M^{(|\gamma^{(p)}|+|\beta^{(p)}|)j}\Big]\sup_{\mathbf{k},\mathbf{t}}\left|\partial_{\mathbf{k}}^{\gamma'}\partial_{\mathbf{t}}^{\beta'}u\right|$$

$$\le const' \tfrac{|k_0|^m}{|\imath k_0-E|^{m+2}} \prod_{p=1}^m \Big[\mathfrak{l}_j M^{(|\gamma^{(p)}|+|\beta^{(p)}|)j}\Big] M^{i(|\beta'|+|\gamma'|)}$$
$$\times \max_{|\beta+\gamma|\le|\alpha|} \sup_{\mathbf{k},\mathbf{t}} \tfrac{1}{M^{i|\beta+\gamma|}} \big|\partial_{\mathbf{t}}^\beta \partial_{\mathbf{k}}^\gamma u(\mathbf{k},\mathbf{t})\big|$$
$$\le const' \tfrac{|k_0|^m}{|\imath k_0-E|^{m+2}} M^{|\alpha-\beta''|j} M^{|\beta'+\gamma'|(i-j)} \mathfrak{l}_j^m \max_{|\beta+\gamma|\le|\alpha|} \sup_{\mathbf{k},\mathbf{t}} \tfrac{1}{M^{i|\beta+\gamma|}} \big|\partial_{\mathbf{t}}^\beta \partial_{\mathbf{k}}^\gamma u(\mathbf{k},\mathbf{t})\big|$$
$$\le const' \tfrac{1}{|\imath k_0-E|^2} M^{|\alpha|j} \mathfrak{l}_j \max_{|\beta+\gamma|\le|\alpha|} \sup_{\mathbf{k},\mathbf{t}} \tfrac{1}{M^{i|\beta+\gamma|}} \big|\partial_{\mathbf{t}}^\beta \partial_{\mathbf{k}}^\gamma u(\mathbf{k},\mathbf{t})\big|.$$

For the second inequality, we used that $|\beta'|+|\gamma'|+\sum_{p=1}^m[|\gamma^{(p)}|+|\beta^{(p)}|] \le |\alpha-\beta''|$. For the final inequality we used that one of m, $|\gamma'|$, $|\beta'|$, $|\beta''|$ must be nonzero for $|\alpha|$ to be nonzero and we also used the hypothesis that $M^{i-j} \le \mathfrak{l}_j$. The bound is completed by applying

$$\int_{\frac{\rm const}{M^j}}^{\rm const} dk_0 \int_I d\theta \int_{-\rm const}^{\rm const} dE \tfrac{1}{|\imath k_0-E|^2} \le const'\, \mathfrak{l} \int_{\frac{\rm const}{M^j}}^{\rm const} dR\, \tfrac{1}{R} \le {\rm const}\ j\mathfrak{l}.$$

Case ii. $\alpha=0$. Recall from (B.1) that

$$B_{i,j}(\mathbf{t}) = \int dk \int_0^1 ds\, \frac{\nu_0^{(i,j)}(k_0)u(\mathbf{k},\mathbf{t})}{[\imath w(\mathbf{k},\mathbf{t},s)k_0 - E(\mathbf{k},\mathbf{t},s) - \tilde{E}(k_0,\mathbf{k},\mathbf{t},s)]^2}$$

and set

$$B'_{i,j}(\mathbf{t}) = \int dk \int_0^1 ds\, \frac{\nu_0^{(i,j)}(k_0)u(\mathbf{k},\mathbf{t})}{[\imath w(\mathbf{k},\mathbf{t},s)k_0 - E(\mathbf{k},\mathbf{t},s)]^2}.$$

By Lemma A.1.i, (B.7) and the reality of k_0 and $E(\mathbf{k},\mathbf{t},s)$,

$$\tfrac14\big|\imath k_0 - E(\mathbf{k},\mathbf{t},s)\big| \le \big|\imath w(\mathbf{k},\mathbf{t},s)k_0 - E(\mathbf{k},\mathbf{t},s) - \tilde{E}(k_0,\mathbf{k},\mathbf{t},s)\big| \le 2\big|\imath k_0 - E(\mathbf{k},\mathbf{t},s)\big|.$$

Hence, by (B.7),

$$\big|B_{i,j}(\mathbf{t}) - B'_{i,j}(\mathbf{t})\big|$$
$$\le \int dk \int_0^1 ds\, |\nu_0^{(i,j)}(k_0)u(\mathbf{k},\mathbf{t})|$$
$$\times \Big| \tfrac{[\imath w(\mathbf{k},\mathbf{t},s)k_0-E(\mathbf{k},\mathbf{t},s)]^2-[\imath w(\mathbf{k},\mathbf{t},s)k_0-E(\mathbf{k},\mathbf{t},s)-\tilde{E}(k_0,\mathbf{k},\mathbf{t},s)]^2}{[\imath w(\mathbf{k},\mathbf{t},s)k_0-E(\mathbf{k},\mathbf{t},s)]^2[\imath w(\mathbf{k},\mathbf{t},s)k_0-E(\mathbf{k},\mathbf{t},s)-\tilde{E}(k_0,\mathbf{k},\mathbf{t},s)]^2} \Big|$$
$$= \int dk \int_0^1 ds\, |\nu_0^{(i,j)}(k_0)u(\mathbf{k},\mathbf{t})|$$
$$\times \Big| \tfrac{\tilde{E}(k_0,\mathbf{k},\mathbf{t},s)[2\imath w(\mathbf{k},\mathbf{t},s)k_0-2E(\mathbf{k},\mathbf{t},s)-\tilde{E}(k_0,\mathbf{k},\mathbf{t},s)]}{[\imath w(\mathbf{k},\mathbf{t},s)k_0-E(\mathbf{k},\mathbf{t},s)]^2[\imath w(\mathbf{k},\mathbf{t},s)k_0-E(\mathbf{k},\mathbf{t},s)-\tilde{E}(k_0,\mathbf{k},\mathbf{t},s)]^2} \Big|$$
$$\le \int dk \int_0^1 ds\, {\rm const}\, |k_0|^{\aleph} |\nu_0^{(i,j)}(k_0)u(\mathbf{k},\mathbf{t})| \tfrac{|\tilde{E}(k_0,\mathbf{k},\mathbf{t},s)|}{|\imath k_0-E(\mathbf{k},\mathbf{t},s)|^3}$$
$$\le \int dk \int_0^1 ds\, {\rm const}\, |k_0|^{\aleph} |\nu_0^{(i,j)}(k_0)u(\mathbf{k},\mathbf{t})| \tfrac{1}{|\imath k_0-E(\mathbf{k},\mathbf{t},s)|^2}$$
$$\le {\rm const} \sup_{\mathbf{k},\mathbf{t}} |u(\mathbf{k},\mathbf{t})| \int_I d\theta \int_{-const}^{\rm const} dk_0 \int_{-\rm const}^{\rm const} dE\, \tfrac{|k_0|^{\aleph}}{|\imath k_0-E|^2}$$
$$\le {\rm const} \sup_{\mathbf{k},\mathbf{t}} |u(\mathbf{k},\mathbf{t})| \int_I d\theta \int_{-const}^{\rm const} dk_0\, |k_0|^{\aleph-1} \int dE'\, \tfrac{1}{|\imath-E'|^2}$$

$$\le \text{const}\, \mathfrak{l} \sup_{\mathbf{k},\mathbf{t}} |u(\mathbf{k},\mathbf{t})|,$$

and it suffices to consider $B'_{i,j}(\mathbf{t})$.

Make, for each fixed s, the change of variables from $\mathbf{k}$ to $E = E(\mathbf{k},\mathbf{t},s)$ and an "angular" variable θ. Then

$$B'_{i,j}(\mathbf{t}) = \textstyle\int_0^1 ds \int d\theta \int dk_0 dE\ \nu_0^{(i,j)}(k_0) \dfrac{u\big(\mathbf{k}(E,\mathbf{t},\theta,s),\mathbf{t}\big) J(E,\mathbf{t},\theta,s)}{[\imath w(\mathbf{k}(E,\mathbf{t},\theta,s),\mathbf{t},s)k_0 - E]^2}.$$

Integrating by parts with respect to k_0,

$$\begin{aligned} B'_{i,j}(\mathbf{t}) &= \int_0^1 ds \int d\theta dE \int_{-\infty}^{\infty} dk_0\ \nu_0^{(i,j)}(k_0)\ \tfrac{uJ}{w}\ \imath \tfrac{d}{dk_0} \tfrac{1}{\imath w k_0 - E} \\ &= \int_0^1 ds \int d\theta dE \int_{-\infty}^{\infty} dk_0\ \frac{\frac{uJ}{w}\ (-\imath) \frac{d}{dk_0}\nu_0^{(i,j)}(k_0)}{\imath w k_0 - E}. \end{aligned}$$

Since $\frac{d}{dk_0}\nu_0^{(i,j)}(k_0)$ is odd under $k_0 \to -k_0$,

$$\begin{aligned} B'_{i,j}(\mathbf{t}) &= \int_0^1 ds \int d\theta dE \int_0^{\infty} dk_0\ \tfrac{uJ}{w}\ (-\imath)\Big[\tfrac{d}{dk_0}\nu_0^{(i,j)}(k_0)\Big]\Big[\tfrac{1}{\imath w k_0 - E} - \tfrac{1}{-\imath w k_0 - E}\Big] \\ &= \int_0^1 ds \int d\theta dE \int_0^{\infty} dk_0 \tfrac{uJ}{w}\ (-\imath)\Big[\tfrac{d}{dk_0}\nu_0^{(i,j)}(k_0)\Big] \tfrac{-2\imath w k_0}{w^2 k_0^2 + E^2} \\ &= -2 \int_0^1 ds \int d\theta \int_{-\infty}^{\infty} dE \int_0^{\infty} dk_0\ \Big[\tfrac{d}{dk_0}\nu_0^{(i,j)}(k_0)\Big] \tfrac{u(\mathbf{k}(E,\mathbf{t},\theta,s),\mathbf{t}) J(E,\mathbf{t},\theta,s) k_0}{w(\mathbf{k}(E,\mathbf{t},\theta,s),\mathbf{t},s)^2 k_0^2 + E^2}. \end{aligned}$$

Hence, since $\big|w(\mathbf{k}(E,\mathbf{t},\theta,s),\mathbf{t},s) - 1\big| \le \rho \le \frac{1}{5}$, $\big|w(\mathbf{k}(E,\mathbf{t},\theta,s),\mathbf{t},s)^2 - 1\big| \le \frac{1}{2}$ and

$$\big|B'_{i,j}(\mathbf{t})\big| \le 4 \sup |uJ| \int_0^1 ds \int_{I'} d\theta \int_0^{\infty} dk_0 \int_{-\infty}^{\infty} dE\ \Big|\tfrac{d}{dk_0}\nu_0^{(i,j)}(k_0)\Big| \tfrac{k_0}{k_0^2 + E^2},$$

where I' is some interval of length const $\mathfrak{l}$. Finally,

$$\begin{aligned} \big|B'_{i,j}(\mathbf{t})\big| &\le \text{const}\, \mathfrak{l} \sup |u| \int_0^{\infty} dk_0\ \Big|\tfrac{d}{dk_0}\nu_0^{(i,j)}(k_0)\Big| \\ &\le \text{const}\, \mathfrak{l} \sup_{\mathbf{k},\mathbf{t}} |u(\mathbf{k},\mathbf{t})|, \end{aligned}$$

since $\int_0^{\infty} dk_0\ \big|\frac{d}{dk_0}\nu_0^{(i,j)}(k_0)\big| = 2$. □

Theorem B.2. *Let $1 \le i \le j$ obey $M^i \le \mathfrak{l}_j M^j$. Let $\mathfrak{t}$ and $\mathfrak{n}$ be mutually perpendicular unit vectors in $\mathbb{R}^2$ and $\rho(\mathbf{k})$ be a function that is supported in a rectangle in $\mathbf{k}$ having one side of length $\frac{\text{const}}{M^j}$ parallel to $\mathfrak{n}$ and one side of length const $\mathfrak{l}_j$ parallel to $\mathfrak{t}$. Furthermore assume that, for all $\alpha_1, \alpha_2 \le 2$,*

$$\Big|\big(\mathfrak{n}\cdot\partial_{\mathbf{k}}\big)^{\alpha_1}\big(\mathfrak{t}\cdot\partial_{\mathbf{k}}\big)^{\alpha_2}\rho(\mathbf{k})\Big| \le \text{const}\, M^{\alpha_1 j} \tfrac{1}{\mathfrak{l}_j^{\alpha_2}}.$$

Let $a = \frac{1/2}{1-\aleph}$ and $\hat{B}_{i,j}(\mathbf{x})$ be the Fourier transform of $\rho(\mathbf{k}) B_{i,j}(\mathbf{k})$. Then

$$\big|\hat{B}_{i,j}(\mathbf{x})\big| \le \text{const}\, \tfrac{\mathfrak{l}\,\mathfrak{l}_j}{M^j} \tfrac{1}{(1+|\mathfrak{n}\cdot\mathbf{x}/M^j|^{3/2})(1+|\mathfrak{l}_j\,\mathfrak{t}\cdot\mathbf{x}|^{(1+a)/2})} \max_{|\beta+\gamma|\le 3} \sup_{\mathbf{k},\mathbf{t}} \tfrac{1}{M^{i|\beta+\gamma|}} \big|\partial_{\mathbf{t}}^{\beta}\partial_{\mathbf{k}}^{\gamma} u(\mathbf{k},\mathbf{t})\big|.$$

Proof. Denote

$$U = \max_{|\beta+\gamma|\le 3} \sup_{\mathbf{k},\mathbf{t}} \tfrac{1}{M^{i|\beta+\gamma|}} \left|\partial_{\mathbf{t}}^{\beta}\partial_{\mathbf{k}}^{\gamma} u(\mathbf{k},\mathbf{t})\right|.$$

Note that $1 < a < \frac{3}{2}$. The first step is to prove that for all $\alpha_1 \in \{0,1\}$ and $0 \le \alpha_2 \le a$,

$$\left|\big(\mathfrak{n}\cdot\partial_{\mathbf{k}}\big)^{\alpha_1}\big(\mathfrak{t}\cdot\partial_{\mathbf{k}}\big)^{[\alpha_2]}\big[B_{i,j}(\mathbf{k}+q\mathfrak{t}) - B_{i,j}(\mathbf{k})\big]\right| \le C\,\mathfrak{l}U\,M^{\alpha_1 j}\tfrac{1}{\mathfrak{l}_j^{\alpha_2}}|q|^{\alpha_2-[\alpha_2]}, \tag{B.8}$$

where $[\alpha_2]$ is the integer part of α_2. Here C is a constant that is independent of i, j, $\mathbf{k}$ and q. To prove (B.8) when $[\alpha_2]=0$, apply Lemma B.1 twice to obtain

$$\begin{aligned}
\left|\big(\mathfrak{n}\cdot\partial_{\mathbf{k}}\big)^{\alpha_1}\big[B_{i,j}(\mathbf{k}+q\mathfrak{t})-B_{i,j}(\mathbf{k})\big]\right| &\le \left|\big(\mathfrak{n}\cdot\partial_{\mathbf{k}}\big)^{\alpha_1}B_{i,j}(\mathbf{k}+q\mathfrak{t})\right| + \left|\big(\mathfrak{n}\cdot\partial_{\mathbf{k}}\big)^{\alpha_1}B_{i,j}(\mathbf{k})\right| \\
&\le 2\,\mathrm{const}\,\mathfrak{l}U\,M^{\alpha_1 j} \\
\left|\big(\mathfrak{n}\cdot\partial_{\mathbf{k}}\big)^{\alpha_1}\big[B_{i,j}(\mathbf{k}+q\mathfrak{t})-B_{i,j}(\mathbf{k})\big]\right| &\le |q|\sup_{\mathbf{p}}\left|\big(\mathfrak{n}\cdot\partial_{\mathbf{p}}\big)^{\alpha_1}\big(\mathfrak{t}\cdot\partial_{\mathbf{p}}\big)B_{i,j}(\mathbf{p})\right| \\
&\le \mathrm{const}\,\mathfrak{l}U\,|q|\,j\mathfrak{l}_j\,M^{(\alpha_1+1)j}.
\end{aligned}$$

Multiplying the $(1-\alpha_2)^{\mathrm{th}}$ power of the first bound by the α_2^{th} power of the second gives

$$\begin{aligned}
\left|\big(\mathfrak{n}\cdot\partial_{\mathbf{k}}\big)^{\alpha_1}\big[B_{i,j}(\mathbf{k}+q\mathfrak{t})-B_{i,j}(\mathbf{k})\big]\right| &\le 2^{1-\alpha_2}\mathrm{const}\,\mathfrak{l}U\,|q|^{\alpha_2}M^{\alpha_1 j}\big(j\mathfrak{l}_j\,M^{j}\big)^{\alpha_2} \\
&= 2^{1-\alpha_2}\mathrm{const}\,\mathfrak{l}U\,M^{\alpha_1 j}\tfrac{1}{\mathfrak{l}_j^{\alpha_2}}|q|^{\alpha_2}\big(jM^{(1-2\aleph)j}\big)^{\alpha_2} \\
&\le C\,\mathfrak{l}U\,M^{\alpha_1 j}\tfrac{1}{\mathfrak{l}_j^{\alpha_2}}|q|^{\alpha_2}.
\end{aligned}$$

To prove (B.8) when $[\alpha_2]=1$, again apply Lemma B.1 twice to obtain

$$\begin{aligned}
\left|\big(\mathfrak{n}\cdot\partial_{\mathbf{k}}\big)^{\alpha_1}\big(\mathfrak{t}\cdot\partial_{\mathbf{k}}\big)\big[B_{i,j}(\mathbf{k}+q\mathfrak{t})-B_{i,j}(\mathbf{k})\big]\right| &\le 2\,\mathrm{const}\,\mathfrak{l}U\,j\mathfrak{l}_j\,M^{(\alpha_1+1)j}, \\
\left|\big(\mathfrak{n}\cdot\partial_{\mathbf{k}}\big)^{\alpha_1}\big(\mathfrak{t}\cdot\partial_{\mathbf{k}}\big)\big[B_{i,j}(\mathbf{k}+q\mathfrak{t})-B_{i,j}(\mathbf{k})\big]\right| &\le |q|\sup_{\mathbf{p}}\left|\big(\mathfrak{n}\cdot\partial_{\mathbf{p}}\big)^{\alpha_1}\big(\mathfrak{t}\cdot\partial_{\mathbf{p}}\big)^2B_{i,j}(\mathbf{p})\right| \\
&\le \mathrm{const}\,\mathfrak{l}U\,|q|\,j\mathfrak{l}_j\,M^{(\alpha_1+2)j}.
\end{aligned}$$

Multiplying the $(2-\alpha_2)^{\mathrm{th}}$ power of the first bound by the $(\alpha_2-1)^{\mathrm{th}}$ power of the second gives

$$\begin{aligned}
&\left|\big(\mathfrak{n}\cdot\partial_{\mathbf{k}}\big)^{\alpha_1}\big(\mathfrak{t}\cdot\partial_{\mathbf{k}}\big)\big[B_{i,j}(\mathbf{k}+q\mathfrak{t})-B_{i,j}(\mathbf{k})\big]\right| \\
&\quad\le 2^{2-\alpha_2}\mathrm{const}\,\mathfrak{l}U\,|q|^{\alpha_2-1}M^{(\alpha_1+1)j}\,j\mathfrak{l}_j\,M^{(\alpha_2-1)j} \\
&\quad= 2^{2-\alpha_2}\mathrm{const}\,\mathfrak{l}U\,M^{\alpha_1 j}\tfrac{1}{\mathfrak{l}_j^{\alpha_2}}|q|^{\alpha_2-1}\big(jM^{-(\aleph+\alpha_2\aleph-\alpha_2)j}\big) \\
&\quad\le C\,\mathfrak{l}U\,M^{\alpha_1 j}\tfrac{1}{\mathfrak{l}_j^{\alpha_2}}|q|^{\alpha_2-1},
\end{aligned}$$

since $\aleph+\alpha_2\aleph-\alpha_2 = \aleph-(1-\aleph)\alpha_2 \ge \aleph-(1-\aleph)a = \aleph-\frac{1}{2} > 0$. We also have, for all $\alpha_1\in\{0,1,2\}$ and $\alpha_2\in\{0,1\}$,

$$\left|\big(\mathfrak{n}\cdot\partial_{\mathbf{k}}\big)^{\alpha_1}\big(\mathfrak{t}\cdot\partial_{\mathbf{k}}\big)^{\alpha_2}B_{i,j}(\mathbf{k})\right| \le \mathrm{const}\,\mathfrak{l}U M^{\alpha_1 j}\tfrac{1}{\mathfrak{l}_j^{\alpha_2}}, \tag{B.9}$$

since $j\mathfrak{l}_j\,M^{\alpha_2 j} = \frac{1}{\mathfrak{l}_j^{\alpha_2}}jM^{-(\aleph+\alpha_2\aleph-\alpha_2)j} \le \mathrm{const}\,\frac{1}{\mathfrak{l}_j^{\alpha_2}}$ for $\alpha_2 = 0, 1$.

The next step is to prove that for all $\alpha_1 \in \{0, 1\}$ and $0 \le \alpha_2 \le a$,

$$\begin{aligned}\Big|\big(\mathfrak{n}\cdot\partial_{\mathbf{k}}\big)^{\alpha_1}\big(\mathfrak{t}\cdot\partial_{\mathbf{k}}\big)^{[\alpha_2]}\big[\rho(\mathbf{k}+q\mathfrak{t})B_{i,j}(\mathbf{k}+q\mathfrak{t})-\rho(\mathbf{k})B_{i,j}(\mathbf{k})\big]\Big|&\\ \le C'\,\mathfrak{l}U\, M^{\alpha_1 j}\tfrac{1}{\mathfrak{l}_j^{\alpha_2}}|q|^{\alpha_2-[\alpha_2]}.&\end{aligned} \tag{B.10}$$

Applying the hypothesis on ρ twice,

$$\begin{aligned}\Big|\big(\mathfrak{n}\cdot\partial_{\mathbf{k}}\big)^{\alpha_1}\big(\mathfrak{t}\cdot\partial_{\mathbf{k}}\big)^{[\alpha_2]}\big[\rho(\mathbf{k}+q\mathfrak{t})-\rho(\mathbf{k})\big]\Big| &\le 2\,\mathrm{const}\,M^{\alpha_1 j}\tfrac{1}{\mathfrak{l}_j^{[\alpha_2]}},\\ \Big|\big(\mathfrak{n}\cdot\partial_{\mathbf{k}}\big)^{\alpha_1}\big(\mathfrak{t}\cdot\partial_{\mathbf{k}}\big)^{[\alpha_2]}\big[\rho(\mathbf{k}+q\mathfrak{t})-\rho(\mathbf{k})\big]\Big| &\le \mathrm{const}\,M^{\alpha_1 j}\tfrac{1}{\mathfrak{l}_j^{[\alpha_2]+1}}|q|.\end{aligned}$$

Multiplying the $(1+[\alpha_2]-\alpha_2)^{\text{th}}$ power of the first bound by the $(\alpha_2-[\alpha_2])^{\text{th}}$ power of the second gives

$$\begin{aligned}\Big|\big(\mathfrak{n}\cdot\partial_{\mathbf{k}}\big)^{\alpha_1}\big(\mathfrak{t}\cdot\partial_{\mathbf{k}}\big)^{[\alpha_2]}\big[\rho(\mathbf{k}+q\mathfrak{t})-\rho(\mathbf{k})\big]\Big|&\\ \le 2^{1+[\alpha_2]-\alpha_2}\mathrm{const}\,M^{\alpha_1 j}\tfrac{1}{\mathfrak{l}_j^{\alpha_2}}|q|^{\alpha_2-[\alpha_2]}.&\end{aligned} \tag{B.11}$$

The bound (B.10) follows from the product rule and

$$\begin{aligned}&\Big|\big(\mathfrak{n}\cdot\partial_{\mathbf{k}}\big)^{\alpha_1}\big(\mathfrak{t}\cdot\partial_{\mathbf{k}}\big)^{[\alpha_2]}\big[\rho(\mathbf{k}+q\mathfrak{t})B_{i,j}(\mathbf{k}+q\mathfrak{t})-\rho(\mathbf{k})B_{i,j}(\mathbf{k})\big]\Big|\\ &\quad\le \Big|\big(\mathfrak{n}\cdot\partial_{\mathbf{k}}\big)^{\alpha_1}\big(\mathfrak{t}\cdot\partial_{\mathbf{k}}\big)^{[\alpha_2]}\big[\rho(\mathbf{k}+q\mathfrak{t})B_{i,j}(\mathbf{k}+q\mathfrak{t})-\rho(\mathbf{k})B_{i,j}(\mathbf{k}+q\mathfrak{t})\big]\Big|\\ &\qquad+\Big|\big(\mathfrak{n}\cdot\partial_{\mathbf{k}}\big)^{\alpha_1}\big(\mathfrak{t}\cdot\partial_{\mathbf{k}}\big)^{[\alpha_2]}\big[\rho(\mathbf{k})B_{i,j}(\mathbf{k}+q\mathfrak{t})-\rho(\mathbf{k})B_{i,j}(\mathbf{k})\big]\Big|\end{aligned}$$

by using (B.11) and (B.9) to bound the first line and (B.8) and the hypothesis on the derivatives of ρ, to bound the second.

The lemma will follow from

$$\sup_{\mathbf{x}}\ \big|\mathfrak{n}\cdot\mathbf{x}/M^j\big|^{\alpha_1}|\mathfrak{l}_j\mathfrak{t}\cdot\mathbf{x}|^{\alpha_2}\big|\hat{B}_{i,j}(\mathbf{x})\big| \le \mathrm{const}\ \mathfrak{l}U\,\tfrac{\mathfrak{l}_j}{M^j}$$

for all $\alpha_1 \in \{0, \frac{3}{2}\}$ and $\alpha_2 \in \{0, \frac{1+a}{2}\}$. This in turn will follow from

$$\sup_{\mathbf{x}}\ \big|\mathfrak{n}\cdot\mathbf{x}/M^j\big|^{\alpha_1}|\mathfrak{l}_j\mathfrak{t}\cdot\mathbf{x}|^{\alpha_2}\big|\hat{B}_{i,j}(\mathbf{x})\big| \le \mathrm{const}\ \mathfrak{l}U\,\tfrac{\mathfrak{l}_j}{M^j} \tag{B.12}$$

for all $\alpha_1 \in \{0,1,2\}$, $\alpha_2 \in \{0,1,a\}$, $(\alpha_1,\alpha_2) \ne (2,a)$, by taking various geometric means. In particular, to handle the case $(\alpha_1,\alpha_2) = \big(\frac{3}{2}, \frac{1+a}{2}\big)$, take the geometric mean of the bounds with $(\alpha_1,\alpha_2) = (1,a)$ and $(\alpha_1,\alpha_2) = (2,1)$. For $\alpha_2 = 0, 1$, (B.12) follows, by integration by parts, from

$$\Big|\big(\mathfrak{n}\cdot\partial_{\mathbf{k}}\big)^{\alpha_1}\big(\mathfrak{t}\cdot\partial_{\mathbf{k}}\big)^{\alpha_2}\big[\rho(\mathbf{k})B_{i,j}(\mathbf{k})\big]\Big| \le \mathrm{const}\ \mathfrak{l}U\,M^{\alpha_1 j}\tfrac{1}{\mathfrak{l}_j^{\alpha_2}}$$

and the fact that $\rho(\mathbf{k})B_{i,j}(\mathbf{k})$ is supported in a region of volume const $\frac{\mathfrak{l}_j}{M^j}$. Furthermore, if $|\mathfrak{l}_j\mathfrak{t}\cdot\mathbf{x}| \le 1$,

$$\big|\mathfrak{n}\cdot\mathbf{x}/M^j\big|^{\alpha_1}|\mathfrak{l}_j\mathfrak{t}\cdot\mathbf{x}|^{a}\big|\hat{B}_{i,j}(\mathbf{x})\big| \le \big|\mathfrak{n}\cdot\mathbf{x}/M^j\big|^{\alpha_1}\big|\hat{B}_{i,j}(\mathbf{x})\big| \le \mathrm{const}\ \mathfrak{l}U\,\tfrac{\mathfrak{l}_j}{M^j},$$

so it suffices to consider $\alpha_1 = 0, 1, \alpha_2 = a$ and $|\mathfrak{t}\cdot\mathbf{x}| \ge \frac{1}{\mathfrak{l}_j}$. Let $\tilde{D}_j(\mathbf{x})$ denote the Fourier transform of

$$D_{i,j}(\mathbf{k}) = \tfrac{1}{M^{\alpha_1 j}}\big(\mathfrak{n}\cdot\partial_{\mathbf{k}}\big)^{\alpha_1}\big(\mathfrak{l}_j\,\mathfrak{t}\cdot\partial_{\mathbf{k}}\big)\big[\rho(\mathbf{k})B_{i,j}(\mathbf{k})\big].$$

Then

$$\begin{aligned}
&\big|e^{-\imath q\mathfrak{t}\cdot\mathbf{x}}-1\big|\big|\mathfrak{n}\cdot\mathbf{x}/M^j\big|^{\alpha_1}|\mathfrak{l}_j\mathfrak{t}\cdot\mathbf{x}|\big|\hat{B}_{i,j}(\mathbf{x})\big|\\
&\quad=\big|e^{-\imath q\mathfrak{t}\cdot\mathbf{x}}-1\big|\,\Big|\int \tfrac{d^2\mathbf{k}}{(2\pi)^2}e^{\imath\mathbf{k}\cdot\mathbf{x}}D_{i,j}(\mathbf{k})\Big|\\
&\quad=\Big|\int \tfrac{d^2\mathbf{k}}{(2\pi)^2}\big[e^{\imath(\mathbf{k}-q\mathfrak{t})\cdot\mathbf{x}}-e^{\imath\mathbf{k}\cdot\mathbf{x}}\big]D_{i,j}(\mathbf{k})\Big|\\
&\quad=\Big|\int \tfrac{d^2\mathbf{k}}{(2\pi)^2}e^{\imath\mathbf{k}\cdot\mathbf{x}}\big[D_{i,j}(\mathbf{k}+q\mathfrak{t})-D_{i,j}(\mathbf{k})\big]\Big|.
\end{aligned}$$

By (B.10)

$$\big|D_{i,j}(\mathbf{k}+q\mathfrak{t})-D_{i,j}(\mathbf{k})\big| \le C'\,\mathfrak{l}U\,\tfrac{|q|^{a-1}}{\mathfrak{l}_j^{a-1}}.$$

Furthermore, if $|q| < \mathfrak{l}_j$, $D_{i,j}(\mathbf{k}+q\mathfrak{t}) - D_{i,j}(\mathbf{k})$ is supported in a region of volume const $\frac{\mathfrak{l}_j}{M^j}$, so that

$$\big|e^{-\imath q\mathfrak{t}\cdot\mathbf{x}}-1\big|\big|\mathfrak{n}\cdot\mathbf{x}/M^j\big|^{\alpha_1}|\mathfrak{l}_j\mathfrak{t}\cdot\mathbf{x}|\big|\hat{B}_{i,j}(\mathbf{x})\big| \le \text{const}\;\mathfrak{l}U\,\tfrac{\mathfrak{l}_j}{M^j}\,\tfrac{|q|^{a-1}}{\mathfrak{l}_j^{a-1}}.$$

To finish the proof of (B.12), and the lemma, it now suffices to choose $q = \frac{1}{10\,\mathfrak{t}\cdot\mathbf{x}}$ and observe that then $\big|e^{-\imath q\mathfrak{t}\cdot\mathbf{x}}-1\big| = \big|e^{-\imath/10}-1\big| > 0$. □

Lemma B.3. *Let I be an interval of length $\mathfrak{l}$ on the Fermi surface F, K be a compact subset of $\mathbb{R}^2$, and $u_j(k,t_0,z) = u_j\big((k_0,\mathbf{k}),t_0,z\big)$ and $v_j(k,\mathbf{t},z) = v_j\big((k_0,\mathbf{k}),\mathbf{t},z\big)$, $j > 1$, be functions that vanish unless $\mathbf{k}\in K$ and $\pi_F(\mathbf{k})\in I$, where π_F is projection on the Fermi surface. The variable z runs over $\mathbb{R}^n$ for some $n\in\mathbb{N}$. Let $n_j(\omega)$, $j>1$, be functions that take values in $[0,1]$, vanish in a (j–dependent) neighbourhood of $\omega = 0$, are supported in a compact set that is independent of j and converge pointwise as $j\to\infty$. Set, for $j > 1$,*

$$\begin{aligned}
A_j(t_0,z) &= \int dk\;\frac{n_j(e(\mathbf{k}))u_j(k,t_0,z)}{[\imath k_0 - e'(k_0,\mathbf{k})][\imath(k_0+t_0)-e'(k_0+t_0,\mathbf{k})]},\\
B_j(\mathbf{t},z) &= \int dk\;\frac{n_j(k_0)v_j(k,\mathbf{t},z)}{[\imath k_0 - e'(k_0,\mathbf{k})][\imath k_0 - e'(k_0,\mathbf{k}+\mathbf{t})]},
\end{aligned}$$

and let

$$\begin{aligned}
U_{\tilde\aleph} &= \sup_{j,k,t_0,z}|u_j(k,t_0,z)| + \sup_{j,k,t_0,\kappa,z}\tfrac{|u_j(k-(\kappa,\mathbf{0}),t_0,z)-u_j(k,t_0,z)|}{|\kappa|^{\tilde\aleph}}\\
&\quad+\sup_{j,k,t_0,z}\tfrac{|u_j(k,t_0,z)-u_j(k,0,z)|}{|t_0|^{\tilde\aleph}}+\sup_{j,k,t_0,z,z'}\tfrac{|u_j(k,t_0,z)-u_j(k,t_0,z')|}{|z-z'|^{\tilde\aleph}},\\
V_{\tilde\aleph} &= \sup_{j,k,\mathbf{t},z}|v_j(k,\mathbf{t},z)| + \sup_{j,k_0,\mathbf{k},\mathbf{k}',\mathbf{t},z}\tfrac{|v_j((k_0,\mathbf{k}'),\mathbf{t},z)-v_j((k_0,\mathbf{k}),\mathbf{t},z)|}{|\mathbf{k}-\mathbf{k}'|^{\tilde\aleph}}\\
&\quad+\sup_{j,k,\mathbf{t},z}\tfrac{|v_j(k,\mathbf{t},z)-v_j(k,\mathbf{0},z)|}{|\mathbf{t}|^{\tilde\aleph}}+\sup_{j,k,\mathbf{t},z,z'}\tfrac{|v_j(k,\mathbf{t},z)-v_j(k,\mathbf{t},z')|}{|z-z'|^{\tilde\aleph}}
\end{aligned}$$

be finite.

a) Let $0 \le \aleph' < \tilde{\aleph} < \aleph$. *Then*

$$\begin{aligned} \left|A_j(t_0,z) - A_j(0,z)\right| &\le \text{const}\, \mathfrak{l}\, U_{\tilde{\aleph}}\, |t_0|^{\aleph'}, \\ \left|A_j(t_0,z) - A_j(t_0,z')\right| &\le \text{const}\, \mathfrak{l}\, U_{\tilde{\aleph}}\, |z-z'|^{\aleph'} \end{aligned}$$

for all t_0 *in a* (j*–independent) neighbourhood of the origin.*

b) If, in addition, $u_j(k,t_0,z)$ *converges pointwise as* $j \to \infty$*, the limit* $A(t_0,z) = \lim_{j\to\infty} A_j(t_0,z)$ *exists for* t_0 *in a neighbourhood of the origin and obeys*

$$\begin{aligned} \left|A(t_0,z) - A(0,z)\right| &\le \text{const}\, \mathfrak{l}\, U_{\tilde{\aleph}}\, |t_0|^{\aleph'}, \\ \left|A(t_0,z) - A(t_0,z')\right| &\le \text{const}\, \mathfrak{l}\, U_{\tilde{\aleph}}\, |z-z'|^{\aleph'} \end{aligned}$$

for all t_0 *in a neighbourhood of the origin.*

c) Let $0 \le \aleph' < \tilde{\aleph} < \aleph$. *Then*

$$\begin{aligned} \left|B_j(\mathbf{t},z) - B_j(\mathbf{0},z)\right| &\le \text{const}\, \mathfrak{l}\, V_{\tilde{\aleph}}\, |\mathbf{t}|^{\aleph'} \\ \left|B_j(\mathbf{t},z) - B_j(\mathbf{t},z')\right| &\le \text{const}\, \mathfrak{l}\, V_{\tilde{\aleph}}\, |z-z'|^{\aleph'} \end{aligned}$$

for all $\mathbf{t}$ *in a* (j*–independent) neighbourhood of the origin.*

d) If, in addition, $v_j(k,\mathbf{t},z)$ *converges pointwise as* $j \to \infty$*, the limit* $B(\mathbf{t},z) = \lim_{j\to\infty} B_j(\mathbf{t},z)$ *exists for* $\mathbf{t}$ *in a neighbourhood of the origin and obeys*

$$\begin{aligned} \left|B(\mathbf{t},z) - B(\mathbf{0},z)\right| &\le \text{const}\, \mathfrak{l}\, V_{\tilde{\aleph}}\, |\mathbf{t}|^{\aleph'}, \\ \left|B(\mathbf{t},z) - B(\mathbf{t},z')\right| &\le \text{const}\, \mathfrak{l}\, V_{\tilde{\aleph}}\, |z-z'|^{\aleph'}. \end{aligned}$$

Proof. The proofs of parts (a) and (b) are very similar to the proofs of parts (c) and (d) respectively. So we only give the latter.
c) As in Lemma B.1,

$$B_j(\mathbf{t},z) = \int dk \int_0^1 ds\; \frac{n_j(k_0)v_j(k,\mathbf{t},z)}{[\imath w(\mathbf{k},\mathbf{t},s)k_0 - E(\mathbf{k},\mathbf{t},s) - \tilde{E}(k_0,\mathbf{k},\mathbf{t},s)]^2},$$

where

$$\begin{aligned} E'(k_0,\mathbf{k},\mathbf{t},s) &= se'(k_0,\mathbf{k}) + (1-s)e'(k_0,\mathbf{k}+\mathbf{t}), \\ E(\mathbf{k},\mathbf{t},s) &= E'(0,\mathbf{k},\mathbf{t},s) = se(\mathbf{k}) + (1-s)e(\mathbf{k}+\mathbf{t}), \\ w(\mathbf{k},\mathbf{t},s) &= 1 - \tfrac{1}{i}\tfrac{\partial E'}{\partial k_0}(0,\mathbf{k},\mathbf{t},s), \\ \tilde{E}(k_0,\mathbf{k},\mathbf{t},s) &= E'(k_0,\mathbf{k},\mathbf{t},s) - E'(0,\mathbf{k},\mathbf{t},s) - k_0\tfrac{\partial E'}{\partial k_0}(0,\mathbf{k},\mathbf{t},s). \end{aligned}$$

Set

$$B'_j(\mathbf{t},z) = \int dk \int_0^1 ds\; \frac{n_j(k_0)v_j(k,\mathbf{t},z)}{[\imath w(\mathbf{k},\mathbf{t},s)k_0 - E(\mathbf{k},\mathbf{t},s)]^2}$$

and

$$\begin{aligned} I(k_0,\mathbf{k},\mathbf{t},s) &= \tfrac{1}{[\imath w(\mathbf{k},\mathbf{t},s)k_0 - E(\mathbf{k},\mathbf{t},s) - \tilde{E}(k_0,\mathbf{k},\mathbf{t},s)]^2} - \tfrac{1}{[\imath w(\mathbf{k},\mathbf{t},s)k_0 - E(\mathbf{k},\mathbf{t},s)]^2} \\ &= \tfrac{[\imath w(\mathbf{k},\mathbf{t},s)k_0 - E(\mathbf{k},\mathbf{t},s)]^2 - [\imath w(\mathbf{k},\mathbf{t},s)k_0 - E(\mathbf{k},\mathbf{t},s) - \tilde{E}(k_0,\mathbf{k},\mathbf{t},s)]^2}{[\imath w(\mathbf{k},\mathbf{t},s)k_0 - E(\mathbf{k},\mathbf{t},s)]^2[\imath w(\mathbf{k},\mathbf{t},s)k_0 - E(\mathbf{k},\mathbf{t},s) - \tilde{E}(k_0,\mathbf{k},\mathbf{t},s)]^2} \\ &= \tfrac{\tilde{E}(k_0,\mathbf{k},\mathbf{t},s)[2\imath w(\mathbf{k},\mathbf{t},s)k_0 - 2E(\mathbf{k},\mathbf{t},s) - \tilde{E}(k_0,\mathbf{k},\mathbf{t},s)]}{[\imath w(\mathbf{k},\mathbf{t},s)k_0 - E(\mathbf{k},\mathbf{t},s)]^2[\imath w(\mathbf{k},\mathbf{t},s)k_0 - E(\mathbf{k},\mathbf{t},s) - \tilde{E}(k_0,\mathbf{k},\mathbf{t},s)]^2} \end{aligned}$$

so that

$$B_j(\mathbf{t},z) - B'_j(\mathbf{t},z) = \int dk \int_0^1 ds\, n_j(k_0) v_j(k,\mathbf{t},z) I(k_0,\mathbf{k},\mathbf{t},s). \tag{B.13}$$

By (B.7) and Lemma A.1, using $\tilde{E}(k_0,\mathbf{k},\mathbf{t},s) = \int_0^{k_0} d\kappa \left[\frac{\partial E'}{\partial k_0}(\kappa,\mathbf{k},\mathbf{t},s) - \frac{\partial E'}{\partial k_0}(0,\mathbf{k},\mathbf{t},s)\right]$,

$$|\tilde{E}(k_0,\mathbf{k},\mathbf{t},s)| \le \text{const}\,\rho\,|k_0|^{1+\aleph}, \quad |\tilde{E}(k_0,\mathbf{k},\mathbf{t},s) - \tilde{E}(k_0,\mathbf{k},\mathbf{0},s)| \le \text{const}\,|k_0|\,|\mathbf{t}|^{\aleph}$$
$$|w(\mathbf{k},\mathbf{t},s) - 1| \le \rho, \qquad |w(\mathbf{k},\mathbf{t},s) - w(\mathbf{k},\mathbf{0},s)| \le \text{const}\,\rho\,|\mathbf{t}|^{\aleph},$$
$$|E(\mathbf{k},\mathbf{t},s) - E(\mathbf{k},\mathbf{0},s)| \le \text{const}\,|\mathbf{t}|$$

Consequently

$$|I(k_0,\mathbf{k},\mathbf{t},s)| \le \text{const}\,\frac{|k_0|^{1+\aleph}|\imath k_0 - E(\mathbf{k},\mathbf{t},s)|}{|\imath k_0 - E(\mathbf{k},\mathbf{t},s)|^4} \le \text{const}\,\frac{|k_0|^{\aleph}}{|\imath k_0 - E(\mathbf{k},\mathbf{t},s)|^2} \tag{B.14}$$

and, we claim, for $0 \le \aleph' \le \aleph$,

$$|I(k_0,\mathbf{k},\mathbf{t},s) - I(k_0,\mathbf{k},\mathbf{0},s)| \le \text{const}\,|k_0|^{\aleph-\aleph'}|\mathbf{t}|^{\aleph'}\left[\frac{1}{|\imath k_0 - E(\mathbf{k},\mathbf{0},s)|^2} + \frac{1}{|\imath k_0 - E(\mathbf{k},\mathbf{t},s)|^2}\right].$$

We prove the last bound in the case $|E(\mathbf{k},\mathbf{t},s)| \le |E(\mathbf{k},\mathbf{0},s)|$. The other case is similar. Set

$$\begin{aligned}
c &= \imath w(\mathbf{k},\mathbf{t},s)k_0 - E(\mathbf{k},\mathbf{t},s), & C &= \imath w(\mathbf{k},\mathbf{0},s)k_0 - E(\mathbf{k},\mathbf{0},s),\\
d &= \imath w(\mathbf{k},\mathbf{t},s)k_0 - E(\mathbf{k},\mathbf{t},s) - \tilde{E}(k_0,\mathbf{k},\mathbf{t},s), & D &= \imath w(\mathbf{k},\mathbf{0},s)k_0 - E(\mathbf{k},\mathbf{0},s)\\
& & &\quad - \tilde{E}(k_0,\mathbf{k},\mathbf{0},s),\\
a &= \tilde{E}(k_0,\mathbf{k},\mathbf{t},s), & A &= \tilde{E}(k_0,\mathbf{k},\mathbf{0},s),\\
b &= c + d, & B &= C + D.
\end{aligned}$$

Then, for $|k_0|, |\mathbf{t}| \le \text{const}$,

$$\begin{aligned}
|a|, |A| &\le \text{const}\,|k_0|^{1+\aleph},\\
|b|, |c|, |d| &\le \text{const}\,|\imath k_0 - E(\mathbf{k},\mathbf{t},s)| \le \text{const}\,|\imath k_0 - E(\mathbf{k},\mathbf{0},s)|,\\
|c|, |d| &\ge \text{const}\,|\imath k_0 - E(\mathbf{k},\mathbf{t},s)|,\\
|B|, |C|, |D| &\le \text{const}\,|\imath k_0 - E(\mathbf{k},\mathbf{0},s)|,\\
|C|, |D| &\ge \text{const}\,|\imath k_0 - E(\mathbf{k},\mathbf{0},s)|,\\
|a - A| &\le \text{const}\,\min\left\{|k_0|^{1+\aleph}, |k_0||\mathbf{t}|^{\aleph}\right\} \le \text{const}\,|k_0|^{1+\aleph-\aleph'}|\mathbf{t}|^{\aleph'},\\
|c - C|, |d - D|, |b - B| &\le \text{const}\,|k_0||\mathbf{t}|^{\aleph} + \text{const}\,\min\{|E(\mathbf{k},\mathbf{0},s)| + |E(\mathbf{k},\mathbf{t},s)|, |\mathbf{t}|\}\\
&\le \text{const}\,|\mathbf{t}|^{\aleph'}|\imath k_0 - E(\mathbf{k},\mathbf{0},s)|^{1-\aleph'}.
\end{aligned}$$

Applying these to

$$\begin{aligned}
\frac{AB}{C^2D^2} - \frac{ab}{c^2d^2} &= \frac{(A-a)B}{C^2D^2} + \frac{a(B-b)}{C^2D^2} + \frac{ab}{c^2d^2}\frac{c^2d^2 - C^2D^2}{C^2D^2}\\
&= \frac{(A-a)B}{C^2D^2} + \frac{a(B-b)}{C^2D^2} + \frac{ab}{c^2d^2}\frac{c^2(d-D)(d+D)}{C^2D^2} + \frac{ab}{c^2d^2}\frac{(c-C)(c+C)}{C^2}
\end{aligned}$$

gives

$$\begin{aligned}
|I(k_0,\mathbf{k},\mathbf{t},s) - I(k_0,\mathbf{k},\mathbf{0},s)| &\le \text{const}\,\frac{|k_0|^{\aleph-\aleph'}|\mathbf{t}|^{\aleph'}}{|\imath k_0 - E(\mathbf{k},\mathbf{0},s)|^2}\\
&\quad + \text{const}\,\frac{|k_0|^{\aleph}}{|\imath k_0 - E(\mathbf{k},\mathbf{t},s)|^2}\frac{|\mathbf{t}|^{\aleph'}}{|\imath k_0 - E(\mathbf{k},\mathbf{0},s)|^{\aleph'}}\\
&\le \text{const}\,|k_0|^{\aleph-\aleph'}|\mathbf{t}|^{\aleph'}\left[\frac{1}{|\imath k_0 - E(\mathbf{k},\mathbf{0},s)|^2} + \frac{1}{|\imath k_0 - E(\mathbf{k},\mathbf{t},s)|^2}\right],
\end{aligned}$$

which is the desired bound.

Using these bounds gives, for $0 \le \aleph' < \tilde{\aleph} < \aleph$,

$$\begin{aligned}
&\left|B_j(\mathbf{t}, z) - B'_j(\mathbf{t}, z) - B_j(\mathbf{0}, z) + B'_j(\mathbf{0}, z)\right| \\
&\le \int dk \int_0^1 ds\, n_j(k_0) \Big| v_j(k, \mathbf{t}, z) I(k_0, \mathbf{k}, \mathbf{t}, s) - v_j(k, \mathbf{0}, z) I(k_0, \mathbf{k}, \mathbf{0}, s)| \Big| \\
&\le \text{const} \int dk \int_0^1 ds\, n_j(k_0) \Big[\tfrac{|v_j(k,\mathbf{t},z) - v_j(k,\mathbf{0},z)||k_0|^{\aleph}}{|\imath k_0 - E(\mathbf{k},\mathbf{t},s)|^2} \\
&\quad + \tfrac{|v_j(k,\mathbf{0},z)||k_0|^{\aleph-\aleph'}|\mathbf{t}|^{\aleph'}}{|\imath k_0 - E(\mathbf{k},\mathbf{0},s)|^2} + \tfrac{|v_j(k,\mathbf{0},z)||k_0|^{\aleph-\aleph'}|\mathbf{t}|^{\aleph'}}{|\imath k_0 - E(\mathbf{k},\mathbf{t},s)|^2} \Big] \\
&\le \text{const} \int_I d\theta \int_{-\text{const}}^{\text{const}} dk_0 \int_{-\text{const}}^{\text{const}} dE \Big[\tfrac{|k_0|^{\aleph} \sup_k |v_j(k,\mathbf{t},z) - v_j(k,\mathbf{0},z)|}{|\imath k_0 - E|^2} \\
&\quad + \tfrac{|k_0|^{\aleph-\aleph'}|\mathbf{t}|^{\aleph'} \sup_k |v_j(k,\mathbf{0},z)|}{|\imath k_0 - E|^2} \Big] \\
&\le \text{const}\, \mathfrak{l} \Big[\sup_k |v_j(k, \mathbf{t}, z) - v_j(k, \mathbf{0}, z)| + |\mathbf{t}|^{\aleph'} \sup_k |v_j(k, \mathbf{0}, z)| \Big] \\
&\le \text{const}\, \mathfrak{l}\, V_{\tilde{\aleph}}\, |\mathbf{t}|^{\aleph'}
\end{aligned}$$

and

$$\begin{aligned}
&\left|B_j(\mathbf{t}, z) - B'_j(\mathbf{t}, z) - B_j(\mathbf{t}, z') + B'_j(\mathbf{t}, z')\right| \\
&\le \int dk \int_0^1 ds\, n_j(k_0) \Big| v_j(k, \mathbf{t}, z) I(k_0, \mathbf{k}, \mathbf{t}, s) - v_j(k, \mathbf{t}, z') I(k_0, \mathbf{k}, \mathbf{t}, s)| \Big| \\
&\le \text{const} \int dk \int_0^1 ds\, n_j(k_0) \tfrac{|v_j(k,\mathbf{t},z) - v_j(k,\mathbf{t},z')||k_0|^{\aleph}}{|\imath k_0 - E(\mathbf{k},\mathbf{t},s)|^2} \\
&\le \text{const} \int_I d\theta \int_{-\text{const}}^{\text{const}} dk_0 \int_{-\text{const}}^{\text{const}} dE\ \tfrac{|k_0|^{\aleph} \sup_k |v_j(k,\mathbf{t},z) - v_j(k,\mathbf{t},z')|}{|\imath k_0 - E|^2} \\
&\le \text{const}\, \mathfrak{l} \sup_k |v_j(k, \mathbf{t}, z) - v_j(k, \mathbf{t}, z')| \\
&\le \text{const}\, \mathfrak{l}\, V_{\tilde{\aleph}}\, |z - z'|^{\aleph'}.
\end{aligned}$$

Hence it suffices to consider $B'_j(\mathbf{t}, z)$.

Making, as in Lemma B.1, for each fixed s, the change of variables from $\mathbf{k}$ to $E = E(\mathbf{k}, \mathbf{t}, s)$ and an "angular" variable θ

$$B'_j(\mathbf{t}, z) = \textstyle\int_0^1 ds \int d\theta \int dk_0 dE\, n_j(k_0) \dfrac{v_j\big((k_0, \mathbf{k}(E, \mathbf{t}, \theta, s)), \mathbf{t}, z\big) J(E, \mathbf{t}, \theta, s)}{[\imath w(\mathbf{k}(E, \mathbf{t}, \theta, s), \mathbf{t}, s) k_0 - E]^2}.$$

Set

$$B''_j(\mathbf{t}, z) = \textstyle\int_0^1 ds \int d\theta \int dk_0 dE\, n_j(k_0) \dfrac{v_j\big((k_0, \mathbf{k}(0, \mathbf{t}, \theta, s)), \mathbf{t}, z\big) J(0, \mathbf{t}, \theta, s)}{[\imath w(\mathbf{k}(0, \mathbf{t}, \theta, s), \mathbf{t}, s) k_0 - E]^2}.$$

Using

$$\partial_E \mathbf{k}_\ell(E, \mathbf{t}, \theta, s) = \tfrac{\partial_{\mathbf{k}_2} \theta(\mathbf{k}) \delta_{\ell,1} - \partial_{\mathbf{k}_1} \theta(\mathbf{k}) \delta_{\ell,2}}{\partial_{\mathbf{k}_1} E(\mathbf{k},\mathbf{t},s)\, \partial_{\mathbf{k}_2} \theta(\mathbf{k}) - \partial_{\mathbf{k}_2} E(\mathbf{k},\mathbf{t},s)\, \partial_{\mathbf{k}_1} \theta(\mathbf{k})} \Big|_{\mathbf{k} = \mathbf{k}(E, \mathbf{t}, \theta, s)}$$

one proves, by induction on n, that, for $n \le r$,

$$\left|\partial_E^n \mathbf{k}(E, \mathbf{t}, \theta, s)\right| \le \mathit{const}''. \tag{B.15}$$

Using this bound and the observation that the Jacobian

$$J(E,\mathbf{t},\theta,s)=\frac{1}{|\partial_{\mathbf{k}_1}E(\mathbf{k},\mathbf{t},s)\,\partial_{\mathbf{k}_2}\theta(\mathbf{k})-\partial_{\mathbf{k}_2}E(\mathbf{k},\mathbf{t},s)\,\partial_{\mathbf{k}_1}\theta(\mathbf{k})|}\Big|_{\mathbf{k}=\mathbf{k}(E,\mathbf{t},\theta,s)},$$

one proves that, for $n\le r-1$,

$$\big|\partial_E^n J(E,\mathbf{t},\theta,s)\big|\le const''. \tag{B.16}$$

Since

$$\begin{aligned}|w(\mathbf{k},\mathbf{t},s)-w(\mathbf{k}',\mathbf{t},s)|&\le s\big|\tfrac{\partial e'}{\partial k_0}(0,\mathbf{k})-\tfrac{\partial e'}{\partial k_0}(0,\mathbf{k}')\big|\\&\quad+(1-s)\big|\tfrac{\partial e'}{\partial k_0}(0,\mathbf{k}+\mathbf{t})-\tfrac{\partial e'}{\partial k_0}(0,\mathbf{k}'+\mathbf{t})\big|\\&\le \text{const}\,|\mathbf{k}-\mathbf{k}'|^{\aleph},\end{aligned}\tag{B.17}$$

(B.15), (B.16) and the fact, from Lemma A.1.i, that $|w(\mathbf{k},\mathbf{t},s)-1|\le\rho\le\frac12$ imply that

$$\begin{aligned}&\Big|\tfrac{v_j((k_0,\mathbf{k}(E,\mathbf{t},\theta,s)),\mathbf{t},z)J(E,\mathbf{t},\theta,s)}{[\imath w(\mathbf{k}(E,\mathbf{t},\theta,s),\mathbf{t},s)k_0-E]^2}-\tfrac{v_j((k_0,\mathbf{k}(0,\mathbf{t},\theta,s)),\mathbf{t},z)J(0,\mathbf{t},\theta,s)}{[\imath w(\mathbf{k}(0,\mathbf{t},\theta,s),\mathbf{t},s)k_0-E]^2}\Big|\\&\quad\le\text{const}\,\big|v_j\big((k_0,\mathbf{k}(E,\mathbf{t},\theta,s)),\mathbf{t},z\big)-v_j\big((k_0,\mathbf{k}(0,\mathbf{t},\theta,s)),\mathbf{t},z\big)\big|\tfrac{1}{k_0^2+E^2}\\&\qquad+\text{const}\sup_{k,\mathbf{t}}\big|v_j(k,\mathbf{t},z)\big|\tfrac{|E|}{k_0^2+E^2}+\text{const}\sup_{k,\mathbf{t}}\big|v_j(k,\mathbf{t},z)\big|\tfrac{[|k_0|+|E|]|k_0||E|^{\aleph}}{[k_0^2+E^2]^2}\\&\quad\le\text{const}\Big[\sup_{\mathbf{k},\mathbf{k}'}\tfrac{|v_j((k_0,\mathbf{k}'),\mathbf{t},z)-v_j((k_0,\mathbf{k}),\mathbf{t},z)|}{|\mathbf{k}-\mathbf{k}'|^{\tilde\aleph}}+\sup_{k,\mathbf{t}}\big|v_j(k,\mathbf{t},z)\big|\Big]\tfrac{|E|^{\tilde\aleph}}{k_0^2+E^2}\le\text{const}\,V_{\tilde\aleph}\tfrac{|E|^{\tilde\aleph}}{k_0^2+E^2}\end{aligned}$$

for all small E. For E bounded away from zero

$$\begin{aligned}&\Big|\tfrac{v_j((k_0,\mathbf{k}(E,\mathbf{t},\theta,s)),\mathbf{t},z)J(E,\mathbf{t},\theta,s)}{[\imath w(\mathbf{k}(E,\mathbf{t},\theta,s),\mathbf{t},s)k_0-E]^2}-\tfrac{v_j((k_0,\mathbf{k}(0,\mathbf{t},\theta,s)),\mathbf{t},z)J(0,\mathbf{t},\theta,s)}{[\imath w(\mathbf{k}(0,\mathbf{t},\theta,s),\mathbf{t},s)k_0-E]^2}\Big|\\&\quad\le\text{const}\sup_{k,\mathbf{t}}\big|v_j(k,\mathbf{t},z)\big|\tfrac{1}{k_0^2+E^2}\end{aligned}$$

so

$$\Big|\tfrac{v_j((k_0,\mathbf{k}(E,\mathbf{t},\theta,s)),\mathbf{t},z)J(E,\mathbf{t},\theta,s)}{[\imath w(\mathbf{k}(E,\mathbf{t},\theta,s),\mathbf{t},s)k_0-E]^2}-\tfrac{v_j((k_0,\mathbf{k}(0,\mathbf{t},\theta,s)),\mathbf{t},z)J(0,\mathbf{t},\theta,s)}{[\imath w(\mathbf{k}(0,\mathbf{t},\theta,s),\mathbf{t},s)k_0-E]^2}\Big|\le\text{const}\,V_{\tilde\aleph}\tfrac{\min\{|E|^{\tilde\aleph},1\}}{k_0^2+E^2}. \tag{B.18}$$

Since

$$|w(\mathbf{k},\mathbf{t},s)-w(\mathbf{k},\mathbf{0},s)|=|1-s|\big|\tfrac{\partial e'}{\partial k_0}(0,\mathbf{k}+\mathbf{t})-\tfrac{\partial e'}{\partial k_0}(0,\mathbf{k})\big|\le\text{const}\,|\mathbf{t}|^{\aleph}$$

(B.17), (B.4) and (B.5) imply that

$$\begin{aligned}&\Big|\tfrac{v_j((k_0,\mathbf{k}(E,\mathbf{t},\theta,s)),\mathbf{t},z)J(E,\mathbf{t},\theta,s)}{[\imath w(\mathbf{k}(E,\mathbf{t},\theta,s),\mathbf{t},s)k_0-E]^2}-\tfrac{v_j((k_0,\mathbf{k}(E,\mathbf{0},\theta,s)),\mathbf{0},z)J(E,\mathbf{0},\theta,s)}{[\imath w(\mathbf{k}(E,\mathbf{0},\theta,s),\mathbf{0},s)k_0-E]^2}\Big|\\&\quad\le\text{const}\,\big|v_j((k_0,\mathbf{k}(E,\mathbf{t},\theta,s)),\mathbf{t},z)-v_j((k_0,\mathbf{k}(E,\mathbf{0},\theta,s)),\mathbf{0},z)\big|\tfrac{1}{k_0^2+E^2}\\&\qquad+\text{const}\,|\mathbf{t}|\sup_k\big|v_j(k,\mathbf{0},z)\big|\tfrac{1}{k_0^2+E^2}+\text{const}\,|\mathbf{t}|^{\aleph}\sup_k\big|v_j(k,\mathbf{0},z)\big|\tfrac{|k_0|[|k_0|+|E|]}{[k_0^2+E^2]^2}\\&\quad\le\text{const}\Big[\sup_{\substack{\mathbf{k},\mathbf{k}'\\|\mathbf{k}-\mathbf{k}'|\le\text{const}\,|\mathbf{t}|}}\big|v_j\big((k_0,\mathbf{k}'),\mathbf{t},z\big)-v_j\big((k_0,\mathbf{k}),\mathbf{0},z\big)\big|\\&\qquad+|\mathbf{t}|^{\aleph}\sup_{\mathbf{k}}\big|v_j(k,\mathbf{0},z)\big|\Big]\tfrac{1}{k_0^2+E^2}\\&\quad\le\text{const}\,V_{\tilde\aleph}|\mathbf{t}|^{\tilde\aleph}\tfrac{1}{k_0^2+E^2}\end{aligned}\tag{B.19}$$

and that

$$\begin{aligned}
&\Big| \tfrac{v_j((k_0,\mathbf{k}(0,\mathbf{t},\theta,s)),\mathbf{t},z)J(0,\mathbf{t},\theta,s)}{[\imath w(\mathbf{k}(0,\mathbf{t},\theta,s),\mathbf{t},s)k_0-E]^2} - \tfrac{v_j((k_0,\mathbf{k}(0,\mathbf{0},\theta,s)),\mathbf{0},z)J(0,\mathbf{0},\theta,s)}{[\imath w(\mathbf{k}(0,\mathbf{0},\theta,s),\mathbf{0},s)k_0-E]^2} \Big| \\
&\quad \le \text{const} \left| v_j((k_0,\mathbf{k}(0,\mathbf{t},\theta,s)),\mathbf{t},z) - v_j((k_0,\mathbf{k}(0,\mathbf{0},\theta,s)),\mathbf{0},z)\right| \tfrac{1}{k_0^2+E^2} \\
&\qquad + \text{const}\, |\mathbf{t}| \sup_k \left| v_j(k,\mathbf{0},z)\right| \tfrac{1}{k_0^2+E^2} + \text{const}\, |\mathbf{t}|^{\aleph} \sup_k \left| v_j(k,\mathbf{0},z)\right| \tfrac{|k_0|[|k_0|+|E|]}{[k_0^2+E^2]^2} \\
&\quad \le \text{const}\, V_{\tilde\aleph} |\mathbf{t}|^{\tilde\aleph} \tfrac{1}{k_0^2+E^2}. \qquad\qquad (B.20)
\end{aligned}$$

Combining, (B.18), a second copy of (B.18) with $\mathbf{t} = \mathbf{0}$, (B.19) and (B.20), gives, for all $0 \le \aleph' < \tilde\aleph < \aleph$,

$$\begin{aligned}
&\Big| \tfrac{v_j((k_0,\mathbf{k}(E,\mathbf{t}',\theta,s)),\mathbf{t}',z)J(E,\mathbf{t}',\theta,s)}{[\imath w(\mathbf{k}(E,\mathbf{t}',\theta,s),\mathbf{t}',s)k_0-E]^2}\Big|_{\mathbf{t}'=\mathbf{0}}^{\mathbf{t}'=\mathbf{t}} - \tfrac{v_j((k_0,\mathbf{k}(0,\mathbf{t}',\theta,s)),\mathbf{t}',z)J(0,\mathbf{t}',\theta,s)}{[\imath w(\mathbf{k}(0,\mathbf{t}',\theta,s),\mathbf{t}',s)k_0-E]^2}\Big|_{\mathbf{t}'=\mathbf{0}}^{\mathbf{t}'=\mathbf{t}} \Big| \\
&\quad \le \text{const}\, V_{\tilde\aleph} |\mathbf{t}|^{\aleph'} \tfrac{\min\{|E|^{\tilde\aleph-\aleph'},1\}}{k_0^2+E^2}
\end{aligned}$$

and

$$\begin{aligned}
&\left| B'_j(\mathbf{t},z) - B''_j(\mathbf{t},z) - B'_j(\mathbf{0},z) + B''_j(\mathbf{0},z)\right| \\
&\quad \le \text{const}\, V_{\tilde\aleph} |\mathbf{t}|^{\aleph'} \int_I d\theta \int_{-\text{const}}^{\text{const}} dk_0 \int dE\ \tfrac{\min\{|E|^{\tilde\aleph-\aleph'},1\}}{k_0^2+E^2} \\
&\quad \le \text{const}\, \mathfrak{l}\, V_{\tilde\aleph}\, |\mathbf{t}|^{\aleph'}.
\end{aligned}$$

Similarly, combining, (B.18), a second copy of (B.18) with $z \to z'$ and

$$\begin{aligned}
&\Big| \tfrac{v_j((k_0,\mathbf{k}(E',\mathbf{t},\theta,s)),\mathbf{t},z)J(E',\mathbf{t},\theta,s)}{[\imath w(\mathbf{k}(E',\mathbf{t},\theta,s),\mathbf{t},s)k_0-E]^2} - \tfrac{v_j((k_0,\mathbf{k}(E',\mathbf{t},\theta,s)),\mathbf{t},z')J(E',\mathbf{t},\theta,s)}{[\imath w(\mathbf{k}(E',\mathbf{t},\theta,s),\mathbf{t},s)k_0-E]^2} \Big| \\
&\quad \le \text{const}\, V_{\tilde\aleph} |z-z'|^{\tilde\aleph} \tfrac{1}{k_0^2+E^2}
\end{aligned}$$

gives, for all $0 \le \aleph' < \tilde\aleph < \aleph$,

$$\begin{aligned}
&\Big| \tfrac{v_j((k_0,\mathbf{k}(E,\mathbf{t},\theta,s)),\mathbf{t},\tilde z)J(E,\mathbf{t},\theta,s)}{[\imath w(\mathbf{k}(E,\mathbf{t},\theta,s),\mathbf{t},s)k_0-E]^2}\Big|_{\tilde z=z'}^{\tilde z=z} - \tfrac{v_j((k_0,\mathbf{k}(0,\mathbf{t},\theta,s)),\mathbf{t},\tilde z)J(0,\mathbf{t},\theta,s)}{[\imath w(\mathbf{k}(0,\mathbf{t},\theta,s),\mathbf{t},s)k_0-E]^2}\Big|_{\tilde z=z'}^{\tilde z=z} \Big| \\
&\quad \le \text{const}\, V_{\tilde\aleph} |z-z'|^{\aleph'} \tfrac{\min\{|E|^{\tilde\aleph-\aleph'},1\}}{k_0^2+E^2}
\end{aligned}$$

and

$$\begin{aligned}
&\left| B'_j(\mathbf{t},z) - B''_j(\mathbf{t},z) - B'_j(\mathbf{t},z') + B''_j(\mathbf{t},z')\right| \\
&\quad \le \text{const}\, V_{\tilde\aleph} |z-z'|^{\aleph'} \int_I d\theta \int_{-\text{const}}^{\text{const}} dk_0 \int dE\ \tfrac{\min\{|E|^{\tilde\aleph-\aleph'},1\}}{k_0^2+E^2} \\
&\quad \le \text{const}\, \mathfrak{l}\, V_{\tilde\aleph}\, |z-z'|^{\aleph'}.
\end{aligned}$$

Fix s, $\mathbf{t}$ and θ and write w for $w(\mathbf{k}(0,\mathbf{t},\theta,s),\mathbf{t},s)$. Then, for all $k_0 \ne 0$,

$$\int_{-\infty}^{\infty} dE\ \tfrac{1}{[\imath w k_0 - E]^2} = \int_{-\infty}^{\infty} dE\ \tfrac{d}{dE} \tfrac{1}{\imath w k_0 - E} = 0.$$

Thus $B''_j(\mathbf{t},z) \equiv 0$.

d) By part (a), it suffices to prove that $B_j(\mathbf{t}, z)$ converges as $j \to \infty$. By (B.14), $\sup_j \big| v_j(k, \mathbf{t}, z) I(k_0, \mathbf{k}, \mathbf{t}, s) \big|$ is locally L^1 in k and s. Hence, by the Lebesgue dominated convergence theorem applied to the integral in (B.13), $B_j(\mathbf{t}, z) - B'_j(\mathbf{t}, z)$ converges as $j \to \infty$. So it suffices to prove that $B'_j(\mathbf{t}, z)$ converges. By (B.18), the Lebesgue dominated convergence theorem also implies that $B'_j(\mathbf{t}, z) - B''_j(\mathbf{t}, z)$ converges as $j \to \infty$. We have already observed that $B''_j(\mathbf{t}, z) \equiv 0$. □

The function $A(t_0, z)$ of Lemma B.3.b was constructed in such a way that the cutoff in the k_0 direction was removed first (it does not even appear in the definition of $A_j(t_0, z)$) and the cutoff in the $e(\mathbf{k})$ direction was removed second (in the limit $j \to \infty$). On the other hand, for the function $B(\mathbf{t}, z)$ of Lemma B.3.d, the cutoff in the $e(\mathbf{k})$ direction was removed before the cutoff in the k_0 direction. The following lemma illustrates, in a simplified setting, that the order of removal of the two cutoffs matters when $t = 0$.

Lemma B.4. *Let, for* $0 < a, b < 1$ *and* $w > 0$*,*

$$B^{(1)}_{a,b} = \int_{a<|k_0|\le 1} dk_0 \int_{b<|E|\le 1} dE \, \tfrac{1}{[\imath w k_0 - E]^2}.$$

Then $B^{(1)}_{a,b}$ *is bounded uniformly on* $w \ge \epsilon > 0$, $0 < a, b < 1$ *and*

$$\lim_{a\to 0} \lim_{b \to 0} B^{(1)}_{a,b} = -\tfrac{4}{w} \tan^{-1} w, \qquad \lim_{b\to 0} \lim_{a \to 0} B^{(1)}_{a,b} = -\tfrac{4}{w}\big[\tan^{-1} w - \tfrac{\pi}{2}\big].$$

Proof.

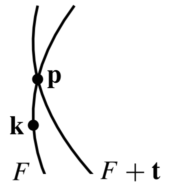
$$\begin{aligned}
B^{(1)}_{a,b} &= \int_{a<|k_0|\le 1} dk_0 \left[\tfrac{1}{\imath w k_0 - E}\Big|_b^1 + \tfrac{1}{\imath w k_0 - E}\Big|_{-1}^{-b} \right] \\
&= -2 \int_{a<|k_0|\le 1} dk_0 \left[\tfrac{1}{1+w^2k_0^2} - \tfrac{b}{b^2+w^2k_0^2} \right] \\
&= -4 \int_a^1 dk_0 \left[\tfrac{1}{1+w^2k_0^2} - \tfrac{b}{b^2+w^2k_0^2} \right] \\
&= -\tfrac{4}{w} \int_{wa}^{w} dx \, \tfrac{1}{1+x^2} + \tfrac{4}{w} \int_{wa}^{w} dx \, \tfrac{b}{b^2+x^2} \\
&= -\tfrac{4}{w} \int_{wa}^{w} dx \, \tfrac{1}{1+x^2} + \tfrac{4}{w} \int_{wa/b}^{w/b} dy \, \tfrac{1}{1+y^2} \\
&= -\tfrac{4}{w} \Big[\tan^{-1} w - \tan^{-1} wa - \tan^{-1} \tfrac{w}{b} + \tan^{-1} \tfrac{wa}{b} \Big].
\end{aligned}$$

Appendix C. Sector Counting with Specified Transfer Momentum

As pointed out in the introduction to §III, we are interested in translates $F + \mathbf{t} = \{ \mathbf{k} + \mathbf{t} \mid \mathbf{k} \in F \}$ of the Fermi surface F, and in particular in the distances from points of $F + \mathbf{t}$ to F.

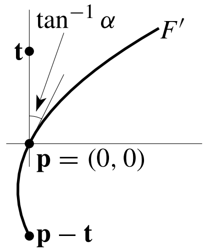
Lemma C.1. *There are constants* δ, $\mathit{const} > 0$ *that depend only on the Fermi curve* F *such that the following holds: Let* $\mathbf{p} \in F$, $|\mathbf{t}| \le \delta$ *such that* $\mathbf{p} - \mathbf{t} \in F$. *Denote by* U *the disc of radius* δ *around* $\mathbf{p}$. *Then for any* $\mathbf{k} \in F \cap U$.

$$|\mathbf{k} - \mathbf{p}| \le \tfrac{\mathit{const}}{|\mathbf{t}|} \operatorname{dist}\big(\mathbf{k}, F + \mathbf{t}\big)$$



Proof. If $\operatorname{dist}\big(\mathbf{k}, F + \mathbf{t}\big) \ge \frac{1}{2}|\mathbf{t}|$ or if $\mathbf{k} = \mathbf{p}$ there is nothing to prove. So assume that $\operatorname{dist}\big(\mathbf{k}, F + \mathbf{t}\big) \le \frac{1}{2}|\mathbf{t}|$. If δ was chosen small enough, the angle between the chords $-\mathbf{t} = (\mathbf{p} - \mathbf{t}) - \mathbf{p}$ and $\mathbf{k} - \mathbf{p}$ of F is sufficiently small. In particular, $(\mathbf{k} - \mathbf{p}) \cdot \mathbf{t} \ne 0$.

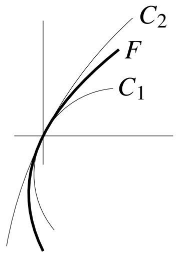
Case 1. $(\mathbf{k} - \mathbf{p}) \cdot \mathbf{t} > 0$. We may assume without loss of generality that $\mathbf{p} = (0, 0)$, that $\mathbf{t} = (0, t_2)$ with $t_2 > 0$ and that the tangent direction of F at $\mathbf{p}$ is $(\alpha, 1)$ with some $\alpha > 0$. As F is strictly convex, and both $\mathbf{p} = (0, 0)$ and $\mathbf{t} = (0, t_2)$ are on F, $F' = \big\{ \mathbf{k}' \in F \cap U \,\big|\, (\mathbf{k}' - \mathbf{p}) \cdot \mathbf{t} \ge 0 \big\}$ is contained in the first quadrant, if δ was chosen small enough. By the implicit function theorem, F' can be parametrized in the form $F' = \big\{ \big(x,\ y(x)\big) \,\big|\, 0 \le x < \mathit{const} \big\}$ with an $r_e + 3$ times differentiable function satisfying $y'(0) = \frac{1}{\alpha}$, $y'' < 0$.



Since the curvature of F is bounded above and below, there are constants const_1, $\mathit{const}_2 > 0$ such that

$$\mathit{const}_1 |\mathbf{t}| \le \alpha \le \mathit{const}_2 |\mathbf{t}|.$$

If δ was chosen small enough, $y' > 1$. Let c_1 resp. c_2 be the maximal resp. minimal curvature of F, and let C_1 resp. C_2 be the circles of curvature c_1 resp. c_2 that are tangent to F at $\mathbf{p}$ and curved in the same direction as F at $\mathbf{p}$. Then F' lies between C_1 and C_2, and the slope of F' at a point $(x,\ y(x))$ lies between the slopes of C_1 resp. C_2 at the points with the same x–coordinate in the first quadrant.



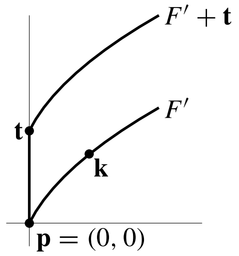
If C is a circle of a radius $r > 0$ that is tangent to F at $\mathbf{p}$ and curved in the same direction as F at $\mathbf{p}$ then the slope of C at any of its points (x, y) in the first quadrant is equal to

$$\frac{\frac{r}{\sqrt{1+\alpha^2}} - x}{y + \frac{\alpha r}{\sqrt{1+\alpha^2}}}.$$

Therefore, for any point $\big(x,\ y(x)\big)$ of F'

$$y'(x) \le \frac{\frac{1}{c_2\sqrt{1+\alpha^2}} - x}{y(x) + \frac{\alpha}{c_2\sqrt{1+\alpha^2}}} \le \frac{const_3}{y(x) + const_4|\mathbf{t}|}.$$

Let F'' be the union of $\mathbf{t} + F'$ and the segment joining $\mathbf{p} = (0, 0)$ to $\mathbf{t}$.



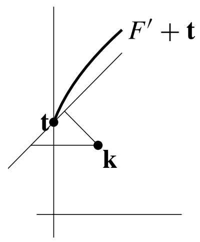
Then

$$\mathrm{dist}(\mathbf{k}, F + \mathbf{t}) \ge \mathrm{dist}(\mathbf{k}, F'') \ge \min\big\{k_1, \mathrm{dist}(\mathbf{k}, F' + \mathbf{t})\big\}.$$

As $k_1 \ge \alpha k_2 \ge const|\mathbf{t}|\,|\mathbf{k}|$, we get that

$$\tfrac{const}{|\mathbf{t}|}\mathrm{dist}(\mathbf{k}, F + \mathbf{t}) \ge \min\big\{|\mathbf{k} - \mathbf{p}|, \tfrac{1}{|\mathbf{t}|}\mathrm{dist}(\mathbf{k}, F' + \mathbf{t})\big\}. \qquad \text{(C.1)}$$

If $k_2 \le |\mathbf{t}|$ then the distance from $\mathbf{k}$ to $F' + \mathbf{t}$ is larger than the distance from $\mathbf{k}$ to the ray through $\mathbf{t}$ in the direction $(1, 1)$, since $y' > 1$.

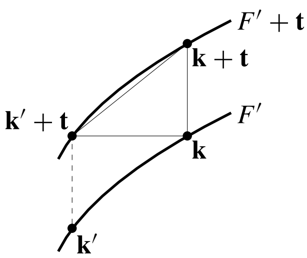


Consequently

$$\mathrm{dist}(\mathbf{k}, F' + \mathbf{t}) \ge \tfrac{1}{\sqrt{2}}k_1 \ge const|\mathbf{t}|\,|\mathbf{k} - \mathbf{p}|.$$

Together with (C.1) this gives the claim of the lemma in the situation that $k_2 \le |\mathbf{t}|$.

Now assume that $k_2 \ge |\mathbf{t}|$. Let $\mathbf{k}' = (k_1', k_2 - |\mathbf{t}|)$ be the point of F' with y–coordinate $k_2' = k_2 - |\mathbf{t}|$ that lies to the left of $\mathbf{k}$, i.e. $k_1' < k_1$.



By the convexity of F, the distance of $\mathbf{k}$ to $\mathbf{t}+F$ is bounded below by the distance of $\mathbf{k}$ to the line segment joining $\mathbf{k}'+\mathbf{t}$ and $\mathbf{k}+\mathbf{t}$. Thus

$$\operatorname{dist}(\mathbf{k}, F'+\mathbf{t}) \geq \tfrac{1}{\sqrt{2}} \min\{|\mathbf{t}|, k_1 - k_1'\}.$$

Since F' is strictly convex

$$k_1 - k_1' \geq \tfrac{|\mathbf{t}|}{y'(k_1')} \geq const_5|\mathbf{t}|(y(k_1') + const_4|\mathbf{t}|) = const_5|\mathbf{t}|(k_2' + const_4|\mathbf{t}|).$$

If $k_2 \leq 2|\mathbf{t}|$ then $|\mathbf{t}| \geq const\, |\mathbf{k}| = const\, |\mathbf{k}-\mathbf{p}|$ and

$$k_1 - k_1' \geq const|\mathbf{t}|^2 \geq const|\mathbf{t}|\,|\mathbf{k}-\mathbf{p}|$$

and if $k_2 \geq 2|\mathbf{t}|$

$$k_1 - k_1' \geq const\, |\mathbf{t}|\, k_2' \geq const\, |\mathbf{t}|\, k_2 \geq const\, |\mathbf{t}|\, |\mathbf{k}| = const\, |\mathbf{t}|\, |\mathbf{k}-\mathbf{p}|.$$

Therefore

$$\tfrac{1}{|\mathbf{t}|}\operatorname{dist}(\mathbf{k}, F'+\mathbf{t}) \geq const\, \min\{1, |\mathbf{k}-\mathbf{p}|\} \geq const|\mathbf{k}-\mathbf{p}|.$$

Again, (C.1) implies the claim of the lemma in the situation that $k_2 \geq |\mathbf{t}|$.

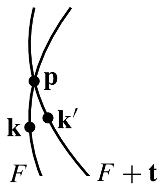
Case 2. $(\mathbf{k}-\mathbf{p})\cdot\mathbf{t} < 0$. Let $\mathbf{k}' \in F+\mathbf{t}$ such that $\operatorname{dist}(\mathbf{k}, F+\mathbf{t}) = |\mathbf{k}-\mathbf{k}'|$. Then

$$\operatorname{dist}\big(\mathbf{k}', (F+\mathbf{t})-\mathbf{t}\big) \leq |\mathbf{k}'-\mathbf{k}| = \operatorname{dist}(\mathbf{k}, F+\mathbf{t}).$$

and

$$|\mathbf{k}-\mathbf{p}| \leq |\mathbf{k}-\mathbf{k}'| + |\mathbf{k}'-\mathbf{p}| \leq \operatorname{dist}(\mathbf{k}, F+\mathbf{t}) + |\mathbf{k}'-\mathbf{p}|. \tag{C.2}$$

Observe that the point $\mathbf{p}$ of $F+\mathbf{t}$ has the property that $\mathbf{p}-(-\mathbf{t})$ lies also in $F+\mathbf{t}$. Also, if δ was chosen small enough, the angle between the tangents to F at $\mathbf{p}$ and to $F+\mathbf{t}$ at $\mathbf{p}$ (which is parallel to the tangent to F at $\mathbf{p}-\mathbf{t}$) is very small and $(\mathbf{k}'-\mathbf{p})\cdot(-\mathbf{t}) > 0$.



Thus we can apply the results of Case 1, with F replaced by $F+\mathbf{t}$, $\mathbf{t}$ replaced by $-\mathbf{t}$ and $\mathbf{k}$ replaced by $\mathbf{k}'$ and get

$$|\mathbf{k}'-\mathbf{p}| \leq \tfrac{const}{|\mathbf{t}|}\operatorname{dist}\big(\mathbf{k}', (F+\mathbf{t})-\mathbf{t}\big) \leq \tfrac{const}{|\mathbf{t}|}\operatorname{dist}(\mathbf{k}, F+\mathbf{t}).$$

This, together with (C.2), proves the lemma in Case 2. □

Lemma C.2. *There are constants* δ_F, const *that depend only on F and M such that the following holds:*

Let $\boldsymbol{\tau} \in \mathbb{R}^2$, $\epsilon > 0$ and D the disc centered at $\boldsymbol{\tau}$ with radius ϵ. Let $m \geq 1$ be a scale with $\mathfrak{l}_m \geq \frac{1}{2}\epsilon$. Define

$$N = \#\big\{ (s_1, s_2) \in \Sigma_m \times \Sigma_m \;\big|\; (s_1 - s_2) \cap D \neq \emptyset \big\},$$

where $D \subset \mathbb{R}^2$ is viewed as $\big\{ (0, \mathbf{t}) \;\big|\; \mathbf{t} \in D \big\} \subset \mathbb{R} \times \mathbb{R}^2$.

a) If $|\boldsymbol{\tau}| \geq \delta_F$*, then* $N \leq \frac{\text{const}}{\sqrt{\mathfrak{l}_m}}$.
b) If $|\boldsymbol{\tau}| \leq \delta_F$*, then*

$$N \leq \tfrac{\text{const}}{\mathfrak{l}_m|\boldsymbol{\tau}|}\Big(\tfrac{1}{M^m} + \epsilon\Big) + \text{const}\,.$$

Proof. We first observe that, given any fixed $s_1 \in \Sigma_m$, $(s_1 - s_2) \cap D \neq \emptyset$ only if $s_2 \cap (s_1 - D) \neq \emptyset$. As $s_1 - D$ is contained in a ball of radius at most $3\mathfrak{l}_m$, there are at most five sectors $s_2 \in \Sigma_m$ that intersect it. Hence

$$\begin{aligned} N &\leq \text{const}\,\#\big\{\, s_1 \in \Sigma_m \;\big|\; \exists s_2 \in \Sigma_m \text{ such that } s_2 \cap (s_1 - D) \neq \emptyset \,\big\} \\ &\leq \text{const}\,\#\big\{\, s_1 \in \Sigma_m \;\big|\; \exists \mathbf{k} \in s_1 \cap F \text{ such that } \operatorname{dist}\big(\mathbf{k} - \boldsymbol{\tau}, F\big) \leq \text{const}_1 \left(\tfrac{1}{M^m} + \epsilon\right) \big\}. \end{aligned}$$

Define

$$I = \big\{\, \mathbf{k} \in F \;\big|\; \operatorname{dist}(\mathbf{k} - \boldsymbol{\tau}, F) \leq \text{const}_1 \left(\tfrac{1}{M^m} + \epsilon\right) \big\}.$$

Then

$$N \leq \text{const}\,\#\big\{\, s \in \Sigma_m \;\big|\; s \cap I \neq \emptyset \,\big\}. \tag{C.3}$$

Clearly $I \subset I'$, where

$$I' = \big\{\, \mathbf{k} \in F \;\big|\; \operatorname{dist}(\mathbf{k}, F + \boldsymbol{\tau}) \leq \text{const } \mathfrak{l}_m \,\big\},$$

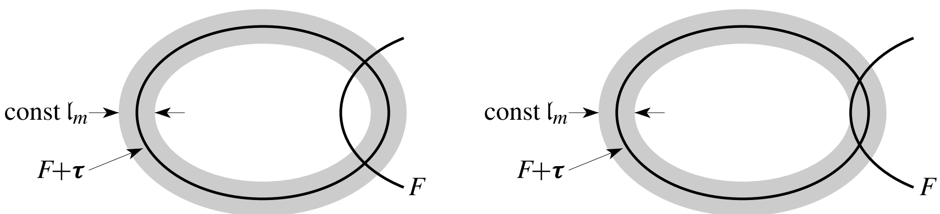
and hence

$$N \leq \text{const}\,\#\big\{\, s \in \Sigma_m \;\big|\; s \cap I' \neq \emptyset \,\big\} \leq \tfrac{\text{const}}{\mathfrak{l}_m} \operatorname{length}(I') + \text{const}\,. \tag{C.4}$$

We choose δ_F to be smaller than the constant δ of Lemma C.1.

a) If $|\boldsymbol{\tau}| \geq \delta_F$, we use (C.4) and that

$$\operatorname{length}\big(I'\big) \leq \text{const } \sqrt{\mathfrak{l}_m}$$



b) Assume that $|\boldsymbol{\tau}| \leq \delta_F$. Since F is strictly convex, $F \cap (F + \boldsymbol{\tau})$ consists of two points, say $\mathbf{p}_1, \mathbf{p}_2$. Let U_1 and U_2 be the discs of radius δ_F around $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. If δ_F is small enough, U_1 and U_2 are disjoint. By Lemma C.1, for $i = 1, 2$, $U_i \cap I$ is contained in an interval of length $\frac{const}{|\boldsymbol{\tau}|}\big(\frac{1}{M^m} + \epsilon\big)$. Also by Lemma C.1, for $i = 1, 2$, the distance between $F + \boldsymbol{\tau}$ and any endpoint of $F \cap U_i$ is bigger than $const\, \delta_F |\boldsymbol{\tau}|$. If $\frac{1}{|\boldsymbol{\tau}|}\big(\frac{1}{M^m} + \epsilon\big) \geq \frac{const}{2\text{const}_1}\delta_F$, the desired bound follows immediately from (C.3) and the fact that $\#\Sigma_m \leq \frac{const}{\mathfrak{l}_m}$. So we may assume that, for $i = 1, 2$ the distance between $F + \boldsymbol{\tau}$ and any endpoint of $F \cap U_i$ is bigger than $2\text{const}_1 \left(\frac{1}{M^m} + \epsilon\right)$.
We now show that $I \subset U_1 \cup U_2$. For this purpose, let $\mathbf{k} \in I$. Then there is $\mathbf{v} \in \mathbb{R}^2$ with $|\mathbf{v}| \leq \text{const}_1 \left(\frac{1}{M^m} + \epsilon\right)$ such that $\mathbf{k} \in F + \boldsymbol{\tau} + \mathbf{v}$. Now for $i = 1, 2$ there is point $\mathbf{k}_i \in F \cap U_i$ such that $\mathbf{k}_i - \mathbf{v} \in F + \boldsymbol{\tau}$. (At the two endpoints $\mathbf{k}'$ of $F \cap U_i$, the points

$\mathbf{k}' - \mathbf{v}$ lie on opposite sides of $F + \boldsymbol{\tau}$.) Since $F \cap (F + \boldsymbol{\tau} + \mathbf{v})$ consists of only two points, $\mathbf{k} = \mathbf{k}_1$ or $\mathbf{k} = \mathbf{k}_2$; in particular $\mathbf{k} \in U_1 \cup U_2$.
Therefore I is contained in two intervals of length $\frac{const}{|\boldsymbol{\tau}|}\big(\frac{1}{M^m} + \epsilon\big)$ and, by (C.3),

$$N \le \tfrac{\text{const}}{\mathfrak{l}_m}\,\tfrac{const}{|\boldsymbol{\tau}|}\big(\tfrac{1}{M^m} + \epsilon\big) + \text{const}\,. \quad \square$$

Recall that $\boldsymbol{\pi} : k = (k_0, \mathbf{k}) \mapsto \mathbf{k}$ is the projection of $\mathbb{M} = \mathbb{R} \times \mathbb{R}^2$ onto its second factor.

Lemma C.3. *Let* $1 \le \ell \le m \le r$ *and* $\kappa' \in \mathfrak{K}_\ell,\ \kappa_1, \kappa_2 \in \mathfrak{K}_r$. *Then the number of 4–tuples* $(u_1, u_2, s_1, s_2) \in \Sigma_\ell \times \Sigma_\ell \times \Sigma_m \times \Sigma_m$ *fulfilling*

$$\begin{aligned} &\boldsymbol{\pi}(s_1 - s_2) \cap \boldsymbol{\pi}(\kappa_1 - \kappa_2) \ne \emptyset,\\ &\boldsymbol{\pi}(u_1 - u_2) \cap \boldsymbol{\pi}(\kappa' - s_1) \ne \emptyset \end{aligned} \tag{C.5}$$

is bounded by $\frac{\text{const}}{\mathfrak{l}_m\sqrt{\mathfrak{l}_\ell}}$ *with the constant* const *independent of* $\kappa_1, \kappa_2, \kappa',\ \ell, m$ *and* r.

Proof. Observe that for each fixed $s_1 \in \Sigma_m$ there are at most const sectors $s_2 \in \Sigma_m$ fulfilling $\boldsymbol{\pi}(s_1 - s_2) \cap \boldsymbol{\pi}(\kappa_1 - \kappa_2) \ne \emptyset$. Recall that for any sector s, $\mathbf{k}_s$ denotes the center of $F \cap s$. When $\kappa' \in \mathbb{M}$, set $\mathbf{k}_{\kappa'} = \boldsymbol{\kappa}'$. We bound each of the three terms in

$$\begin{aligned} &\#\{(u_1, u_2, s_1, s_2)\big|(\text{C.5) holds}\}\\ &\quad\le \#\{(u_1, u_2, s_1, s_2)\big|(\text{C.5) holds},\ |\mathbf{k}_{\kappa'} - \mathbf{k}_{s_1}| \le \text{const}\,\mathfrak{l}_\ell\}\\ &\qquad+\#\{(u_1, u_2, s_1, s_2)\big|(\text{C.5) holds},\ \text{const}\,\mathfrak{l}_\ell \le |\mathbf{k}_{\kappa'} - \mathbf{k}_{s_1}| \le \delta_F\}\\ &\qquad+\#\{(u_1, u_2, s_1, s_2)\big|(\text{C.5) holds},\ \delta_F \le |\mathbf{k}_{\kappa'} - \mathbf{k}_{s_1}|\} \end{aligned}$$

separately. For the first term observe that there are at most const $\big[\frac{\mathfrak{l}_\ell}{\mathfrak{l}_m} + 1\big]$ sectors s_1 with $|\mathbf{k}_{s_1} - \mathbf{k}_{\kappa'}| \le \text{const}\,\mathfrak{l}_\ell$, and that for any given s_1, there are at most $\frac{\text{const}}{\mathfrak{l}_\ell}$ pairs (u_1, u_2) such that $\boldsymbol{\pi}(u_1 - u_2) \cap \boldsymbol{\pi}(\kappa' - s_1) \ne \emptyset$. Hence

$$\#\{(u_1, u_2, s_1, s_2)\big|(\text{C.5) holds},\ |\mathbf{k}_{\kappa'} - \mathbf{k}_{s_1}| \le \text{const}\,\mathfrak{l}_\ell\} \le \text{const}\,\big[\tfrac{\mathfrak{l}_\ell}{\mathfrak{l}_m} + 1\big]\tfrac{1}{\mathfrak{l}_\ell} \le \tfrac{\text{const}}{\mathfrak{l}_m}.$$

We next bound the third term. There are at most $\frac{\text{const}}{\mathfrak{l}_m}$ pairs (s_1, s_2) obeying $\boldsymbol{\pi}(s_1 - s_2) \cap \boldsymbol{\pi}(\kappa_1 - \kappa_2) \ne \emptyset$ and $|\mathbf{k}_{\kappa'} - \mathbf{k}_{s_1}| \ge \delta_F$. For each such a pair (s_1, s_2), $\boldsymbol{\pi}(\kappa' - s_1)$ is contained in a disc of radius $2\mathfrak{l}_\ell$, centered a distance at least δ_F from the origin, so, by Lemma C.2a, with m replaced by ℓ, there are at most $\frac{\text{const}}{\sqrt{\mathfrak{l}_\ell}}$ pairs (u_1, u_2) such that (C.5) holds. Hence

$$\#\{(u_1, u_2, s_1, s_2)\big|(\text{C.5) holds},\ \delta_F \le |\mathbf{k}_{\kappa'} - \mathbf{k}_{s_1}|\} \le \tfrac{\text{const}}{\mathfrak{l}_m\sqrt{\mathfrak{l}_\ell}}.$$

Finally, for the second term, we observe that, for each fixed (s_1, s_2) satisfying $\text{const}\,\mathfrak{l}_\ell \le |\mathbf{k}_{\kappa'} - \mathbf{k}_{s_1}| \le \delta_F$ there are, by Lemma C.2b with $\epsilon = 2\mathfrak{l}_\ell$ and m replaced by ℓ, at most $\frac{\text{const}}{\mathfrak{l}_\ell|\mathbf{k}_{\kappa'} - \mathbf{k}_{s_1}|}\big[\frac{1}{M^\ell} + \mathfrak{l}_\ell\big] + \text{const} \le \frac{\text{const}}{|\mathbf{k}_{\kappa'} - \mathbf{k}_{s_1}|}$ pairs (u_1, u_2) such that (C.5) holds. Furthermore, we may order the allowed s_1's so that the μ^{th} obeys $|\mathbf{k}_{s_1} - \mathbf{k}_{\kappa'}| \ge \text{const}\,(\mathfrak{l}_\ell + \mu\mathfrak{l}_m)$. Hence

$$\#\{(u_1,u_2,s_1,s_2)\ \big|\ \text{(C.5) holds, const}\,\mathfrak{l}_\ell \le |\mathbf{k}_{\kappa'}-\mathbf{k}_{s_1}| \le \delta_F\} \le \sum_{\mu=1}^{\mathrm{const}/\mathfrak{l}_m} \tfrac{\mathrm{const}}{\mathfrak{l}_\ell+\mu\mathfrak{l}_m}$$

$$\le \tfrac{\mathrm{const}}{\mathfrak{l}_m}\sum_{\mu=1}^{\mathrm{const}/\mathfrak{l}_m}\tfrac{1}{\frac{\mathfrak{l}_\ell}{\mathfrak{l}_m}+\mu} \le \tfrac{\mathrm{const}}{\mathfrak{l}_m}\ln\tfrac{\frac{\mathfrak{l}_\ell}{\mathfrak{l}_m}+\frac{\mathrm{const}}{\mathfrak{l}_m}}{\frac{\mathfrak{l}_\ell}{\mathfrak{l}_m}} \le \tfrac{\mathrm{const}}{\mathfrak{l}_m}\ln\big(1+\tfrac{\mathrm{const}}{\mathfrak{l}_\ell}\big)$$

$$\le \tfrac{\mathrm{const}}{\mathfrak{l}_m}\ell \le \tfrac{\mathrm{const}}{\mathfrak{l}_m\sqrt{\mathfrak{l}_\ell}}. \quad \square$$

Lemma C.4. *Let $1\le\ell\le m$ and $\kappa'\in\mathfrak{K}_\ell$. Let D be the disc of radius ϵ centered at $\boldsymbol{\tau}$, with $\boldsymbol{\tau}\in\mathbb{R}^2$ and $0\le\epsilon\le 2\mathfrak{l}_m$. Let N be the number of 4–tuples $(u_1,u_2,s_1,s_2)\in\Sigma_\ell\times\Sigma_\ell\times\Sigma_m\times\Sigma_m$ fulfilling*

$$\begin{aligned}\boldsymbol{\pi}(s_1-s_2)\cap D &\ne \emptyset,\\ \boldsymbol{\pi}(u_1-u_2)\cap\boldsymbol{\pi}(\kappa'-s_1) &\ne \emptyset.\end{aligned}\tag{C.6}$$

a) If $|\boldsymbol{\tau}|\ge\delta_F$, then $N\le\frac{\mathrm{const}}{\mathfrak{l}_\ell\sqrt{\mathfrak{l}_m}}$.
b) If $|\boldsymbol{\tau}|\le\delta_F$, then

$$N\le\tfrac{\mathrm{const}}{\mathfrak{l}_m\mathfrak{l}_\ell}\Big[\min\big(\ell\mathfrak{l}_\ell,\tfrac{1+M^m\epsilon}{M^m|\boldsymbol{\tau}|}\big)+\tfrac{\sqrt{\mathfrak{l}_\ell}}{|\boldsymbol{\tau}|}\big(\tfrac{1}{M^m}+\epsilon\big)+\mathfrak{l}_m\Big].$$

Proof. a) By Lemma C.2.a, $\#\{(s_1,s_2)\in\Sigma_m^2\big|\boldsymbol{\pi}(s_1-s_2)\cap D\ne\emptyset\}\le\frac{\mathrm{const}}{\sqrt{\mathfrak{l}_m}}$. For each fixed (s_1,s_2) there are at most $\frac{\mathrm{const}}{\mathfrak{l}_\ell}$ pairs (u_1,u_2) such that (C.6) holds. The desired bound follows.

b) As in the proof of Lemma C.3, we bound each of the three terms in

$$\begin{aligned}&\#\{(u_1,u_2,s_1,s_2)\big|\text{(C.6) holds}\}\\ &\quad\le\#\{(u_1,u_2,s_1,s_2)\big|\text{(C.6) holds, } |\mathbf{k}_{\kappa'}-\mathbf{k}_{s_1}|\le\mathrm{const}\,\mathfrak{l}_\ell\}\\ &\qquad+\#\{(u_1,u_2,s_1,s_2)\big|\text{(C.6) holds, const}\,\mathfrak{l}_\ell\le|\mathbf{k}_{\kappa'}-\mathbf{k}_{s_1}|\le\delta_F\}\\ &\qquad+\#\{(u_1,u_2,s_1,s_2)\big|\text{(C.6) holds, } \delta_F\le|\mathbf{k}_{\kappa'}-\mathbf{k}_{s_1}|\}\end{aligned}$$

separately.
By Lemma C.2.b

$$\#\{(s_1,s_2)\in\Sigma_m^2\big|\boldsymbol{\pi}(s_1-s_2)\cap D\ne\emptyset\}\le\mathrm{const}\big[1+\tfrac{1+M^m\epsilon}{M^m\mathfrak{l}_m|\boldsymbol{\tau}|}\big].\tag{C.7}$$

As well, for each fixed s_1 there are at most const $s_2\in\Sigma_m$ such that $\boldsymbol{\pi}(s_1-s_2)\cap D\ne\emptyset$. Hence

$$\begin{aligned}&\#\{(s_1,s_2)\in\Sigma_m^2\big|\boldsymbol{\pi}(s_1-s_2)\cap D\ne\emptyset,\ |\mathbf{k}_{s_1}-\mathbf{k}_{\kappa'}|\le\mathrm{const}\,\mathfrak{l}_\ell\}\\ &\quad\le\mathrm{const}\ \min\big\{1+\tfrac{1+M^m\epsilon}{M^m\mathfrak{l}_m|\boldsymbol{\tau}|},\ \tfrac{\mathfrak{l}_\ell}{\mathfrak{l}_m}\big\}.\end{aligned}$$

Also, for each fixed (s_1,s_2) there are at most $\frac{\mathrm{const}}{\mathfrak{l}_\ell}$ pairs (u_1,u_2) such that (C.6) holds. Hence

$$\begin{aligned}&\#\{(u_1,u_2,s_1,s_2)\big|\text{(C.6) holds, } |\mathbf{k}_{\kappa'}-\mathbf{k}_{s_1}|\le\mathrm{const}\,\mathfrak{l}_\ell\}\\ &\quad\le\tfrac{\mathrm{const}}{\mathfrak{l}_m\mathfrak{l}_\ell}\ \min\big\{\mathfrak{l}_m+\tfrac{1+M^m\epsilon}{M^m|\boldsymbol{\tau}|},\ \mathfrak{l}_\ell\big\}\\ &\quad\le\tfrac{\mathrm{const}}{\mathfrak{l}_m\mathfrak{l}_\ell}\big[\min\big\{\tfrac{1+M^m\epsilon}{M^m|\boldsymbol{\tau}|},\ \mathfrak{l}_\ell\big\}+\mathfrak{l}_m\big].\end{aligned}$$

This gives the desired bound for the first term.

We next bound the third term. For each fixed (s_1, s_2) with $|\mathbf{k}_{\kappa'} - \mathbf{k}_{s_1}| \ge \delta_F$, there are, by Lemma C.2a, at most $\frac{\rm const}{\sqrt{\mathfrak{l}_\ell}}$ pairs (u_1, u_2) such that (C.6) holds. Hence

$$\begin{aligned}\#\{(u_1,u_2,s_1,s_2)\big|(\text{C.6})\text{ holds},\ \delta_F \le |\mathbf{k}_{\kappa'}-\mathbf{k}_{s_1}|\} &\le \tfrac{\rm const}{\sqrt{\mathfrak{l}_\ell}}\big[1+\tfrac{1+M^m\epsilon}{M^m\mathfrak{l}_m|\boldsymbol{\tau}|}\big]\\ &\le \tfrac{\rm const}{\mathfrak{l}_m\mathfrak{l}_\ell}\big[\mathfrak{l}_m\sqrt{\mathfrak{l}_\ell}+\tfrac{\sqrt{\mathfrak{l}_\ell}}{|\boldsymbol{\tau}|}\big(\tfrac{1}{M^m}+\epsilon\big)\big]\end{aligned}$$

which is smaller than the desired bound.

Finally, for the second term, we observe that, for each fixed (s_1, s_2) satisfying $\text{const}\,\mathfrak{l}_\ell \le |\mathbf{k}_{\kappa'} - \mathbf{k}_{s_1}| \le \delta_F$ there are, by Lemma C.2b, with $\epsilon = 2\mathfrak{l}_\ell$ and m replaced by ℓ, at most $\frac{\rm const}{\mathfrak{l}_\ell|\mathbf{k}_{\kappa'}-\mathbf{k}_{s_1}|}\big[\frac{1}{M^\ell}+\mathfrak{l}_\ell\big] \le \frac{\rm const}{|\mathbf{k}_{\kappa'}-\mathbf{k}_{s_1}|} \le \frac{\rm const}{\mathfrak{l}_\ell}$ pairs (u_1, u_2) such that (C.6) holds. Hence, by (C.7) and the last argument of Lemma C.3,

$$\begin{aligned}&\#\{(u_1,u_2,s_1,s_2)\big|(\text{C.6})\text{ holds},\ \text{const}\,\mathfrak{l}_\ell \le |\mathbf{k}_{\kappa'}-\mathbf{k}_{s_1}| \le \delta_F\}\\ &\le \min\Big\{\sum_{\mu=1}^{\text{const}/\mathfrak{l}_m}\tfrac{\rm const}{\mathfrak{l}_\ell+\mu\mathfrak{l}_m},\ \tfrac{\rm const}{\mathfrak{l}_\ell}\big[1+\tfrac{1+M^m\epsilon}{M^m\mathfrak{l}_m|\boldsymbol{\tau}|}\big]\Big\}\\ &\le \tfrac{\rm const}{\mathfrak{l}_m\mathfrak{l}_\ell}\min\Big\{\mathfrak{l}_\ell\sum_{\mu=1}^{\text{const}/\mathfrak{l}_m}\tfrac{1}{\frac{\mathfrak{l}_\ell}{\mathfrak{l}_m}+\mu},\ \mathfrak{l}_m+\tfrac{1+M^m\epsilon}{M^m|\boldsymbol{\tau}|}\Big\}\\ &\le \tfrac{\rm const}{\mathfrak{l}_m\mathfrak{l}_\ell}\min\Big\{\mathfrak{l}_\ell\ell,\ \mathfrak{l}_m+\tfrac{1+M^m\epsilon}{M^m|\boldsymbol{\tau}|}\Big\}\\ &\le \tfrac{\rm const}{\mathfrak{l}_m\mathfrak{l}_\ell}\big[\mathfrak{l}_m+\min\{\ell\mathfrak{l}_\ell,\ \tfrac{1+M^m\epsilon}{M^m|\boldsymbol{\tau}|}\}\big].\qquad\square\end{aligned}$$

Notation

Configuration Spaces

Symbol	Interpretation	Reference
$\mathbb{M}$	momentum	after Definition I.3
$\mathfrak{Y}$	momentum or position	before Definition III.1
$\mathfrak{Y}_\Sigma$	momentum or (position, sector)	after Definition I.3
$\mathfrak{Y}_{0,\Sigma}$	momentum	(I.2)
$\mathfrak{Y}_{1,\Sigma}$	(position, sector)	(I.2)
$\mathfrak{Y}_{2,\Sigma}$	(momentum, sector)	Definition I.5
$\mathfrak{Y}_\Sigma^\updownarrow$	(momentum, spin) or (position, spin, sector)	after Definition I.3
$\mathfrak{X}_\Sigma$	(momentum, spin, creation/annihilation index)	after Definition I.3
	or (position, spin, creation/annihilation index, sector)	after Definition I.3
$\mathcal{B}^\updownarrow$	(position, spin)	after Definition I.3
$\check{\mathcal{B}}^\updownarrow$	(momentum, spin)	after Definition I.3
$\mathcal{B}$	(position, spin, creation/annihilation index)	after Definition I.3
$\check{\mathcal{B}}$	(momentum, spin, creation/annihilation index)	after Definition I.3
$\mathfrak{Y}^{(4)}_{\Sigma,\Sigma'}$	$\mathfrak{Y}_\Sigma^2 \times \mathfrak{Y}_{\Sigma'}^2$	(I.2)
$\mathfrak{Y}_{\ell,r}$	$\mathfrak{Y}^{(4)}_{\Sigma_\ell,\Sigma_r}$	Convention II.13
$\mathfrak{K}_r$	momentum or sector	Definition III.7

Norms

Norm	Characteristics	Reference
$\lvert\lvert\lvert\,\cdot\,\rvert\rvert\rvert_{1,\infty}$	no derivatives, external positions only	Definition I.11
$\lvert\lvert\lvert\,\cdot\,\rvert\rvert\rvert$	no derivatives, external positions and momenta	Definition III.1
$\lVert\,\cdot\,\rVert_{\rm bubble}$	operator norm for bubble propagators	Definition III.1
$\lvert\,\cdot\,\rvert^{\delta}_{1,\Sigma}$	two–legged kernel, δ derivatives, sectors	Definition I.12
$\lvert\,\cdot\,\rvert^{(\delta_{\rm l},\delta_{\rm c},\delta_{\rm r})}_{\Sigma,\Sigma'}$	four–legged kernel, $(\delta_{\rm l},\delta_{\rm c},\delta_{\rm r})$ derivatives, sectors	Definition I.14
$\lvert\,\cdot\,\rvert_{1,\Sigma}$	two–legged kernel, all derivatives, sectors	Definition I.15
$\lvert\,\cdot\,\rvert_{\Sigma}$	four–legged kernel, all derivatives, sectors	Definition I.15
$\lVert\,\cdot\,\rVert^{(\delta_{\rm l},\delta_{\rm c},\delta_{\rm r})}_{\ell,r}$	$(\delta_{\rm l},\delta_{\rm c},\delta_{\rm r})$ scaled derivatives, sectors Σ_ℓ, Σ_r	Definition II.14
$\lvert\,\cdot\,\rvert^{[\delta_{\rm l},\delta_{\rm c},\delta_{\rm r}]}_{\ell,r}$	$\le(\delta_{\rm l},\delta_{\rm c},\delta_{\rm r})$ scaled derivatives, sectors Σ_ℓ, Σ_r	Def'ns II.14,II.16
$\lvert\,\cdot\,\rvert^{[\![\delta]\!]}_{j}$	$\delta_{\rm l}+\delta_{\rm c}+\delta_{\rm r}\le\delta$ scaled derivatives, sectors Σ_j	Def'ns II.14,II.16
$\lvert\,\cdot\,\rvert_{\ell,r}$	no derivatives, sectors Σ_ℓ, Σ_r	Definition III.6
$\lVert\,\cdot\,\rVert_{\kappa_1,\kappa_2}$	no derivatives, specified right hand momenta/ sectors	Definition III.7

Propagators and Ladders

Symbol	Interpretation	Reference
$C_v^{(j)}$	$C_v^{(j)}(k)=\frac{\nu^{(j)}(k)}{ik_0-e(\mathbf{k})-v(k)}$, single scale propagator	before Definition I.17
$C_v^{(\ge j)}$	$C_v^{(\ge j)}(k)=\frac{\nu^{(\ge j)}(k)}{ik_0-e(\mathbf{k})-v(k)}$, multi scale propagator	before Definition I.17
$\mathcal{C}(A,B)$	$A\otimes A^t+A\otimes B^t+B\otimes A^t$, bubble propagator	Definition I.8
$\mathcal{C}^{(j)}$	$\sum\limits_{\substack{i_1,i_2\ge 1\\ \min(i_1,i_2)=j}} C_v^{(i_1)}\otimes C_v^{(i_2)\,t}$, single scale bubble propagator	before Convention II.1
$\mathcal{C}^{[i,j]}$	$\sum_{i\le\ell\le j}\mathcal{C}^{(\ell)}$, multi scale bubble propagator	before Convention II.1
$\mathcal{C}^{[i,j]}_{\rm top}$	$\sum_{\substack{i\le i_t\le j\\ i_b>j}} C_v^{(i_t)}\otimes C_v^{(i_b)\,t}$	Definition II.18
$\mathcal{C}^{[i,j]}_{\rm mid}$	$\sum_{\substack{i\le i_t\le j\\ i\le i_b\le j}} C_v^{(i_t)}$	Definition II.18
$\mathcal{C}^{[i,j]}_{\rm bot}$	$\sum_{\substack{i_t>j\\ i\le i_b\le j}} C_v^{(i_t)}\otimes C_v^{(i_b)\,t}$	Definition II.18
$\mathcal{D}^{(\ell)}_{v,{\rm up}}$	$\frac{1}{M^{\lvert\nu\rvert\ell}}\sum_{m=\ell}^\infty {\rm D}^\nu_{1;3}C_v^{(\ell)}\otimes C_v^{(m)\,t}$	Theorem II.20
$\mathcal{D}^{(\ell)}_{v,{\rm dn}}$	$\frac{1}{M^{\lvert\nu\rvert\ell}}\sum_{m=\ell+1}^\infty C_v^{(m)}\otimes {\rm D}^\nu_{2;4}C_v^{(\ell)\,t}$	Theorem II.20
$\mathcal{M}$	model particle–hole bubble propagator	(III.22)
$\mathcal{L}_v^{(j)}(\vec F)$	compound particle–hole ladder	Definition I.19
$L^{(j)}$	single scale compound particle–hole ladder	Definition II.2

Scales and Sectors

Symbol	Interpretation	Reference
M	scale parameter, $M > 1$, large enough	Lemma I.1
ν, φ	used in constructing scale functions	Definition I.2
$\nu^{(j)},\ j \geq 1$	partition of unity that implements scales	Definition I.2
$\nu^{(\geq j)}$	basically $\sum_{i \geq j} \nu^{(i)}$	Definition I.2
$\nu_0(\omega)\nu_1(\mathbf{p}, \mathbf{k})$	factorized cutoff for model bubble propagator	before (III.22)
$\aleph$	$\frac{1}{2} < \aleph < \frac{2}{3}$, parameter controlling sector length	before Definition I.17
$\mathfrak{l}_j$	$\mathfrak{l}_j = \frac{1}{M^{\aleph j}}$, sector length for scale j	before Definition I.17
$\chi_s,\ s \in \Sigma$	partition of unity that implements sectorization	before Definition I.17
Σ_j	set of sectors of scale j	before Definition I.17
$p_\Sigma,\ f_{\Sigma,\Sigma'}$	resectorization	Definition I.17

Miscellaneous

Symbol	Interpretation	Reference
const	generic constant, independent of scale	
const	generic constant, independent of scale and M	
F	Fermi curve $= \left\{ \mathbf{k} \in \mathbb{R}^2 \mid e(\mathbf{k}) = 0 \right\}$	before Definition I.2
r_0, r_e	$r_0, r_e \geq 6$, number of derivatives controlled	before Definition I.2
π_F	projection on the Fermi surface	Definition I.3
$\boldsymbol{\pi}$	$\boldsymbol{\pi}(k_0, \mathbf{k}) = \mathbf{k}$	Remark III.13
$<k, x>_-$	$-k_0 x_0 + \mathbf{k}_1 \mathbf{x}_1 + \mathbf{k}_2 \mathbf{x}_2$	Definition I.4
b_v	$b_1 = b_4 = 0, b_1 = b_2 = 1$	Definition I.4
K^f	flipped vertex	(I.5)
$\mathfrak{c}_j$	$\sum_{\delta \in \Delta} M^{j\|\delta\|} t^\delta + \sum_{\delta \notin \Delta} \infty t^\delta$	(I.4)
Δ	$\left\{ \delta \in \mathbb{N}_0 \times \mathbb{N}_0^2 \mid \delta_0 \leq r_0,\ \delta_1 + \delta_2 \leq r_e \right\}$	(II.1)
$\vec{\Delta}$	$\left\{ \vec{\delta} = (\delta_{\rm l}, \delta_{\rm c}, \delta_{\rm r}) \in \left(\mathbb{N}_0 \times \mathbb{N}_0^2\right)^3 \mid \delta_{\rm l} + \delta_{\rm c} + \delta_{\rm r} \in \Delta \right\}$	(II.1)
$\bullet$	convolution with sector sums	Definition I.8
$\circ$	convolution without sector sums	before (III.2)
$f_C,\ f_S$	charge and spin components	Lemma II.8
W_R	$W_R(p, k) = W(p, k)R(p - k)$, transfer momentum cutoff	Definition III.3
$\mathcal{R}(d)$	set of functions $R(t)$ that are identically one on d	Definition III.10
$\mathcal{Z},\ \mathcal{Z}^t$	zero component localization operator and transpose $\big(\mathcal{Z} \circ W \circ \mathcal{Z}^t\big)(p, k) = \delta(k_0) \int d\omega\ W\big((\omega, \mathbf{0}) + p, (\omega, \mathbf{k})\big)$	(III.18)
$\widetilde{W}$	$\widetilde{W}(p, k) = \delta(k_0) \int d\omega\ W\big((\omega, \mathbf{p}), (\omega, \mathbf{k})\big)$	(III.20)
$\bar{j}$	boundary between large and small transfer momentum	before Prop'n III.16

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Communicated by J.Z. Imbrie