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A Two Dimensional Fermi Liquid. Part 2: Convergence

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Abstract: Using results established in other papers in our series, we prove the existence of the infinite volume, temperature zero, thermodynamic Green's functions of a two dimensional, weakly coupled fermion gas with an asymmetric Fermi curve and short range interactions. This is done by showing that our sequence of renormalization group maps converges.

Contents

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IV. Introduction

This paper, together with [FKTf1] and [FKTf3] provides a construction of a two dimensional Fermi liquid at zero temperature. It contains Sects. IV through X and Appendix B. Sects. I through III and Appendix A are in [FKTf1] and Sects. XI through XV and Appendices C and D are in [FKTf3]. Cumulative notation tables are provided at the end of each part. The main goal of this part is the proof of convergence of the Green's functions stated in Theorem I.4. In the proof of this theorem, which follows the statement of Theorem VIII.5, we compare $\mathcal{G}_{\not{j}}(\phi,\bar\phi)$ with a generating functional $\mathcal{G}_j^{\rm rg}(\phi,\bar\phi)$ constructed by iterating a renormalization group map j times for some $\not{j}-2<j<\not{j}$. See also §III. To aid in the derivation of bounds on the renormalization group map, we fix a scale parameter M that is sufficiently big (depending on the dispersion relation $e(\mathbf{k})$ and the ultraviolet cutoff $U(\mathbf{k})$). This M is used throughout the rest of this paper, with the exception of the proof of Theorem I.4 from Theorem VIII.5, where we also explain that fixing M gives no loss of generality.

V. The First Scales

In §III, we outlined the algebraic aspects of our strategy for proving Theorem I.4. To state and prove the convergence of $\delta e_j(0)$ and $\tilde{\mathcal{G}}_j(\phi,0)$, we clearly have to introduce norms for these and various related objects. There are at least two places where control over derivatives will be needed. The analog, Lemma IX.7, of the formal power series Lemma III.11 will involve an application of the implicit function theorem and will require control of derivatives with respect to K. Secondly, we need to control the size of $\check u\big((k_0,\mathbf{k});K\big)$ in a neighbourhood of the Fermi surface when $k_0\neq 0$, using the fact that this quantity is small when $k_0=0$. This is done using the k_0–derivatives. For this reason we shall also control momentum space derivatives, through position space decay, of quantities appearing in the strategy outlined in the last section. The notations in Definitions V.1 and V.2, below, are introduced to aid in keeping track of the effect of the chain rule and Leibniz's rule on the estimates of derivatives (see [FKTo1, §II]). We use these notations to formulate the bounds of Theorem V.8, in which all scales up to some fixed index j_0 are integrated out. These bounds would deteriorate badly with j_0 and not permit the limit $j_0\to\infty$ to be taken. Theorem V.8 (which will be reformulated in Theorem VI.12) provides the starting point for the renormalization group analysis of Chapter IX, in which scale dependent power counting is crucial.

Definition V.1 (Decay operators).

i) Recall that, for a multiindex δ, $x=(x_0,\mathbf{x},\sigma),\ x'=(x_0',\mathbf{x}',\sigma')\in\mathbb{R}\times\mathbb{R}^d\times\{\uparrow,\downarrow\}$,

$$(x-x')^\delta=(x_0-x_0')^{\delta_0}\,(\mathbf{x}_1-\mathbf{x}_1')^{\delta_1}\cdots(\mathbf{x}_d-\mathbf{x}_d')^{\delta_d}.$$

If $\xi=(x,a),\ \xi'=(x',a')\in\mathcal{B}$, we define $(\xi-\xi')^\delta=(x-x')^\delta$.

ii) Let n be a positive integer. For a function $f(\xi_1,\cdots,\xi_n)$ on $\mathcal{B}^n$, a multiindex δ, and $1\le i,j\le n;\ i\neq j$ set

$$\mathcal{D}^\delta_{i,j}f\,(\xi_1,\cdots,\xi_n)=(\xi_i-\xi_j)^\delta f(\xi_1,\cdots,\xi_n).$$

A decay operator $\mathcal{D}$ on the set of functions on $\mathcal{B}^n$ is an operator of the form

$$\mathcal{D}=\mathcal{D}^{\delta^{(1)}}_{u_1,v_1}\cdots\mathcal{D}^{\delta^{(k)}}_{u_k,v_k}$$

with multiindices $\delta^{(1)}, \cdots, \delta^{(k)}$ *and* $1 \le u_j, v_j \le n,\ u_j \ne v_j$. *The indices* u_j, v_j *are called variable indices. The total order of* $\mathcal{D}$ *is*

$$\delta(\mathcal{D}) = \delta^{(1)} + \cdots + \delta^{(k)}.$$

In a similar way, we define the action of a decay operator on the set of functions on $\big(\mathbb{R} \times \mathbb{R}^d\big)^n$ *or on* $\big(\mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\}\big)^n$.

Definition V.2. *i) On* $\mathbb{R}_+ \cup \{\infty\} = \big\{\, x \in \mathbb{R} \,\big|\, x \ge 0 \,\big\} \cup \{+\infty\}$, *addition and the total ordering* $\le$ *are defined in the standard way. With the convention that* $0 \cdot \infty = \infty$, *multiplication is also defined in the standard way.*

ii) Let $d \ge 0$. *The* $(d+1)$*–dimensional norm domain* $\mathfrak{N}_{d+1}$ *is the set of all formal power series*

$$X = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^d} X_\delta\, t_0^{\delta_0} t_1^{\delta_1} \cdots t_d^{\delta_d}$$

in the variables $t_0, t_1, \cdots, t_d$ *with coefficients* $X_\delta \in \mathbb{R}_+ \cup \{\infty\}$. *To shorten notation, we set* $t^\delta = t_0^{\delta_0} t_1^{\delta_1} \cdots t_d^{\delta_d}$. *Addition and partial ordering on* $\mathfrak{N}_{d+1}$ *are defined componentwise. Multiplication is defined by*

$$(X \cdot X')_\delta = \sum_{\beta+\gamma=\delta} X_\beta X'_\gamma.$$

The max and min of two elements of $\mathfrak{N}_{d+1}$ *are again defined componentwise. We identify* $\mathbb{R}_+ \cup \{\infty\}$ *with the set of all* $X \in \mathfrak{N}_{d+1}$ *having* $X_\delta = 0$ *for all* $\delta \ne \mathbf{0} = (0, \cdots, 0)$. *If* $a > 0$, $X_{\mathbf{0}} \ne \infty$ *and* $a - X_{\mathbf{0}} > 0$ *then* $(a - X)^{-1}$ *is defined as*

$$(a - X)^{-1} = \tfrac{1}{a - X_{\mathbf{0}}} \sum_{n=0}^{\infty} \big(\tfrac{X - X_{\mathbf{0}}}{a - X_{\mathbf{0}}}\big)^n.$$

For an element $X = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^d} X_\delta\, t^\delta$ *of* $\mathfrak{N}_{d+1}$ *and* $0 \le j \le d$ *the formal derivative* $\frac{\partial}{\partial t_j} X$ *is defined as*

$$\tfrac{\partial}{\partial t_j} X = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^d} (\delta_j + 1) X_{\delta + \epsilon_j}\, t^\delta,$$

where ϵ_j *is the* j^{th} *unit vector.*

iii) For $j \ge 0$ *we set*

$$\mathfrak{c}_j = \sum_{\substack{|\boldsymbol{\delta}| \le r \\ |\delta_0| \le r_0}} M^{j|\delta|}\, t^\delta + \sum_{\substack{|\boldsymbol{\delta}| > r \\ \text{or } |\delta_0| > r_0}} \infty\, t^\delta \in \mathfrak{N}_{d+1},$$

and for $X \in \mathfrak{N}_{d+1}$ *with* $X_{\mathbf{0}} < \frac{1}{M^j}$,

$$\mathfrak{e}_j(X) = \tfrac{\mathfrak{c}_j}{1 - M^j X}.$$

Definition V.3. *For a function f on $\mathcal{B}^m \times \mathcal{B}^n$ we define the (scalar valued) L_1–L_∞–norm as*

$$|||f|||_{1,\infty} = \begin{cases} \max\limits_{1\le j_0\le n} \sup\limits_{\xi_{j_0}\in\mathcal{B}} \int \prod\limits_{\substack{j=1,\cdots,n\\ j\ne j_0}} d\xi_j \, |f(\xi_1,\cdots,\xi_n)| & \text{if } m = 0 \\ \sup\limits_{\eta_1,\cdots,\eta_m\in\mathcal{B}} \int \prod\limits_{j=1,\cdots,n} d\xi_j \, |f(\eta_1,\cdots,\eta_m;\ \xi_1,\cdots,\xi_n)| & \text{if } m\ne 0 \end{cases}$$

and the $(d+1)$–dimensional L_1–L_∞ seminorm

$$\|f\|_{1,\infty} = \begin{cases} \sum\limits_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^d} \frac{1}{\delta!}\Big(\max\limits_{\substack{\mathcal{D}\text{ decay operator}\\ \text{with } \delta(\mathcal{D})=\delta}} |||\mathcal{D} f|||_{1,\infty}\Big)\, t^\delta & \text{if } m=0 \\ |||f|||_{1,\infty} & \text{if } m\ne 0 \end{cases}.$$

Here $|||f|||_{1,\infty}$ stands for the formal power series with constant coefficient $|||f|||_{1,\infty}$ and all other coefficients zero and $\int d\xi\ g(\xi) = \sum_{a\in\{0,1\}}\sum_{\sigma\in\{\uparrow,\downarrow\}}\int dx_0 d\mathbf{x}\, g\big((x_0,\mathbf{x},\sigma,a)\big)$.

Given a function on momentum space, we apply the above norms using the

Notation V.4. If $\chi(k)$ is a function on $\mathbb{R}\times\mathbb{R}^d$, we define the Fourier transform $\hat\chi$ by

$$\hat\chi(\xi,\xi') = \delta_{\sigma,\sigma'}\delta_{a,a'} \int e^{(-1)^a \imath <k,x-x'>_-}\, \chi(k)\, \tfrac{d^{d+1}k}{(2\pi)^{d+1}}$$

for $\xi = (x,a) = (x_0,\mathbf{x},\sigma,a)$, $\xi' = (x',a') = (x_0',\mathbf{x}',\sigma',a') \in \mathcal{B}$.

Remark V.5.

i) Let $V(x_1,x_2,x_3,x_4)$ be an interaction kernel as in Theorem I.4 and define, by abuse of notation, the function V on $\mathcal{B}^4$ by

$$V\big((x_1,b_1),(x_2,b_2),(x_3,b_3),(x_4,b_4)\big) = \delta_{b_1,1}\delta_{b_2,0}\delta_{b_3,1}\delta_{b_4,0}\, V(x_1,x_2,x_3,x_4).$$

Then the hypothesis of Theorem I.4 is equivalent to $\|V\|_{1,\infty} \le \varepsilon\mathfrak{c}_0$ for some sufficiently small ε.

ii) The constants $\mathfrak{c}_j$ will be used to describe the behaviour of momentum space derivatives of the covariance $C^{(j)}$. The quantities $\mathfrak{e}_j(X)$ are used in bounding the differentiability properties of various kernels depending on a counterterm whose norm is bounded by X. This allows us to take into account the fact that the characteristics, as regards both size and smoothness, of counterterms are very different from the characteristics of kernels built purely from $C^{(j)}$ and various smooth functions. The characteristics of the counterterms are a consequence of their construction from various $C^{(j')}$'s, including those with $j' \gg j$. As j' increases, the contribution to the counterterm from $C^{(j')}$ becomes smaller and smaller, and more and more concentrated near the Fermi surface, but less and less smooth.

We also wish to use our norms to control the coupling constant dependence of various kernels. This is done using

Definition V.6. *Fix* $0 < \upsilon < \frac{1}{4}$. *Set, for a coupling constant* $0 < \lambda < 1$,

$$\rho_{m;n}(\lambda) = \frac{1}{\lambda^{(1-\upsilon)\max\{m+n-2,2\}/2}} = \begin{cases} \lambda^{-(1-\upsilon)(m+n-2)/2} & \text{if } m+n \geq 4 \\ \lambda^{-(1-\upsilon)} & \text{if } m+n = 2 \end{cases}.$$

Remark V.7. The exponent of Definition V.6 is motivated by the following considerations. For this discussion, introduce a coupling constant λ and replace $\mathcal{V}(\psi)$ by $\lambda\mathcal{V}(\psi)$.

The exponent of the initial generating functional contains, aside from the counterterm, two vertices with ψ fields. One, $\lambda\mathcal{V}(\psi)$, has four ψ fields and is proportional to the coupling constant λ. The other, $\psi\phi$, has one ψ field, one ϕ field and is independent of λ. Consider any connected graph G with m external ϕ legs, n external ψ legs, $\mathrm{v} \geq 1$ of the $\lambda\mathcal{V}(\psi)$ vertices and m of the $\psi\phi$ vertices. Since the ϕ field is always external, G must have precisely m $\psi\phi$ vertices to have m external ϕ legs. The graph has $\frac{4\mathrm{v}+2m-(m+n)}{2}$ internal lines. To be connected, G must have at least $\mathrm{v}+m-1$ internal lines, so that

$$\tfrac{4\mathrm{v}+2m-(m+n)}{2} \geq \mathrm{v}+m-1 \qquad \Longrightarrow \qquad \mathrm{v} \geq \tfrac{m+n-2}{2}.$$

Thus G is proportional to λ^{v} with

$$\mathrm{v} \geq \max\left\{\tfrac{m+n-2}{2}, 1\right\}.$$

We set aside $\lambda^{\upsilon\max\{m+n-2,2\}/2}$, which we bound by $\lambda^{\upsilon n/10}$ to achieve good inductive behaviour, i.e. to control various constants that arise in the course of the expansion. Ultimately, we choose a maximum allowed coupling constant λ_0, rename $\frac{1}{\lambda_0^{\upsilon/10}} = \alpha_0$ and consider $|\lambda| < \lambda_0$ and $\alpha \geq \alpha_0$. Then, our bound on the m ϕ–legged, n ψ–legged part of the effective interaction will be proportional to

$$\tfrac{1}{\alpha^n}\lambda_0^{(1-\upsilon)\max\{m+n-2,2\}/2}.$$

We now further explain the phrase "good inductive behaviour" used in the last paragraph. Consider, more generally, a connected graph G with m external ϕ legs, n external ψ legs, $\tilde{m}$ of the $\psi\phi$ vertices and $\mathrm{v} \geq 1$ other vertices. Suppose that the i^{th} other vertex has m_i ϕ–legs and n_i ψ–legs. The number $\frac{\Sigma_i(m_i+n_i)+2\tilde{m}-(m+n)}{2}$ of internal lines must be at least $\mathrm{v}+\tilde{m}-1$ so

$$\tfrac{\Sigma_i(m_i+n_i)+2\tilde{m}-(m+n)}{2} \geq \mathrm{v}+\tilde{m}-1 \quad \Rightarrow \quad \sum_{i=1}^{\mathrm{v}} \tfrac{m_i+n_i-2}{2} \geq \tfrac{m+n-2}{2}.$$

As $\mathrm{v} \geq 1$,

$$\sum_{i=1}^{\mathrm{v}} \max\left\{\tfrac{m_i+n_i-2}{2}, 1\right\} \geq \max\left\{\tfrac{m+n-2}{2}, 1\right\}.$$

We thus have

$$\frac{\alpha^n}{\lambda_0^{(1-\upsilon)\max\{m+n-2,2\}/2}} \leq \frac{1}{\alpha^{\Sigma n_i - n}} \prod_{i=1}^{\mathrm{v}} \frac{\alpha^{n_i}}{\lambda_0^{(1-\upsilon)\max\{m_i+n_i-2,2\}/2}}.$$

The small factors $\frac{1}{\alpha^{\Sigma n_i - n}}$ are available for controlling various constants that arise in the course of the expansion. Observe that, as $m = \tilde{m} + \Sigma m_i$ and $\tilde{m} \leq \Sigma n_i - n$, the number of internal lines of G, $\frac{\Sigma_i(m_i+n_i)+2\tilde{m}-(m+n)}{2}$ is bounded by $\Sigma n_i - n$.

We choose an arbitrary but fixed scale, $j_0 \geq 2$, and integrate the first scales, between 1 and j_0, in one fell swoop.

Theorem V.8. *There are (M and j_0–dependent) constants μ, $\bar\lambda$ and β_0 such that, for all $\lambda < \bar\lambda$ and $\beta_0 \leq \beta \leq \frac{1}{\lambda^{\upsilon/5}}$, the following holds: Let $X \in \mathfrak{N}_{d+1}$ with $X_{\mathbf{0}} < \mu$, $\delta e \in \mathcal{E}$ with $\|\delta\hat e\|_{1,\infty} \leq X$ and*

$$\mathcal{V}(\psi) = \int_{\mathcal{B}^4} d\xi_1\cdots d\xi_4\, V(\xi_1,\cdots,\xi_4)\, \psi(\xi_1)\cdots\psi(\xi_1)$$

with an antisymmetric function V fulfilling

$$\|V\|_{1,\infty} \leq \lambda\, \mathfrak{e}_0(X).$$

Write

$$\begin{aligned}\tilde\Omega_{C^{(\leq j_0)}_{-\delta e}}\big(\mathcal{V}(\psi)\big)(\phi,\psi) = \mathcal{V}(\psi) &+ \tfrac12 \phi J C^{(\leq j_0)}_{-\delta e} J\phi \\ &+ \sum_{\substack{m,n\geq 0\\ m+n \text{ even}}} \int_{\mathcal{B}^{m+n}} d\eta_1\cdots d\eta_m\, d\xi_1\cdots d\xi_n\, W_{m,n}(\eta_1,\cdots,\eta_m,\xi_1,\cdots,\xi_n;\delta e) \\ &\times \phi(\eta_1)\cdots\phi(\eta_m)\,\psi(\xi_1)\cdots\psi(\xi_n)\end{aligned}$$

with kernels $W_{m,n}$ that are separately antisymmetric under permutations of their η and ξ arguments. Then

$$\sum_{\substack{m+n\geq 2\\ m+n \text{ even}}} \beta^n\, \rho_{m;n}(\lambda)\, \|W_{m,n}(\delta e)\|_{1,\infty} \leq \text{const}\, \beta^3 \lambda^{\upsilon}\, \mathfrak{e}_0(X)$$

and

$$\sum_{\substack{m+n\geq 2\\ m+n \text{ even}}} \beta^n\, \rho_{m;n}(\lambda)\, \Big\| \tfrac{d}{ds} W_{m,n}(\delta e + s\delta e')\big|_{s=0}\Big\|_{1,\infty} \leq \text{const}\, \beta^3\lambda^{\upsilon}\, \mathfrak{e}_0(X)\, \|\delta\hat e'\|_{1,\infty}.$$

Furthermore, each $W_{m,n}$ is jointly analytic[1] in V and δe. If V fulfills the reality condition of (I.1) and $\delta e(\mathbf{k})$ is real valued, then $\tilde\Omega_{C^{(\leq j_0)}_{-\delta e}}\big(\mathcal{V}(\psi)\big)(\phi,\psi)$ is k_0–reversal real, in the sense of Definition B.1.R of [FKTo2][2].

Proof. Apply Theorem VIII.6 of [FKTo2] with $\rho_{m,n} = \rho_{m;n}(\lambda)$ and $\varepsilon = {\scriptstyle const}\,\beta^4\lambda^{\upsilon}$. Observe that, by Remark VIII.7.iii of [FKTo2], the hypotheses on $\rho_{m;n}$ are fulfilled. If $\bar\lambda$ is chosen small enough, then the hypothesis $\varepsilon < \varepsilon_0$ is also fulfilled.The reality statement is a consequence of Remark B.5 of [FKTo2]. □

In Theorem V.8, we integrated out the part of the field ψ with covariance $C^{(\leq j_0)}$. To recover the full, infrared cutoff covariance $C^{\mathrm{IR}(j)}$ of Theorem I.4, we must also integrate out the part of the field with covariance

$$C_u^{(i,j)}(k) = \frac{\nu^{(>i)}(k) - \nu^{(\geq j)}(k)}{\imath k_0 - e(\mathbf{k}) - \check u(k)[1-\nu^{(\geq j)}(k)]}.$$

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[1] As in the discussion leading up to Theorem I.4, the $W_{m,n}$'s are initially defined as formal Taylor series in V. The conclusions of the theorem implicitly include the convergence of the formal Taylor series for V obeying $\|V\|_{1,\infty} \leq \lambda\,\mathfrak{e}_0(X)$ and δe obeying $\|\delta\hat e\|_{1,\infty} \leq X$.

[2] This definition is the obvious generalization of (I.1) to functions of arbitrarily many variables.

Lemma V.9. *Let $\rlap{\,/}j \ge j_0+2$ be an infrared cutoff. For $|||V|||_{1,\infty}$ and $|||\delta e|||_{1,\infty}$ sufficiently small*

$$\tilde\Omega_{C_{-\delta e}^{(j_0,\rlap{\,/}j)}}\Big(\tilde\Omega_{C_{-\delta e}^{(\le j_0)}}\big(\mathcal{V}(\psi)\big)\Big) = \tilde\Omega_{C^{\mathrm{IR}(\rlap{\,/}j)}(\delta e)}\big(\mathcal{V}(\psi)\big).$$

Proof. Since $|||V|||_{1,\infty}$ and $|||\delta e|||_{1,\infty}$ are sufficiently small, both sides of the desired identity are well-defined. It is proven by applying the semi–group property (III.4) using

$$C^{\mathrm{IR}(\rlap{\,/}j)}(\delta e) = C_{-\delta e}^{(\le j_0)} + C_{-\delta e}^{(j_0,\rlap{\,/}j)}. \quad \square$$

VI. Sectors and Sectorized Norms

From now on we consider only $d = 2$, so that the Fermi "surface" is a curve in $\mathbb{R}\times\mathbb{R}^2$. We choose a projection π_F from the first extended neighbourhood onto the Fermi surface.

Convention. Generic constants that depend only on the dispersion relation $e(\mathbf{k})$ and the ultraviolet cutoff $U(\mathbf{k})$ will be denoted by "*const*". Generic constants that may also depend on the scale parameter M, but still not on the scale j, will be denoted "const".

To systematically deal with Fourier transforms, we call

$$\check{\mathcal{B}} = \mathbb{R}\times\mathbb{R}^d\times\{\uparrow,\downarrow\}\times\{0,1\}$$

"momentum space". For $\check\xi = (k,\sigma',a') = (k_0,\mathbf{k},\sigma',a') \in \check{\mathcal{B}}$ and $\xi = (x,a) = (x_0,\mathbf{x},\sigma,a) \in \mathcal{B}$ we define the inner product

$$\left\langle\check\xi,\xi\right\rangle = \delta_{\sigma',\sigma}\delta_{a',a}(-1)^a\ \langle k,x\rangle_- = \delta_{\sigma',\sigma}\delta_{a',a}(-1)^a\Big(-k_0x_0+\mathbf{k}_1\mathbf{x}_1+\cdots+\mathbf{k}_d\mathbf{x}_d\Big),$$

"characters"

$$E_+(\check\xi,\xi) = \delta_{\sigma',\sigma}\delta_{a',a}e^{\imath\left\langle\check\xi,\xi\right\rangle} = \delta_{\sigma',\sigma}\delta_{a',a}e^{\imath(-1)^a\left(-k_0x_0+\mathbf{k}_1\mathbf{x}_1+\cdots+\mathbf{k}_d\mathbf{x}_d\right)},$$
$$E_-(\check\xi,\xi) = \delta_{\sigma',\sigma}\delta_{a',a}e^{-\imath\left\langle\check\xi,\xi\right\rangle} = \delta_{\sigma',\sigma}\delta_{a',a}e^{-\imath(-1)^a\left(-k_0x_0+\mathbf{k}_1\mathbf{x}_1+\cdots+\mathbf{k}_d\mathbf{x}_d\right)},$$

and integrals

$$\int d\xi\ \cdot\ = \sum_{\substack{a\in\{0,1\}\\ \sigma\in\{\uparrow,\downarrow\}}}\int_{\mathbb{R}\times\mathbb{R}^d} dx_0\,d^d\mathbf{x}\ \cdot \qquad\qquad \int d\check\xi\ \cdot\ = \sum_{\substack{a\in\{0,1\}\\ \sigma\in\{\uparrow,\downarrow\}}}\int_{\mathbb{R}\times\mathbb{R}^d} dk_0\,d^d\mathbf{k}\ \cdot$$

For $\check\xi = (k,\sigma,a)$, $\check\xi' = (k',\sigma',a') \in \check{\mathcal{B}}$ we set

$$\check\xi + \check\xi' = (-1)^a\,k + (-1)^{a'}\,k' \ \in \mathbb{R}\times\mathbb{R}^d.$$

Definition VI.1 (Fourier transforms). *Let $f(\eta_1,\cdots,\eta_m;\xi_1,\cdots,\xi_n)$ be a translation invariant function on $\mathcal{B}^m\times\mathcal{B}^n$. The total Fourier transform $\check f$ of f is defined by*

$$\begin{aligned}&\check f(\check\eta_1,\cdots,\check\eta_m;\check\xi_1,\cdots,\check\xi_n)\ (2\pi)^{d+1}\delta(\check\eta_1+\cdots+\check\eta_m+\check\xi_1+\cdots+\check\xi_n)\\ &\quad= \int\prod_{i=1}^m E_+(\check\eta_i,\eta_i)\,d\eta_i\ \prod_{j=1}^n E_+(\check\xi_j,\xi_j)\,d\xi_j\ f(\eta_1,\cdots,\eta_m;\xi_1,\cdots,\xi_n)\end{aligned}$$

or, equivalently, by

$$f(\eta_1,\cdots,\eta_m;\xi_1,\cdots,\xi_n) = \int \prod_{i=1}^{m} \frac{E_-(\check\eta_i,\eta_i)\,d\check\eta_i}{(2\pi)^{d+1}} \prod_{j=1}^{n} \frac{E_-(\check\xi_j,\xi_j)\,d\check\xi_j}{(2\pi)^{d+1}}\, \check f(\check\eta_1,\cdots,\check\eta_m;\check\xi_1,\cdots,\check\xi_n)$$
$$\times (2\pi)^{d+1}\delta(\check\eta_1+\cdots+\check\eta_m+\check\xi_1+\cdots+\check\xi_n).$$

$\check f$ *is defined on the set* $\left\{ (\check\eta_1,\cdots,\check\eta_m;\check\xi_1,\cdots,\check\xi_n) \in \check{\mathcal{B}}^m \times \mathcal{B}^n \;\middle|\; \check\eta_1+\cdots+\check\eta_m+\check\xi_1+\cdots+\check\xi_n=0 \right\}$. *If* $m=0$, $n=2$ *and* $f(\xi_1,\xi_2)$ *conserves particle number and is spin independent and antisymmetric, we define* $\check f(k)$ *by*

$$\check f\big((k,\sigma,1),(k,\sigma',0)\big) = \delta_{\sigma,\sigma'}\check f(k).$$

We now introduce sectors.

Definition VI.2 (Sectors and sectorizations).

i) Let I *be an interval on the Fermi surface* F *and* $j \ge 2$. *Then*

$$s = \left\{ k \text{ in the } j^{\text{th}} \text{ neighbourhood} \;\middle|\; \pi_F(k) \in I \right\}$$

is called a sector of length $|I|$ *at scale* j. *Two different sectors* s *and* s' *are called neighbours if* $s' \cap s \ne \emptyset$.

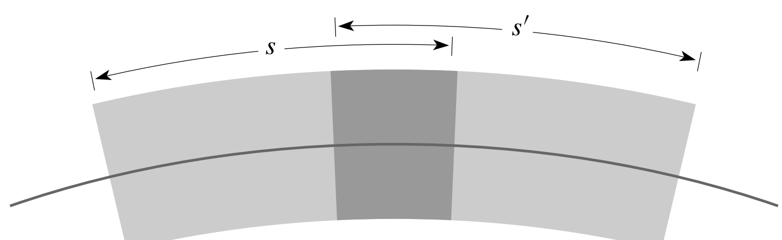
ii) If s *is a sector at scale* j, *its extension is*

$$\tilde s = \left\{ k \text{ in the } j^{\text{th}} \text{ extended neighbourhood} \;\middle|\; \pi_F(k) \in s \right\}.$$

iii) A sectorization of length $\mathfrak{l}$ *at scale* j *is a set* Σ *of sectors of length* $\mathfrak{l}$ *at scale* j *that obeys*

- *the set* Σ *of sectors covers the Fermi surface*
- *each sector in* Σ *has precisely two neighbours in* Σ, *one to its left and one to its right*
- *if* $s,\ s' \in \Sigma$ *are neighbours then* $\frac{1}{16}\mathfrak{l} \le |s \cap s' \cap F| \le \frac{1}{8}\mathfrak{l}$.

Observe that there are at most $2\,\mathrm{length}(F)/\mathfrak{l}$ *sectors in* Σ.



Definition VI.3 (Sectorized representatives). *Let* Σ *be a sectorization at scale* j, *and let* $m, n \ge 0$.

i) The antisymmetrization of a function φ *on* $\mathcal{B}^m \times \big(\mathcal{B}\times\Sigma\big)^n$ *is*

$$\begin{aligned}&\mathrm{Ant}\,\varphi(\eta_1,\cdots,\eta_m;(\xi_1,s_1),\cdots,(\xi_n,s_n))\\ &\quad= \frac{1}{m!\,n!}\sum_{\substack{\pi\in S_m\\ \pi'\in S_n}} \mathrm{sgn}\pi\ \mathrm{sgn}\pi'\,\varphi\big(\eta_{\pi(1)},\cdots,\eta_{\pi(m)};(\xi_{\pi'(1)},s_{\pi'(1)}),\cdots,(\xi_{\pi'(n)},s_{\pi'(n)})\big).\end{aligned}$$

ii) Denote by $\mathcal{F}_m(n;\Sigma)$ *the space of all translation invariant, complex valued functions*

$$\varphi\big(\eta_1,\cdots,\eta_m;\,(\xi_1,s_1),\cdots,(\xi_n,s_n)\big)$$

on $\mathcal{B}^m\times\big(\mathcal{B}\times\Sigma\big)^n$ *that are antisymmetric in their external (*$=\eta$*) variables and whose Fourier transform* $\check\varphi(\check\eta_1,\cdots,\check\eta_m;\,(\check\xi_1,s_1),\cdots,(\check\xi_n,s_n))$ *vanishes unless* $k_i\in\tilde s_i$ *for all* $1\le i\le n$*. Here,* $\check\xi_i=(k_i,\sigma_i,a_i)$*.*

iii) Let f *be a translation invariant, complex valued function on* $\mathcal{B}^m\times\mathcal{B}^n$ *that is antisymmetric in its first m variables. A* Σ*–sectorized representative for* f *is a function* $\varphi\in\mathcal{F}_m(n;\Sigma)$ *obeying*

$$\check f(\check\eta_1,\cdots,\check\eta_m;\check\xi_1,\cdots,\check\xi_n)=\sum_{\substack{s_i\in\Sigma\\ i=1,\cdots,n}}\check\varphi(\check\eta_1,\cdots,\check\eta_m;\,(\check\xi_1,s_1),\cdots,(\check\xi_n,s_n))$$

for all $\check\xi_i=(k_i,\sigma_i,a_i)$ *with* k_i *in the* $j^{\rm th}$ *neighbourhood.*

iv) Let $u((\xi,s),\,(\xi',s'))$ *be a translation invariant, spin independent, particle number conserving function on* $(\mathcal{B}\times\Sigma)^2$*. We define* $\check u(k)$ *by*

$$\delta_{\sigma,\sigma'}\check u(k)=\sum_{s,s'\in\Sigma}\check u\big((k,\sigma,1,s),\,(k,\sigma',0,s')\big).$$

We now fix a constant $\frac12<\aleph<\frac23$, and for each scale $j\ge2$, a sectorization Σ_j of length $\mathfrak{l}_j=\frac{1}{M^{\aleph j}}$. Also, we fix for each $j\ge2$, a system $\chi_s(k),\ s\in\Sigma_j$ of functions that take values in $[0,1]$ such that

i) χ_s is supported in the extended sector $\tilde s$ and

$$\sum_{s\in\Sigma}\chi_s(k)=1\qquad\text{for } k \text{ in the } j^{\rm th}\text{ neighbourhood.}$$

ii)

$$\|\hat\chi_s\|_{1,\infty},\ \le\ {\rm const}\ \mathfrak{c}_{j-1}$$

with a constant *const* that does not depend M, j, or s. The existence of such a "partition of unity" is shown in Lemma XII.3 of [FKTo3]. They are used to construct sectorized representatives.

Definition VI.4. *Let* $j,i\ge2$*. If* $i\ne j$*, define, for functions* φ *on* $\mathcal{B}^m\times\big(\mathcal{B}\times\Sigma_i\big)^n$ *and* f *on* $\check{\mathcal{B}}^m\times\big(\mathcal{B}\times\Sigma_i\big)^n$*,*

$$\begin{aligned}&\varphi_{\Sigma_j}\big(\eta_1,\cdots,\eta_m;\,(\xi_1,s_1),\cdots,(\xi_n,s_n)\big)\\&\quad=\sum_{s'_1,\cdots,s'_n\in\Sigma_i}\int d\xi'_1\cdots d\xi'_n\,\varphi\big(\eta_1,\cdots,\eta_m;\,(\xi'_1,s'_1),\cdots,(\xi'_n,s'_n)\big)\prod_{\ell=1}^n\hat\chi_{s_\ell}(\xi'_\ell,\xi_\ell),\\&f_{\Sigma_j}\big(\check\eta_1,\cdots,\check\eta_m;\,(\xi_1,s_1),\cdots,(\xi_n,s_n)\big)\\&\quad=\sum_{s'_1,\cdots,s'_n\in\Sigma_i}\int d\xi'_1\cdots d\xi'_n\,f\big(\check\eta_1,\cdots,\check\eta_m;\,(\xi'_1,s'_1),\cdots,(\xi'_n,s'_n)\big)\prod_{\ell=1}^n\hat\chi_{s_\ell}(\xi'_\ell,\xi_\ell).\end{aligned}$$

If φ is antisymmetric under permutation of its η arguments, then $\varphi_{\Sigma_j} \in \mathcal{F}_m(n, \Sigma_j)$. For $i = j$ define $\varphi_{\Sigma_j} = \varphi$ and $f_{\Sigma_j} = f$.
Similarly, define, for functions φ on $\mathcal{B}^m \times \mathcal{B}^n$ and f on $\check{\mathcal{B}}^m \times \mathcal{B}^n$,

$$\varphi_{\Sigma_j}(\eta_1,\cdots,\eta_m;(\xi_1,s_1),\cdots,(\xi_n,s_n)) = \int d\xi_1'\cdots d\xi_n'\,\varphi(\eta_1,\cdots,\eta_m;\xi_1',\cdots,\xi_n')\prod_{\ell=1}^n \hat{\chi}_{s_\ell}(\xi_\ell',\xi_\ell),$$
$$f_{\Sigma_j}(\check{\eta}_1,\cdots,\check{\eta}_m;(\xi_1,s_1),\cdots,(\xi_n,s_n)) = \int d\xi_1'\cdots d\xi_n'\,f(\check{\eta}_1,\cdots,\check{\eta}_m;\xi_1',\cdots,\xi_n')\prod_{\ell=1}^n \hat{\chi}_{s_\ell}(\xi_\ell',\xi_\ell).$$

They are Σ_j–sectorized representatives for φ resp. f.

Definition VI.5. *Let $j \ge 2$ be a scale. We consider fermionic fields $\phi(\eta)$, $\eta \in \mathcal{B}$ and $\psi(\xi,s)$, $\xi \in \mathcal{B}$, $s \in \Sigma_j$.*

i) A Σ_j–sectorized Grassmann function is of the form

$$w = \sum_{m,n\ge 0}\sum_{s_1,\cdots,s_n\in\Sigma_j}\int\prod_{i=1}^m d\eta_i\prod_{j=1}^n d\xi_j\; w_{m,n}(\eta_1,\cdots,\eta_m;(\xi_1,s_1),\cdots,(\xi_n,s_n))$$
$$\times\,\phi(\eta_1)\cdots\phi(\eta_m)\;\psi(\xi_1,s_1)\cdots\psi(\xi_n,s_n). \qquad \text{(VI.1)}$$

ii) Let

$$\mathcal{W} = \sum_{m,n\ge 0}\int\prod_{i=1}^m d\eta_i\prod_{j=1}^n d\xi_j\, W_{m,n}(\eta_1,\cdots,\eta_m;\xi_1,\cdots,\xi_n)\phi(\eta_1)\cdots\phi(\eta_m)\psi(\xi_1)\cdots\psi(\xi_n)$$

be a Grassmann function with each $W_{m,n}$ a function on $\mathcal{B}^m \times \mathcal{B}^n$ that is separately antisymmetric in its external ($=\phi$) variables and in its internal ($=\psi$) variables. A Σ_j–sectorized representative for $\mathcal{W}$ is a Σ_j–sectorized Grassmann function of the form (VI.1), where, for each m, n, $w_{m,n}$ is a Σ_j–sectorized representative for $W_{m,n}$ that is also antisymmetric in the variables $(\xi_1, s_1), \cdots, (\xi_n, s_n)$.

Definition VI.6 (Norms for sectorized functions). *Let $j \ge 2$ and $m,\ n \ge 0$.*

i) For a function φ on $\mathcal{B}^m \times (\mathcal{B} \times \Sigma_j)^n$ and an integer $p > 0$ we define the seminorm $|\varphi|_{p,\Sigma_j}$ to be zero if $m \ge 1$, $p \ge 2$ or if $m = 0$, $p > n$.
In the case $m \ge 1$, $p = 1$ we set

$$|\varphi|_{p,\Sigma_j} = \sum_{s_i\in\Sigma_j}\|\varphi(\eta_1,\cdots,\eta_m;(\xi_1,s_1),\cdots,(\xi_n,s_n))\|_{1,\infty}.$$

In the case $m = 0$, $p \le n$ we set

$$|\varphi|_{p,\Sigma_j} = \max_{1\le i_1<\cdots<i_p\le n}\ \max_{s_{i_1},\cdots,s_{i_p}\in\Sigma_j}\ \sum_{\substack{s_i\in\Sigma_j \text{ for}\\ i\ne i_1,\cdots,i_p}}\|\varphi((\xi_1,s_1),\cdots,(\xi_n,s_n))\|_{1,\infty}.$$

In both cases, the $\|\cdot\|_{1,\infty}$ norm applies to all the position space variables.

ii) We shall fix a $\lambda_0 > 0$*, sufficiently small depending on* j_0 *and the scale parameter* M*. For* $\varphi \in \mathcal{F}_m(n, \Sigma_j)$ *set*

$$|\varphi|_j = \rho_{m;n} \begin{cases} |\varphi|_{1,\Sigma_j} + \frac{1}{\mathfrak{l}_j} |\varphi|_{3,\Sigma_j} + \frac{1}{\mathfrak{l}_j^2} |\varphi|_{5,\Sigma_j} & \text{if } m = 0 \\ \frac{\mathfrak{l}_j}{M^{2j}} |\varphi|_{1,\Sigma_j} & \text{if } m \neq 0 \end{cases},$$

where

$$\rho_{m;n} = \rho^{(j)}_{m;n} = \frac{1}{\lambda_0^{(1-\upsilon)\max\{m+n-2,2\}/2}} \begin{cases} 1 & \text{if } m = 0 \\ \sqrt[4]{\mathfrak{l}_j M^j} & \text{if } m > 0 \end{cases}$$

and υ *was fixed in Definition V.6.*

Definition VI.7 (Norms for sectorized Grassmann functions).

i) A Σ_j*–sectorized Grassmann function* w *can be uniquely written in the form*

$$w(\phi, \psi) = \sum_{m,n} \sum_{s_1,\cdots,s_n\in\Sigma_j} \int d\eta_1\cdots d\eta_m\, d\xi_1\cdots d\xi_n\ w_{m,n}(\eta_1,\cdots,\eta_m\, (\xi_1,s_1),\cdots,(\xi_n,s_n)) \\ \times \phi(\eta_1)\cdots\phi(\eta_m)\ \psi((\xi_1,s_1))\cdots\psi((\xi_n,s_n))$$

with $w_{m,n}$ *antisymmetric separately in the* η *and in the* ξ *variables. Set, as in Definition XV.1 of [FKTo3], for* $\alpha > 0$ *and* $X \in \mathfrak{N}_{d+1}$*,*

$$N_j(w, \alpha,\ X) = \frac{M^{2j}}{\mathfrak{l}_j}\, \mathfrak{e}_j(X) \sum_{m,n\ge 0} \alpha^n \left(\frac{\mathfrak{l}_j\, \mathrm{B}}{M^j}\right)^{n/2} |w_{m,n}|_j.$$

The constant B *depends on* M*, but not* j *and was specified in Definition XV.1 of [FKTo3].*

ii) A Grassmann function $\mathcal{G}(\phi)$ *can be uniquely written in the form*

$$\mathcal{G}(\phi) = \sum_m \int d\eta_1\cdots d\eta_m\ G_m(\eta_1,\cdots,\eta_m)\, \phi(\eta_1)\cdots\phi(\eta_m)$$

with G_m *antisymmetric. Set*

$$N(\mathcal{G}) = \sum_{m>0} \frac{1}{\lambda_0^{(1-\upsilon)\max\{m-2,2\}/2}}\ |||G_m|||_\infty,$$

where $|||G_m|||_\infty = \sup_{\eta_1,\cdots,\eta_m} \left|G_m(\eta_1,\cdots,\eta_m)\right|$.

Remark VI.8.

i) The system $\vec{\rho} = (\rho_{m;n})$ of Definition VI.6.ii fulfill the inequalities (XV.1) of [FKTo3].

ii) If $w(\phi, \psi)$ is a Σ_j–sectorized Grassmann function, then

$$N\big(w(\phi, 0)\big) \le \frac{1}{\sqrt[4]{\mathfrak{l}_j M^j}} N_j(w, \alpha,\ X)$$

for all α and $X \in \mathfrak{N}_{d+1}$.

iii) The j independent part of the coefficient of $|w_{m,n}|_{1,\Sigma_j}$ in $N_j(w, \alpha, X)$ is, up to a factor of $\mathrm{B}^{n/2}$, equal to $\frac{\alpha^n}{\lambda_0^{(1-\upsilon)\max\{m+n-2,2\}/2}}$. This choice was motivated in Remark V.7.

iv) If $N_j(w,\alpha,\ X)_{\mathbf{0}} \le 1$, then, up to $\sum_{\delta\ne\mathbf{0}} \infty\, t^\delta$,

$$|w_{m,n}|_{1,\Sigma_j} \le \begin{cases} \frac{1}{\mathfrak{l}_j^{\frac{n}{2}-1}} M^{j(\frac{n}{2}-2)} & \text{if } m=0 \\ \frac{1}{\sqrt[4]{M^j \mathfrak{l}_j}} \big(\frac{M^j}{\mathfrak{l}_j}\big)^{n/2} & \text{if } m\ne 0 \end{cases}. \tag{VI.2}$$

The case $m=0$ was motivated in (II.11). Next consider the case that $m+n=4$, $m,n\ge 1$ and $w_{m,n}$ is the coefficient of $\phi^m\psi^n$ in $\mathcal{V}(\phi+\psi)$. Then, allowing a full sector sum for each ψ leg, $|||V|||_{1,\infty} < \infty$ implies that $|w_{m,n}|_{1,\Sigma_j} = O\big(\frac{1}{\mathfrak{l}_j^n}\big)$, which is a tighter bound than (VI.2). An argument similar to that in Subsect. 8 of §II may also be used to show that if $w_{m,n}$ is a graph with vertices obeying (VI.2), then $w'_{m,n}$ obeys a bound of the same order as (VI.2).

In Definition VI.6, we use the norm $\|\cdot\|_{1,\infty}$ of Definition V.3, to measure kernel sizes. When $m=0$, i.e. there are no external legs, this norm takes a "supremum norm in momentum space". When $m\ne 0$ and there are external legs, this norm sups over the positions of external arguments in position space, which corresponds to an L^1 norm in momentum space. Additional integrals in momentum space tend to smooth out kernel singularities and reduce norm sizes. The resulting improved power counting is captured by the factor $(M^j\mathfrak{l}_j)^{1/4}$ in the definition of $\rho_{m;n}$ (part (ii) of Definition VI.6) and correspondingly in (VI.2).

In [FKTf3] we make a more detailed analysis of the two– and four–point functions. There we apply the "supremum norm in momentum space" to external as well as internal arguments. The improved power counting discussed in the previous paragraph does not appear and we cannot include a factor like $(M^j\mathfrak{l}_j)^{1/4}$ in the norms. See Definition XIII.15.ii.

v) The quantity of main interest in the norm $|\varphi|_j$ of Definition VI.6 is $|\varphi|_{1,\Sigma_j}$. As seen in Example A.2 of [FKTf1], we need $|\cdot|_{3,\Sigma_j}$ norms to get improved bounds on $|\varphi|_{1,\Sigma_j}$ by exploiting overlapping loops. For four point functions φ, $|\varphi|_{3,\Sigma_j}$ is also useful because it mimics the supremum in momentum space. To get improved bounds on $|\varphi|_{3,\Sigma_j}$ by exploiting overlapping loops, we need to use the $|\cdot|_{5,\Sigma_j}$ norm.

We now define the space of functions from which the various counterterm kernels will be chosen. The Fourier transforms of counterterms are functions of the spatial momenta components $\mathbf{k}$ only. The norms introduced earlier in this section for k_0 dependent functions have obvious generalizations to k_0 independent functions. For example, if $K\big((\mathbf{x},s),(\mathbf{x}',s')\big)$ is a translation invariant function on $\big(\mathbb{R}^2\times\Sigma_j\big)^2$, we define

$$\|K\|_{1,\Sigma_j} = \max_{\substack{i_1=1,2\\ s_{i_1}\in\Sigma_j}} \sum_{s_{\bar\imath_1}\in\Sigma_j} \sum_{\delta\in\mathbb{N}_0^d} \tfrac{1}{\delta!} \times \Big(\max_{\substack{\mathcal{D}\text{ decay operator}\\ \text{with }\delta(\mathcal{D})=(0,\delta)}} \max_{i_2=1,2} \sup_{\mathbf{x}_{i_2}} \int d\mathbf{x}_{\bar\imath_2}\, \big|\mathcal{D}K\big((\mathbf{x}_1,s_1),(\mathbf{x}'_2,s'_2)\big)\big|\Big)\, t^\delta,$$

where $\bar\imath = 3-i$. Other obvious generalizations of this nature to k_0 independent functions are formulated precisley in Appendix E of [FKTo4].

Definition VI.9. *Let $\mathfrak{K}_j$ be the space of all translation invariant, sectorized functions $K\big((\mathbf{x},s),(\mathbf{x}',s')\big)$ on $\big(\mathbb{R}^2\times\Sigma_j\big)^2$ for which*

i) $\|K\|_{1,\Sigma_j} < \lambda_0^{1-\upsilon}\frac{\mathfrak{l}_{j+1}}{M^{j+1}} + \sum_{\delta\neq\mathbf{0}}\infty\, t^\delta$,
ii) the Fourier transform $\check{K}(\mathbf{k})$ *is supported on* $\operatorname{supp}\nu^{(\geq j+1)}\big((0,\mathbf{k})\big)$.

The counterterm K *is said to be* **real** *if, for each* $s, s' \in \Sigma_j$, *the Fourier transform* $\check{K}\big((\mathbf{k}, s), (\mathbf{k}', s')\big)$ *is real valued.*

Remark VI.10. If $K \in \mathfrak{K}_j$, then $K_{\Sigma_{j-1}} \in \mathfrak{K}_{j-1}$. To see this, observe that, by part (iii) of Proposition XIX.4 of [FKTo4],

$$\|K_{\Sigma_{j-1}}\|_{1,\Sigma_{j-1}} \leq const\,\frac{\mathfrak{l}_{j-1}}{\mathfrak{l}_j}\mathfrak{c}_{j-2}\|K\|_{1,\Sigma_j} < const\,\frac{\mathfrak{l}_{j-1}}{\mathfrak{l}_j}\lambda_0^{1-\upsilon}\frac{\mathfrak{l}_{j+1}}{M^{j+1}} + \sum_{\delta\neq\mathbf{0}}\infty\, t^\delta < \lambda_0^{1-\upsilon}\frac{\mathfrak{l}_j}{M^j} + \sum_{\delta\neq\mathbf{0}}\infty\, t^\delta,$$

if M is large enough.

Remark VI.11. The final counterterm $\delta e(\mathbf{k})$ will be constructed in Theorem VIII.5 using bounds proven in Lemma X.1.

As in Definition III.5, Remark III.7 and Lemma V.9, we have the following covariances.

Definition VI.12. *(i) Let* $u\big((\xi,s), (\xi',s')\big)$ *be a translation invariant, spin independent, particle number conserving function on* $(\mathcal{B}\times\Sigma_\ell)^2$. *Then*

$$C_u^{(j)}(k) = \frac{\nu^{(j)}(k)}{ik_0 - e(\mathbf{k}) - \check{u}(k)},$$
$$C_u^{(\geq j)}(k) = \frac{\nu^{(\geq j)}(k)}{ik_0 - e(\mathbf{k}) - \check{u}(k)},$$
$$C_u^{[i,j)}(k) = \frac{\nu^{(\geq i)}(k) - \nu^{(\geq j)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{u}(k)[1-\nu^{(\geq j)}(k)]}.$$

(ii) Let u *be a function from* $\mathfrak{K}_j$ *to the space of antisymmetric, translation invariant, spin independent, particle number conserving functions on* $(\mathcal{B}\times\Sigma_j)^2$. *Then, for* $K \in \mathfrak{K}_j$,

$$C_j(u;K)(k) = \frac{\nu^{(\geq j)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{u}(k;K) - \check{K}(\mathbf{k})\nu^{(\geq j+2)}(k)},$$
$$D_j(u;K)(k) = \frac{\nu^{(\geq j+1)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{u}(k;K) - \check{K}(\mathbf{k})\nu^{(\geq j+2)}(k)}.$$

(iii) Let $C_u^{(j)}(\xi,\xi')$, $C_u^{(\geq j)}(\xi,\xi')$, $C_u^{[i,j)}(\xi,\xi')$, $D_j(u;K)(\xi,\xi')$ *and* $C_j(u;K)(\xi,\xi')$ *be their Fourier transforms as in (III.1) and (III.2).*

To start the recursive construction of the Green's functions, we reformulate Theorem V.8 in terms of sectorized objects.

Theorem VI.13. *For* $K \in \mathfrak{K}_{j_0}$, *set*

$$u(K) = -\big[K_{\text{ext}}\big]_{\Sigma_{j_0}} \in \mathcal{F}_0(2, \Sigma_{j_0}),$$

where $K_{\rm ext}$ was defined in Definition E.3 of [FKTo4][3]*. Then there exist constants $\bar\lambda, \bar\alpha > 0$ such that for all $0 \le \lambda_0 < \bar\lambda$, $\bar\alpha < \alpha < \frac{1}{\lambda_0^{\upsilon/10}}$ and all*

$$K \in \mathfrak{K}_{j_0} \qquad \|V\|_{1,\infty} \le \lambda_0 \mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big),$$

the Grassmann function

$$\tilde\Omega_{C_{u(K)}^{(\le j_0)}}\big(\mathcal{V}(\psi)\big)(\phi,\psi) - \tfrac12 \phi J C_{u(K)}^{(\le j_0)} J\phi$$

has a Σ_{j_0}–sectorized representative

$$w(\phi,\psi;K) = \sum_{m,n} \sum_{s_1,\cdots,s_n \in \Sigma_{j_0}} \int d\eta_1\cdots d\eta_m\, d\xi_1\cdots d\xi_n\ w_{m,n}\big(\eta_1,\cdots,\eta_m\,(\xi_1,s_1),\cdots,(\xi_n,s_n);\,K\big)$$
$$\times\, \phi(\eta_1)\cdots\phi(\eta_m)\ \psi\big((\xi_1,s_1)\big)\cdots\psi\big((\xi_n,s_n)\big)$$

with $w_{m,n}$ antisymmetric separately in the η and in the ξ variables, $w_{0,0} = 0$ and

$$N_{j_0}\big(w(K),\alpha,\|K\|_{1,\Sigma_{j_0}}\big) \le \text{const}\ \alpha^4\lambda_0^{\upsilon}\ \mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big),$$
$$N_{j_0}\big(\tfrac{d}{ds} w(K+sK')\big|_{s=0},\alpha,\ \|K\|_{1,\Sigma_{j_0}}\big) \le M^{j_0}\ \mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big)\ \|K'\|_{1,\Sigma_{j_0}}$$

for all K'. w is analytic in V and K. If V fulfills the reality condition of (I.1) and K is real, then $w(\phi,\psi;K)$ is k_0–reversal real, in the sense of Definition B.1.R of [FKTo2].

Proof. Write

$$\tilde\Omega_{C_{u(K)}^{(\le j_0)}}\big(\mathcal{V}(\psi)\big)(\phi,\psi) = \mathcal{V}(\psi) + \tfrac12\phi J C_{u(K)}^{(\le j_0)} J\phi + \sum_{\substack{m,n\ge 0\\ m+n\ \text{even}}} \int_{\mathcal{B}^{m+n}} d\eta_1\cdots d\eta_m\ d\xi_1\cdots d\xi_n$$
$$\times\, W_{m,n}(\eta_1\cdots\eta_m,\xi_1,\cdots,\xi_n)\phi(\eta_1)\cdots\phi(\eta_m)\ \psi(\xi_1)\cdots\psi(\xi_n)$$

and set

$$w_{m,n} = \begin{cases} \big(W_{m,n}\big)_{\Sigma_{j_0}} & \text{if } (m,n)\ne(0,4)\\ \big(W_{0,4}+V\big)_{\Sigma_{j_0}} & \text{if } (m,n)=(0,4)\end{cases}$$

using the sectorization f_Σ of Definition VI.4. By Proposition XIX.15 of [FKTo4]

$$\begin{aligned} N_{j_0}\big(w(K),\alpha,\|K\|_{1,\Sigma_{j_0}}\big) &= \tfrac{M^{2j_0}}{\mathfrak{l}_{j_0}}\ \mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big) \sum_{m,n\ge 0} \alpha^n \big(\tfrac{\mathfrak{l}_{j_0}\,\mathrm{B}}{M^{j_0}}\big)^{n/2}\ |w_{m,n}|_{j_0}\\ &\le \text{const}\ \mathfrak{c}_{j_0}\ \mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big)\Big[\tfrac{\alpha^4}{\lambda_0^{1-\upsilon}}\ \|V\|_{1,\infty}\\ &\qquad + \sum_{m,n\ge 0} (\text{const}\,\alpha)^n\ \rho_{m;n}(\lambda_0)\ \|W_{m,n}\|_{1,\infty}\Big]\end{aligned}$$

since $\rho_{m;n}^{(j_0)} \le \text{const}\ \rho_{m;n}(\lambda_0)$. By hypothesis

$$\tfrac{\alpha^4}{\lambda_0^{1-\upsilon}}\ \|V\|_{1,\infty} \le \alpha^4\lambda_0^{\upsilon}\mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big)$$

[3] By Remark E.4.i of [FKTo4], under this definition, $\check K_{\rm ext}\,((k_0,\mathbf{k})) = \check K(\mathbf{k})$.

and by Theorem V.8, with $\delta e = -\check{u}$, $X = \text{const}\,\|K\|_{1,\Sigma_{j_0}}$ and $\beta = \text{const}\,\alpha$,

$$\sum_{m,n\geq 0} (\text{const}\,\alpha)^n \rho_{m;n}(\lambda_0)\|W_{m,n}\|_{1,\infty} \leq \text{const}\,\beta^3\lambda^\upsilon \mathfrak{e}_0(X) \leq \text{const}\,\alpha^3\lambda_0^\upsilon \mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big).$$

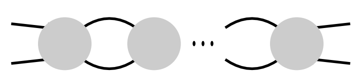
Therefore, by Corollary A.5.ii of [FKTo1],

$$\begin{aligned} N_{j_0}\big(w(K),\alpha,\,\|K\|_{1,\Sigma_{j_0}}\big) &\leq \text{const}\ \alpha^4\lambda_0^\upsilon\ \mathfrak{c}_{j_0}\ \mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big)^2 \\ &\leq \text{const}\ \alpha^4\lambda_0^\upsilon\ \mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big). \end{aligned}$$

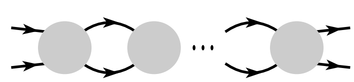
The proof of the bound on $N_{j_0}\big(\frac{d}{ds}w(K+sK')\big|_{s=0},\alpha,\ \|K\|_{1,\Sigma_{j_0}}\big)$ is similar. □

VII. Ladders

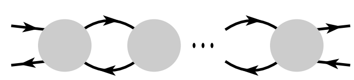
In naive power counting for our model, four-legged vertices are neutral. So there is a danger that the size of four legged kernels after j steps of the renormalization group flow is of order j. We shall show that this logarithmic divergence does not occur. More precisely, let $(\mathcal{W},u,\mathcal{G})\in D_{\text{in}}^{(j)}$ be an input datum before integrating out the j$^{\text{th}}$ scale (see Definition IX.1) and let $(\mathcal{W}',\mathcal{G},u,\vec{p}) = \Omega_j(\mathcal{W},\mathcal{G},u,\vec{p})$ be the result of integrating out scale j. Assume that $(\mathcal{W},\mathcal{G},u,\vec{p})$ is bounded – the precise hypothesis is given in Definition IX.1. We shall show that the norm of the four point part of $\mathcal{W}'$ does not exceed the norm of the four point part of $\mathcal{W}$ by more than $\frac{\text{const}}{M^{\epsilon j}}$ with constants ϵ, const independent of j. Most contributions to the four point part of $\mathcal{W}'-\mathcal{W}$ are controlled using "overlapping loops", see [FKTr1]. The only exceptions are ladders. A ladder consists of a sequence of four legged kernel "rungs" connected by pairs of propagators.



For a formal definition, see §XIV of [FKTo3]. Taking creation and annihilation indices into account, such a ladder is either a "particle–particle ladder"



or a "particle–hole ladder".

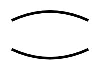


The strong asymmetry of the Fermi curve (see Definition I.10) allows us to bound particle–particle ladders of scale j by $\frac{\text{const}}{M^{\epsilon j}}$. This estimate is stated precisely in Proposition VII.6 below and proven in Theorem XXII.7 of [FKTo4]. This is in contrast to the case of a symmetric Fermi curve, where the particle–particle ladders generate the Cooper instability (see [FW, Chapter 10], [FMRT, §4]). The main estimates on particle–hole ladders are stated in Theorem VII.8 below. They are proven in [FKTl] for arbitrary strictly convex Fermi curves.

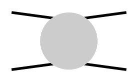
Before we formulate the estimates on particle–hole and particle–particle ladders we give a precise definition of ladders. To treat "undirected ladders", particle–particle and particle–hole ladders with or without spin and with or without external momenta simultaneously, we first work over arbitrary measure spaces, like, for example, $\mathbb{R} \times \mathbb{R}^2$ or $\mathbb{R} \times \mathbb{R}^2 \times \{\uparrow, \downarrow\}$ or $\mathcal{B} = \mathbb{R} \times \mathbb{R}^2 \times \{\uparrow, \downarrow\} \times \{0, 1\}$. See also §XIV of [FKTo3].

Definition VII.1. *Let $\mathfrak{X}$ be a measure space.*

i) A complex valued function on $\mathfrak{X} \times \mathfrak{X}$ is called a propagator over $\mathfrak{X}$.
ii) A four legged kernel over $\mathfrak{X}$ is a complex valued function on $\mathfrak{X}^2 \times \mathfrak{X}^2$. We sometimes consider it as a bubble propagator over $\mathfrak{X}$, graphically depicted by



or as a rung over $\mathfrak{X}$, graphically depicted by



iii) If A and B are propagators over $\mathfrak{X}$ then the tensor product

$$A \otimes B(x_1, x_2, x_3, x_4) = A(x_1, x_3)B(x_2, x_4)$$

is a bubble propagator over $\mathfrak{X}$. We set

$$\mathcal{C}(A, B) = A \otimes A + A \otimes B + B \otimes A.$$

iv) Let F be a four legged kernel over $\mathfrak{X}$. The antisymmetrization of F is the four legged kernel

$$\big(\mathrm{Ant}\, F\big)(x_1, x_2, x_3, x_4) = \tfrac{1}{4!} \sum_{\pi \in S_4} \mathrm{sign}(\pi)\, F(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, x_{\pi(4)}).$$

F is called antisymmetric if $F = \mathrm{Ant}\, F$.

We will consider ladders with rungs taking values in the measure space $\mathcal{B} \times \Sigma$, where Σ is a sectorization. As propagators, we will use the unsectorized propagators $A = C^{(j)}_{u(K)}$ and $B = D_j(u(K); K)$ of Definition III.6.

Definition VII.2. *Let $\mathfrak{X}$ be a measure space and let S be a finite set*[4]*. It is endowed with the counting measure. Then $\mathfrak{X} \times S$ is also a measure space.*

i) Let P be a propagator over $\mathfrak{X}$, f a four legged kernel over $\mathfrak{X} \times S$ and F a function on $(\mathfrak{X} \times S)^2 \times \mathfrak{X}^2$. We define

$$\begin{aligned}
&(f \bullet P)\big((x_1,s_1),(x_2,s_2);x_3,x_4\big) \\
&\quad = \sum_{s_1',s_2' \in S} \int dx_1' dx_2'\; f\big((x_1,s_1),(x_2,s_2),(x_1',s_1'),(x_2',s_2')\big)\; P(x_1',x_2';x_3,x_4), \\
&(F \bullet f)\big((x_1,s_1),\cdots,(x_4,s_4)\big) \\
&\quad = \sum_{s_1',s_2' \in S} \int dx_1' dx_2'\; F\big((x_1,s_1),(x_2,s_2);x_1',x_2'\big)\; f\big((x_1',s_1'),(x_2',s_2'),(x_3,s_3),(x_4,s_4)\big)
\end{aligned}$$

whenever the integrals are well–defined. Observe that $(f \bullet P)$ is a function on $(\mathfrak{X} \times S)^2 \times \mathfrak{X}^2$ and $F \bullet f$ is a four legged kernel over $\mathfrak{X} \times S$.

[4] In practice, S will be a set of sectors and $\mathfrak{X}$ will be $\mathcal{B}$ or $\mathbb{R} \times \mathbb{R}^2 \times \{\uparrow, \downarrow\}$ or $\mathbb{R} \times \mathbb{R}^2$.

ii) Let $\ell \geq 1$, $r_1, \cdots, r_{\ell+1}$ rungs over $\mathfrak{X} \times S$ and $P_1, \cdots, P_\ell$ bubble propagators over $\mathfrak{X}$. The ladder with rungs $r_1, \cdots, r_{\ell+1}$ and bubble propagators $P_1, \cdots, P_\ell$ is defined to be

$$r_1 \bullet P_1 \bullet r_2 \bullet P_2 \bullet \cdots \bullet r_\ell \bullet P_\ell \bullet r_{\ell+1}.$$

If r is a rung over $\mathfrak{X} \times S$ and A, B are propagators over $\mathfrak{X}$, we define $L_\ell(r; A, B)$ as the ladder with $\ell + 1$ rungs r and ℓ bubble propagators $\mathcal{C}(A, B)$.

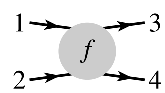
When we integrate out scale j in our model, the contributions to the four legged kernel that are not controlled by "overlapping loops" are antisymmetrizations of ladders that are defined over $\mathcal{B} \times \Sigma$, where Σ is a sectorization. Such ladders decompose into particle–particle ladders and particle–hole ladders that are defined over smaller spaces that do not have creation/annihilation components.

Definition VII.3. *Set $\mathcal{B}^\updownarrow = \left\{ (x_0, \mathbf{x}, \sigma) \,\middle|\, x_0 \in \mathbb{R},\ \mathbf{x} \in \mathbb{R}^2,\ \sigma \in \{\uparrow, \downarrow\} \right\}$. If Σ is a sectorization and $z = (x, \sigma, b, s) \in \mathcal{B} \times \Sigma$, we define its undirected part $u(z) \in \mathcal{B}^\updownarrow \times \Sigma$ and its creation/annihilation index $b(z) \in \{0, 1\}$ by $u(z) = (x, \sigma, s)$ and $b(z) = b$, respectively. If $z' = (x, \sigma, s) \in \mathcal{B}^\updownarrow \times \Sigma$ and $b \in \{0, 1\}$, we define $\iota_b(z') = (x, \sigma, b, s) \in \mathcal{B} \times \Sigma$.*

Definition VII.4.

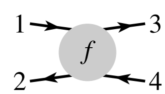
i) Let f be a four legged kernel over $\mathcal{B} \times \Sigma$. When f is a rung, its particle–particle reduction is the four legged kernel over $\mathcal{B}^\updownarrow \times \Sigma$ given by

$$f^{\mathrm{pp}}(z_1',z_2',z_3',z_4') = f(\iota_0(z_1'),\iota_0(z_2'),\iota_1(z_3'),\iota_1(z_4')) =$$



and its particle–hole reduction is

$$f^{\mathrm{ph}}(z_1',z_2',z_3',z_4') = f(\iota_0(z_1'),\iota_1(z_2'),\iota_1(z_3'),\iota_0(z_4')) =$$



When f is a bubble propagator, the corresponding reductions are

$$\begin{aligned} {}^{\mathrm{pp}}f(z_1',z_2',z_3',z_4') &= f(\iota_1(z_1'),\iota_1(z_2'),\iota_0(z_3'),\iota_0(z_4')), \\ {}^{\mathrm{ph}}f(z_1',z_2',z_3',z_4') &= f(\iota_1(z_1'),\iota_0(z_2'),\iota_0(z_3'),\iota_1(z_4')). \end{aligned}$$

ii) Let f' be a four legged kernel over $\mathcal{B}^\updownarrow \times \Sigma$. Its particle–particle value is the four legged kernel over $\mathcal{B} \times \Sigma$ given by

$$\begin{aligned} V_{\mathrm{pp}}(f')(z_1,z_2,z_3,z_4) = {}& \delta_{b(z_1),0}\delta_{b(z_2),0}\delta_{b(z_3),1}\delta_{b(z_4),1}\, f'(u(z_1),u(z_2),u(z_3),u(z_4)) \\ & +\delta_{b(z_1),1}\delta_{b(z_2),1}\delta_{b(z_3),0}\delta_{b(z_4),0}\, f'(u(z_3),u(z_4),u(z_1),u(z_2)) \end{aligned}$$

and its particle–hole value is

$$\begin{aligned} V_{\mathrm{ph}}(f')(z_1,z_2,z_3,z_4) = {}& \delta_{b(z_1),0}\delta_{b(z_2),1}\delta_{b(z_3),1}\delta_{b(z_4),0}\, f'(u(z_1),u(z_2),u(z_3),u(z_4)) \\ & +\delta_{b(z_1),1}\delta_{b(z_2),0}\delta_{b(z_3),0}\delta_{b(z_4),1}\, f'(u(z_2),u(z_1),u(z_4),u(z_3)) \\ & -\delta_{b(z_1),1}\delta_{b(z_2),0}\delta_{b(z_3),1}\delta_{b(z_4),0}\, f'(u(z_2),u(z_1),u(z_3),u(z_4)) \\ & -\delta_{b(z_1),0}\delta_{b(z_2),1}\delta_{b(z_3),0}\delta_{b(z_4),1}\, f'(u(z_1),u(z_2),u(z_4),u(z_3)). \end{aligned}$$

The decomposition of ladders over $\mathcal{B}$ into particle–particle and particle–hole ladders is given by the following lemma, whose proof is trivial.

Lemma VII.5.

i) Let $f_1, \cdots, f_{\ell+1}$ be particle number preserving four legged kernels over $\mathcal{B} \times \Sigma$ that are separately antisymmetric in their first two and their last two arguments. Let $P_1, \cdots, P_\ell$ be particle number preserving bubble propagators over $\mathcal{B}$ that satisfy $P_i(\xi_1, \xi_2, \xi_3, \xi_4) = P_i(\xi_2, \xi_1, \xi_4, \xi_3)$ for $i = 1, \cdots, \ell$. Then

$$\big(f_1 \bullet P_1 \bullet \cdots \bullet P_\ell \bullet f_{\ell+1}\big)^{\mathrm{pp}} = f_1^{\mathrm{pp}} \bullet {}^{\mathrm{pp}}P_1 \bullet \cdots \bullet {}^{\mathrm{pp}}P_\ell \bullet f_{\ell+1}^{\mathrm{pp}},$$
$$\big(f_1 \bullet P_1 \bullet \cdots \bullet P_\ell \bullet f_{\ell+1}\big)^{\mathrm{ph}} = 2^\ell \, f_1^{\mathrm{ph}} \bullet {}^{\mathrm{ph}}P_1 \bullet \cdots \bullet {}^{\mathrm{ph}}P_\ell \bullet f_{\ell+1}^{\mathrm{ph}}.$$

ii) Let f be an antisymmetric, particle number preserving, four legged kernel over $\mathcal{B} \times \Sigma$. Then

$$f = V_{\mathrm{pp}}(f^{\mathrm{pp}}) + V_{\mathrm{ph}}(f^{\mathrm{ph}}).$$

We now state the ladder estimates used in the rest of the paper.

Proposition VII.6. *Let $0 < \Lambda < \frac{\tau_2}{2M^j}$, where τ_2 is the constant of Lemma XIII.6 of [FKTo3]. Let $u((\xi,s),(\xi',s')) \in \mathcal{F}_0(2, \Sigma_j)$ be an antisymmetric, spin independent, particle number conserving function whose Fourier transforms obey $|\check{u}(k)| \le \frac{1}{2}|\imath k_0 - e(k)|$ and such that $|u|_{1,\Sigma_j} \le \Lambda \mathfrak{c}_j$. Furthermore let $f \in \mathcal{F}_0(4, \Sigma_j)$. Then for all $\ell \ge 1$,*

$$\big|L_\ell\big(f; C_u^{(j)}, C_u^{(\ge j+1)}\big)\big|_{3,\Sigma_j} \le \big(\mathrm{const}\ \mathfrak{c}_j\big)^\ell \ |f|_{3,\Sigma_j}^{\ell+1},$$
$$\Big|V_{\mathrm{pp}}\Big(L_\ell\big(f; C_u^{(j)}, C_u^{(\ge j+1)}\big)^{\mathrm{pp}}\Big)\Big|_{3,\Sigma_j} \le \big(\mathrm{const}\ \mathfrak{l}_j^{1/n_0}\ \mathfrak{c}_j\big)^\ell \ |f|_{3,\Sigma_j}^{\ell+1},$$

if the Fermi curve F is strongly asymmetric in the sense of Definition I.10. Here, n_0 is the constant of Definition I.10.

Proposition VII.6 is a special case of Proposition XIV.9. See Remark XIV.10.

The first inequality of Proposition VII.6 is not good enough for the control of the four point function, since replacing $\mathcal{C}_u^{(j)}$ by $\mathcal{C}_u^{(\le j)}$ would give an additional factor of j^n. The second inequality of Proposition VII.6 gives estimates for particle–particle ladders at each individual scale j that are good enough to be summable over j. Particle–hole ladders do not, at least in general, obey such estimates. If they did, they would be continuous in momentum space, even after all cutoffs are removed. Therefore, we bound the sum of all particle–hole ladders of scales up to j together, making use of cancellations between neighbouring scales. Building up such sums of ladders leads to "compound particle–hole ladders".

Definition VII.7. *Let $\vec{F} = \big\{ F^{(i)} \,\big|\, i = 2, 3, \cdots \big\}$ be a family of antisymmetric functions in $\mathcal{F}_0(4, \Sigma_i)$. Let $\vec{p} = \big(p^{(2)}, p^{(3)}, \cdots\big)$ be a sequence of antisymmetric, spin independent, particle number conserving functions $p^{(i)}((\xi,s),(\xi',s')) \in \mathcal{F}_0(2, \Sigma_i)$. We define, recursively on $0 \le j < \infty$, the iterated particle hole (or wrong way) ladders up to scale j, denoted by $\mathcal{L}^{(j)}(\vec{p}, \vec{F})$, as*

$$\mathcal{L}^{(0)}(\vec{p}, \vec{F}) = 0$$
$$\mathcal{L}^{(j+1)}(\vec{p}, \vec{F}) = \mathcal{L}^{(j)}(\vec{p}, \vec{F})_{\Sigma_j} + 2 \sum_{\ell=1}^{\infty} (-1)^\ell (12)^{\ell+1} L_\ell\big(w_j; C_{u_j}^{(j)}, C_{u_j}^{(\ge j+1)}\big)^{\mathrm{ph}},$$

where $u_j = \sum_{i=2}^{j-1} p_{\Sigma_j}^{(i)}$ and $w_j = \sum_{i=2}^{j} F_{\Sigma_j}^{(i)} + \frac{1}{8}\mathrm{Ant}\Big(V_{\mathrm{ph}}\big(\mathcal{L}^{(j)}(\vec{p}, \vec{F})\big)\Big)_{\Sigma_j}$. The resectorization $\mathcal{L}^{(j)}(\vec{p}, \vec{F})_{\Sigma_j}$ is defined by the natural analog of Definition VI.4. For the details, see Definition XIX.6 of [FKTo4].

Observe that $\mathcal{L}^{(j)}(\vec{p},\vec{F})$ *is a four legged kernel over* $\mathcal{B}^{\updownarrow}\times\Sigma_{j-1}$ *and depends only on the components* $F^{(2)},\cdots,F^{(j-1)}$ *of* $\vec{F}$ *and* $p^{(2)},\cdots,p^{(j-2)}$ *of* $\vec{p}$. *Also observe that* w_0, $\mathcal{L}^{(1)}(\vec{p},\vec{F})$, w_1 *and* $\mathcal{L}^{(2)}(\vec{p},\vec{F})$ *all vanish.*

When we apply Definition VII.7, $F^{(i)}$ will be the volume improved part of the contribution to the four–point function generated by integrating out scale i. Furthermore, $p^{(i)}$ will be, roughly speaking, the contribution to the renormalized two–point function at $K=0$ that is moved into the covariance at scale i. In particular, $F^{(2)}$ through $F^{(j_0)}$ and $p^{(2)}$ through $p^{(j_0-1)}$ will be zero.

The main estimate on iterated particle hole ladders is

Theorem VII.8. *For every* $\varepsilon>0$ *there are constants* ρ_0, const *such that the following holds. Let* $\vec{F}=\big(F^{(2)},F^{(3)},\cdots\big)$ *be a sequence of antisymmetric, spin independent, particle number conserving functions* $F^{(i)}\in\mathcal{F}_0(4,\Sigma_i)$ *and* $\vec{p}=\big(p^{(2)},p^{(3)},\cdots\big)$ *be a sequence of antisymmetric, spin independent, particle number conserving functions* $p^{(i)}\in\mathcal{F}_0(2,\Sigma_i)$. *Assume that there is* $\rho\le\rho_0$ *such that for* $i\ge 2$

$$|F^{(i)}|_{3,\Sigma_i}\le\tfrac{\rho}{M^{\varepsilon i}}\mathfrak{c}_i \qquad |p^{(i)}|_{1,\Sigma_i}\le\tfrac{\rho\,\mathfrak{l}_i}{M^i}\mathfrak{c}_i \qquad \check{p}^{(i)}(0,\mathbf{k})=0.$$

Then for all $j\ge 2$

$$\big|V_{\rm ph}\big(\mathcal{L}^{(j)}(\vec{p},\vec{F})_{\Sigma_j}\big)\big|_{3,\Sigma_j}\le \mathrm{const}\,\rho^2\,\mathfrak{c}_j.$$

Theorem VII.8 is a special case of Theorem XIV.12 in part 3. See Remark XIV.13. Both Theorems are consequences of the estimates on iterated particle hole ladders derived in [FKTl].

VIII. Infrared Limit of Finite Scale Green's Functions

The nonperturbative construction of the infrared limit will be similar to the formal construction outlined in §III. We first adapt the notion of a formal interaction triple $(\mathcal{W},\mathcal{G},u)$ at scale j of Definition III.4 to the needs of the nonperturbative construction. The function u modifying the covariance at scale j is built up of contributions created at scales up to $j-1$. To bound u, we keep track of all of these individual contributions. They are encoded in the additional datum $\vec{p}$ of

Definition VIII.1 (Interaction Quadruple). *An interaction quadruple at scale* j *is a quadruple* $(\mathcal{W},\mathcal{G},u,\vec{p})$ *that obeys the following conditions.*

- ∘ $\mathcal{W}$ *is a map from the space* $\mathfrak{K}_j$ *of counterterms to the space of even, translation invariant, spin independent, particle number conserving Grassmann functions in* ϕ *and* ψ, *that obeys* $\mathcal{W}(\phi,0,K)=0$.
- ∘ $\mathcal{G}$ *is a map from* $\mathfrak{K}_j$ *to the space of even, translation invariant, spin independent, particle number conserving Grassmann functions in* ϕ, *that obeys* $\mathcal{G}(0,K)=0$.
- ∘ $\vec{p}=\big(p^{(2)},\cdots,p^{(j-1)}\big)$ *where each* $p^{(i)}((\xi,s),(\xi',s'))$ *is an antisymmetric, spin independent, particle number conserving function in* $\mathcal{F}_0(2,\Sigma_i)$ *that obeys*

$$\big|p^{(i)}\big|_{1,\Sigma_i}\le\lambda_0^{1-\upsilon}\tfrac{\mathfrak{l}_i}{M^i}\mathfrak{c}_i. \qquad \text{(VIII.1)}$$

The Fourier transform $\check{p}^{(i)}(k)$ *of* $p^{(i)}$ *is supported in the* $i^{\rm th}$ *neighbourhood and vanishes at* $k_0=0$.

∘ *u is a map from $\mathfrak{K}_j$ to the space of antisymmetric, spin independent, particle number conserving functions in $\mathcal{F}_0(2,\Sigma_j)$. The function $u(K)$ has a decomposition*

$$u(K) = \sum_{i=2}^{j-1} p_{\Sigma_j}^{(i)} + \big[\delta u(K) - K_{\rm ext}\big]_{\Sigma_j} \tag{VIII.2}$$

with $\delta u\big((\xi,s),(\xi',s');K\big)$ an antisymmetric function in $\mathcal{F}_0(2,\Sigma_{j-1})$ that vanishes at $k_0 = 0$ and when $K = 0$ and obeys

$$\begin{aligned} \big|\tfrac{d}{ds}\delta u(K+sK')\big|_{s=0}\big|_{1,\Sigma_{j-1}} &\le \lambda_0^{1-\upsilon}\,\mathfrak{e}_j\big(\|K\|_{1,\Sigma_j}\big)\,\|K'\|_{1,\Sigma_j} \\ \big|\mathcal{D}_{1,2}^{(1,0,0)}\tfrac{d}{ds}\delta u(K+sK')\big|_{s=0}\big|_{1,\Sigma_{j-1}} &\le \lambda_0^{1-\upsilon}M^{j-3}\,\mathfrak{e}_j\big(\|K\|_{1,\Sigma_j}\big)\,\|K'\|_{1,\Sigma_j} \\ &\quad + \sum_{\delta_0=r_0} \infty\, t^\delta \end{aligned} \tag{VIII.3}$$

for all $K\in\mathfrak{K}_j$ and all K'.

The interaction quadruple $(\mathcal{W},\mathcal{G},u,\vec p)$ is said to be **real**, *if $\mathcal{W}(\phi,\psi,K)$, $\mathcal{G}(\phi,K)$, $u(K)$ and $p^{(1)},\cdots,p^{(j-1)}$ are k_0–reversal real, in the sense of Definition B.1.R of [FKTo2], for all real $K\in\mathfrak{K}_j$. In particular $\check p^{(i)}(-k_0,\mathbf{k}) = \overline{\check p^{(i)}(k_0,\mathbf{k})}$.*

Remark VIII.2.

i) We remind the reader that
The space $\mathfrak{K}_j$ of counterterms was defined in Definition VI.9. The extension $K_{\rm ext}\big((\xi,s),(\xi',s')\big)$ of $K\big((\mathbf{x},s),(\mathbf{x}',s')\big)$ was defined in Definition E.1 of [FKTo4]. The sector length $\mathfrak{l}_j$ was fixed after Definition VI.3, the resectorization p_{Σ_j} was defined in Definition VI.4, the space $\mathcal{F}_0(2,\Sigma_j)$ was defined in Definition VI.3.ii and the seminorm $|\!|\!|\,\cdot\,|\!|\!|_{1,\Sigma_j}$ was defined in Definition VI.6.i.
The decay operator $\mathcal{D}_{i,j}^{\delta}$ was defined in Definition V.1.ii and the elements $\mathfrak{c}_j$ and $\mathfrak{e}_j(X)$ of the norm domain were defined in Definition V.2.iii.

ii) Observe that $\vec p$ is independent of K, so that, in (VIII.2), the only K dependence of u is through $\delta u(K) - K_{\rm ext}$.

iii) The representation (VIII.2) of u implies that

$$\check u(k;K) = \sum_{i=2}^{j-1}\check p^{(i)}(k) \; + \big[\delta\check u(k;K) - \check K(\mathbf{k})\big]$$

for k in the $j^{\rm th}$ neighbourhood. Bounds on $u(K)$ will be provided in Lemma VIII.7.

The relation between counterterms at different scales is formalized in

Definition VIII.3. *A projective system of counterterms consists of analytic maps*

$$\begin{aligned} \mathrm{ren}_{i,j} &: \mathcal{K}_{j+1} \longrightarrow \mathcal{K}_{i+1} \qquad \text{for } j_0 \le i \le j, \\ \delta e_j &: \mathcal{K}_{j+1} \longrightarrow \mathcal{E} \qquad\qquad \text{for } j_0 \le j, \end{aligned}$$

such that

$$\begin{aligned} &\mathrm{ren}_{j,j} \text{ is the identity map of } \mathcal{K}_{j+1}, \\ &\mathrm{ren}_{i,i'}\circ\mathrm{ren}_{i',j} = \mathrm{ren}_{i,j} \qquad \text{for } j_0 \le i \le i' \le j, \\ &\delta e_i \circ \mathrm{ren}_{i,j} = \delta e_j \qquad \text{for } j_0 \le i \le j, \end{aligned}$$

and

$$\sup_{K\in\mathfrak{K}_j} |||\delta\hat{e}_j(K)|||_{1,\infty} \le \lambda_0^{1-\upsilon},$$
$$\left\|\mathrm{ren}_{i,j}(0)-\mathrm{ren}_{i,j'}(0)\right\|_{1,\Sigma_i} \le \lambda_0^{1-\upsilon}\tfrac{1}{2^j}+\sum_{\delta\neq 0}\infty t^\delta,$$
$$|||\delta\hat{e}_j(0)-\delta\hat{e}_{j'}(0)|||_{1,\infty} \le \lambda_0^{1-\upsilon}\tfrac{1}{2^j}$$

for all $j_0 \le i \le j \le j'$.
A projective system is said to be **real** *if* $\mathrm{ren}_{i,j}(K)$ *is real and* $\delta e_j(\mathbf{k};K)$ *is real–valued for all* i, j *and all real* K.

Remark VIII.4. For any projective system of counterterms, the sequence $\delta e_j(K)\big|_{K=0}$ of infrared cutoff counterterms converges in the topology of $\mathcal{E}$.

We shall prove, in §X, the following bounds on the analogs of the formal interaction triple $(\mathcal{W}_j^{\rm out}, \mathcal{G}_j^{\rm out}, u_j)$ of (III.9).

Theorem VIII.5. *Assume that* $d=2$, *that* $e(\mathbf{k})$ *fulfills the Hypotheses I.12 and that the scale parameter* M *has been chosen big enough. Then there exist constants* $\bar\alpha,\ \bar\lambda>0$ *such that for all* $0\le\lambda_0<\bar\lambda$, $\bar\alpha<\alpha<\frac{1}{\lambda_0^{\upsilon/10}}$ *the following holds. For each translation invariant, spin independent interaction kernel* V *obeying*

$$\|V\|_{1,\infty}\le\lambda_0\mathfrak{c}_0$$

there exist

∘ *a projective system of counterterms* $\left(\mathrm{ren}_{i,j},\ \delta e_j\right)$
∘ *a family* $\vec{p}=\left(p^{(2)},p^{(3)}\cdots\right)$ *of functions* $p^{(i)}\in\mathcal{F}_0(2,\Sigma_i)$
∘ *a family* $\vec{F}=\left(F^{(2)},F^{(3)}\cdots\right)$ *of antisymmetric kernels*

$$F^{(i)}((\xi_1,s_1),\cdots,(\xi_4,s_4))\in\mathcal{F}_0(4,\Sigma_i)$$

such that

$$\left|F^{(i)}\right|_{3,\Sigma_i}\le\frac{\lambda_0^{1-\upsilon}}{\alpha^7}\,\mathfrak{l}_i^{1/n_0}\,\mathfrak{c}_i.$$

All of this data depends analytically on V. *Also, for each scale* $j\ge j_0$ *there exist* $\mathcal{W}_j$, $\mathcal{G}_j^{\rm rg}$ *and* u_j, *depending analytically on* V *and* K, *such that* $\left(\mathcal{W}_j,\mathcal{G}_j^{\rm rg},u_j,(p^{(2)},\cdots,p^{(j-1)})\right)$ *is an interaction quadruple at scale* j. *Furthermore*

(R1) $\mathcal{W}_j(K)$ *has a* Σ_j*–sectorized representative,*

$$w(\phi,\psi;K)=\sum_{m,n}\sum_{s_1,\cdots,s_n\in\Sigma_j}\int d\eta_1\cdots d\eta_m\,d\xi_1\cdots d\xi_n\ w_{m,n}(\eta_1,\cdots,\eta_m\,(\xi_1,s_1),\cdots,(\xi_n,s_n);K)$$
$$\times\,\phi(\eta_1)\cdots\phi(\eta_m)\ \psi((\xi_1,s_1))\cdots\psi((\xi_n,s_n))$$

with $w_{m,n}$ *antisymmetric separately in the* η *and in the* (ξ,s) *variables,* $w_{m,0}=0$ *for all* $m\ge 0$ *and*

$$\left|w_{0,2}(K)\right|_{1,\Sigma_j}\le\frac{\lambda_0^{1-\upsilon}}{\alpha^7}\,\frac{\mathfrak{l}_j}{M^j}\,\mathfrak{e}_j\left(\|K\|_{1,\Sigma_j}\right),$$
$$N_j\left(w(K),\alpha,\|K\|_{1,\Sigma_j}\right)\le\mathfrak{e}_j\left(\|K\|_{1,\Sigma_j}\right),$$
$$N_j\left(\tfrac{d}{ds}w(K+sK')\big|_{s=0},\alpha,\|K\|_{1,\Sigma_j}\right)\le M^j\,\mathfrak{e}_j\left(\|K\|_{1,\Sigma_j}\right)\|K'\|_{1,\Sigma_j}$$

for all $K\in\mathfrak{K}_j$ *and all* K'. w *depends analytically on* V *and* K.

(R2) The function $w_{0,4}(K)$ *has a decomposition*

$$w_{0,4}(K) = \delta F^{(j+1)}(K) + \sum_{i=2}^{j} F^{(i)}_{\Sigma_j} + \tfrac{1}{8}\mathrm{Ant}\Big(V_{\rm ph}\big(\mathcal{L}^{(j+1)}(\vec{p},\vec{F})\big)\Big)$$

with an antisymmetric kernel $\delta F^{(j+1)}({\scriptstyle(\xi_1,s_1)},\cdots,{\scriptstyle(\xi_4,s_4)};K) \in \mathcal{F}_0(4,\Sigma_j)$ *such that*

$$\big|\delta F^{(j+1)}(K)\big|_{3,\Sigma_j} \le \frac{\lambda_0^{1-\upsilon}}{\alpha^4}\Big\{\frac{\mathfrak{l}_{j+1}^{1/n_0}}{\alpha^4} + \frac{1}{\mathrm{B}^2} M^j \|K\|_{1,\Sigma_j}\Big\}\mathfrak{e}_j(\|K\|_{1,\Sigma_j}) \quad \text{for all } K \in \mathfrak{K}_j.$$

The particle–hole projector $V_{\rm ph}$ *is defined in Definition VII.4.*

(R3) For each $K \in \mathfrak{K}_j$,

$$N\big(\mathcal{G}^{\rm rg}_j(K) - \tfrac{1}{2}\phi J C^{(\le j)} J\phi\big) \le 4\sum_{i=2}^{j} \frac{1}{\sqrt[4]{\mathfrak{l}_i M^i}} \text{ for all } K \in \mathfrak{K}_j.$$

Let the part of $\mathcal{G}^{\rm rg}_j(K)$ *that is homogeneous of degree two be*

$$\mathcal{G}^{\rm rg}_{j,2}(K) = \int d\eta_1 d\eta_2\ G^{\rm rg}_{j,2}(\eta_1,\eta_2,K)\ \phi(\eta_1)\phi(\eta_2).$$

Then

$$\begin{aligned} N\Big(\tfrac{d}{ds}\big[\mathcal{G}^{\rm rg}_j(K+sK') - \mathcal{G}^{\rm rg}_{j,2}(K+sK')\big]_{s=0}\Big) &\le M^j\|K'\|_{1,\Sigma_j} \\ \Big|\Big|\Big|\tfrac{d}{ds} G^{\rm rg}_{j,2}(K+sK')\big|_{s=0}\Big|\Big|\Big|_\infty &\le M^j\|K'\|_{1,\Sigma_j} \end{aligned}$$

for all $K \in \mathfrak{K}_j$ *and all* K'.

(R4) For $K \in \mathcal{K}_{j+2}$ *and* $K' = \mathrm{ren}_{j,j+1}(K)$,

$$N\big(\mathcal{G}^{\rm rg}_{j+1}(K) - \tfrac{1}{2}\phi J C^{(j+1)} J\phi - \mathcal{G}^{\rm rg}_j(K')\big) \le \frac{4}{\sqrt[4]{\mathfrak{l}_j M^j}}.$$

(R5) For infrared cutoffs $\not{j} \ge j+2$, *the generating function of the connected Green's functions at scale* $\not{j}$ *of Theorem I.4, considered as a formal Taylor series in* V, *fulfills*

$$\mathcal{G}_{\not{j}}(\phi,\bar\phi;\delta e_j(K)) = \mathcal{G}^{\rm rg}_j(\phi;K) + \log\frac{\int e^{\phi J\psi}\, e^{:\mathcal{W}_j(\phi,\psi;K):_{\psi,D_j(u_j;K)}}\, d\mu_{C^{[j+1,\not{j})}_{u_j(K)}}(\psi)}{\int e^{:\mathcal{W}_j(0,\psi;K):_{\psi,D_j(u_j;K)}}\, d\mu_{C^{[j+1,\not{j})}_{u_j(K)}}(\psi)}$$

for $K \in \mathfrak{K}_j$.

If, in addition, V *satisfies the reality condition of (I.1) then*

- ◦ *the projective system* $\big(\mathrm{ren}_{i,j},\ \delta e_j\big)$ *is real*
- ◦ *each* $F^{(i)}$ *is* k_0*–reversal real, in the sense of Definition B.1.R of [FKTo2]*
- ◦ *each interaction quadruple* $\big(\mathcal{W}_j, \mathcal{G}^{\rm rg}_j, u_j, {\scriptstyle(p^{(2)},\cdots,p^{(j-1)})}\big)$ *is real*
- ◦ *for real* K, *the* Σ_j*–sectorized representative* $w(\phi,\psi;K)$ *of* $\mathcal{W}_j(K)$ *is* k_0*–reversal real.*

Proof of Theorem I.4 from Theorem VIII.5. Observe that $\mathcal{G}_{\jmath}(\phi;\delta e)$ depends on M and $\jmath$ only through the combination $M^{\jmath}$. Hence, for constructing $\lim_{\substack{\jmath\to+\infty\\ \jmath\in\mathbb{R}}} \mathcal{G}_{\jmath}(\phi;\delta e)$, we may, without loss of generality, use any $M>1$ we wish.

Choose M, α and λ_0 fulfilling the hypotheses of Theorem VIII.5. By Remark V.5, the conditions on the interaction kernel in Theorems I.4 and VIII.5 agree. By Remark VIII.4,

$$\delta e = \lim_{j\to\infty} \delta e_j(0)$$

exists. If V is k_0–reversal real, as in (I.1), then $\delta e(\mathbf{k})$ is real for all $\mathbf{k}$. By Definition VIII.3,

$$|||\delta\hat{e}|||_{1,\infty} \le \lambda_0^{1-\upsilon} \qquad |||\delta\hat{e}_j(0)-\delta\hat{e}|||_{1,\infty} \le \lambda_0^{1-\upsilon}\tfrac{1}{2^j}$$

for all $j\ge j_0$. By Weierstrass' Theorem, δe is analytic in V.

We now show that the sequence $\mathcal{G}_j^{\rm rg}(0)-\frac12\phi JC^{(\le j)}J\phi$ is a Cauchy sequence. Let $j\ge j_0$ and $K'=\mathrm{ren}_{j,j+1}(0)$. By Definition VIII.3, $\big\|K'\big\|_{1,\Sigma_j}\le \lambda_0^{1-\upsilon}\frac{\mathfrak{l}_j}{M^j}+\sum_{\delta\ne0}\infty t^\delta$. Hence, by (R3), (R4) and the Definition VI.7.ii of the norm $N(\mathcal{G})$,

$$\begin{aligned}
&N\big(\mathcal{G}_{j+1}^{\rm rg}(0)-\tfrac12\phi JC^{(\le j+1)}J\phi-\mathcal{G}_j^{\rm rg}(0)+\tfrac12\phi JC^{(\le j)}J\phi\big)\\
&\quad\le N\big(\mathcal{G}_{j+1}^{\rm rg}(0)-\tfrac12\phi JC^{(j+1)}J\phi-\mathcal{G}_j^{\rm rg}(K')\big)+N\big(\mathcal{G}_j^{\rm rg}(K')-\mathcal{G}_{j,2}^{\rm rg}(K')\\
&\qquad-\mathcal{G}_j^{\rm rg}(0)+\mathcal{G}_{j,2}^{\rm rg}(0)\big)+N\big(\mathcal{G}_{j,2}^{\rm rg}(K')-\mathcal{G}_{j,2}^{\rm rg}(0)\big)\\
&\quad\le \tfrac{4}{\sqrt[4]{\mathfrak{l}_jM^j}}+M^j\|K'\|_{1,\Sigma_j}+\tfrac{1}{\lambda_0^{1-\upsilon}}M^j\|K'\|_{1,\Sigma_j}\\
&\quad\le \tfrac{4}{\sqrt[4]{\mathfrak{l}_jM^j}}+2\mathfrak{l}_j.
\end{aligned}$$

Let

$$\mathcal{G}(\phi)=\sum_{n=1}^{\infty}\tfrac{1}{n!}\int\prod_{i=1}^n d\xi_i\; G_n(\xi_1,\cdots,\xi_n)\prod_{i=1}^n\phi(\xi_i)$$

be the limit of the $\mathcal{G}_j^{\rm rg}(0)$'s. It is analytic in V.

We now show that the generating functionals $\mathcal{G}_{\jmath}(\phi)$ of the connected Green's functions at scale $\jmath$ also converge to $\mathcal{G}(\phi)$. Let $\jmath\ge j_0+3$ and let $j=j(\jmath)$ be the integer with $j+1<\jmath\le j+2$. By Definition VIII.3,

$$K=\lim_{j'\to\infty}\mathrm{ren}_{j,j'}(0)$$

exists in $\mathfrak{K}_j$ and obeys $\delta e=\delta e_j(K)$. Observe that, by (R5),

$$\begin{aligned}
&\mathcal{G}_{\jmath}(\phi;\delta e)-\mathcal{G}_j^{\rm rg}(\phi;0)\\
&\quad=\mathcal{G}_{\jmath}(\phi;\delta e_j(K))-\mathcal{G}_j^{\rm rg}(\phi;K)+\mathcal{G}_j^{\rm rg}(\phi;K)-\mathcal{G}_j^{\rm rg}(\phi;0)\\
&\quad=\log\frac{\int e^{\phi J\psi}\,e^{:\mathcal{W}_j(\phi,\psi;K):_{\psi,D_j(u_j;K)}}\,d\mu_S(\psi)}{\int e^{:\mathcal{W}_j(0,\psi;K):_{\psi,D_j(u_j;K)}}\,d\mu_S(\psi)}+\mathcal{G}_j^{\rm rg}(\phi;K)-\mathcal{G}_j^{\rm rg}(\phi;0)\\
&\quad=\tilde{\Omega}_S\big(:\mathcal{W}_j(K):_{\psi,D_j(u_j;K)}\big)(\phi,0)+\mathcal{G}_j^{\rm rg}(\phi;K)-\mathcal{G}_j^{\rm rg}(\phi;0), \qquad \text{(VIII.4)}
\end{aligned}$$

where

$$\begin{aligned} S &= \frac{\nu^{(\geq j+1)}(k) - \nu^{(\geq \jmath)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{u}_j(k;K)[1-\nu^{(\geq \jmath)}(k)]} \\ &= \frac{\nu^{(\geq j+1)}(k) - \nu^{(\geq \jmath)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{u}_j(k;K)[\nu^{(\geq j)}(k) - \nu^{(\geq \jmath)}(k)]}. \end{aligned}$$

As in the previous paragraph

$$N\big(\mathcal{G}_j^{\mathrm{rg}}(K) - \mathcal{G}_j^{\mathrm{rg}}(0)\big) \leq 2\mathfrak{l}_j. \tag{VIII.5}$$

In Lemma VIII.7, below, we prove that the hypotheses of Proposition XV.10 of [FKTo3], with $\check{u}(k) = \check{u}_j(k;K)[\nu^{(\geq j)}(k) - \nu^{(\geq \jmath)}(k)]$, $\check{v}(k) = \check{u}_j(k;K) + \check{K}(\mathbf{k})\nu^{(\geq j+2)}(k)$ and $X = \|K\|_{1,\Sigma_j}$ are satisfied. Consequently,

$$\sqrt[4]{\mathfrak{l}_j M^j}\; N\Big(\tilde{\Omega}_S\big(:\mathcal{W}_j(K):_{\psi, D_j(u_j;K)}\big)(\phi,0) - \tfrac{1}{2}\phi J S J \phi\Big) \leq 10$$

As

$$\begin{aligned} N\big(\tfrac{1}{2}\phi J S J \phi\big) &= \tfrac{1}{\lambda_0^{1-\upsilon}} |||JSJ|||_\infty \leq \tfrac{1}{\lambda_0^{1-\upsilon}} \|S(k)\|_{L^1} \\ &\leq \tfrac{1}{\lambda_0^{1-\upsilon}} \operatorname{Vol}\big(\text{support } \nu^{(\geq j+1)} - \nu^{(\geq \jmath)}\big) \sup_k |S(k)| \\ &\leq \tfrac{\mathrm{const}}{\lambda_0^{1-\upsilon} M^j}, \end{aligned}$$

we have that

$$N\Big(\tilde{\Omega}_S\big(:\mathcal{W}_j(K):_{\psi, D_j(u_j;K)}\big)(\phi,0)\Big) \leq \tfrac{10}{\sqrt[4]{\mathfrak{l}_j M^j}} + \tfrac{\mathrm{const}}{\lambda_0^{1-\upsilon} M^j}. \tag{VIII.6}$$

Combining (VIII.4), (VIII.5) and (VIII.6),

$$\lim_{\jmath\to\infty} N\big(\mathcal{G}_\jmath(\phi;\delta e) - \mathcal{G}_{j(\jmath)}^{\mathrm{rg}}(\phi;0)\big) = 0$$

so that

$$\lim_{\jmath\to\infty} \mathcal{G}_\jmath = \mathcal{G}$$

in the $N(\,\cdot\,)$ norm. Consequently, for each n, $G_{n;\jmath}$ converges uniformly to G_n. □

Definition VIII.6. *If $u\big((\xi,s),(\xi',s')\big)$ is an antisymmetric, translation invariant, spin independent, particle number conserving function on $(\mathcal{B}\times\Sigma)^2$ and $\mu(k)$ is a function on $\mathbb{R}\times\mathbb{R}^2$, set*

$$\begin{aligned} (u*\hat{\mu})\big((\xi,s),(\xi',s')\big) &= \int_{\mathcal{B}} d\zeta\; u\big((\xi,s),(\zeta,s')\big)\hat{\mu}(\zeta,\xi'), \\ (\hat{\mu}*u)\big((\xi,s),(\xi',s')\big) &= \int_{\mathcal{B}} d\zeta\; u\big((\zeta,s),(\xi',s')\big)\hat{\mu}(\zeta,\xi), \end{aligned}$$

where $\hat{\mu}$ was defined in Notation V.4.

With this definition $(u*\hat{\mu})\check{}(k) = (\hat{\mu}*u)\check{}(k) = \check{u}(k)\mu(k)$.

Lemma VIII.7. *Let $\big(\mathcal{W},\mathcal{G},u,\vec{p}\big)$ be an interaction quadruple at scale j. Then* *i)*

$$\Big|\check u(k;K)+\check K(\mathbf{k})\nu^{(\ge j+2)}(k)\Big|\le \lambda_0^{1-\upsilon}\big|\imath k_0-e(\mathbf{k})\big|\le \tfrac12\big|\imath k_0-e(\mathbf{k})\big|,$$
$$\Big|\tfrac{d}{ds}\check u(k;K+sK')\big|_{s=0}+\check K'(\mathbf{k})\nu^{(\ge j+2)}(k)\Big|\le 4M^{j+\frac32}\|K'\|_{1,\Sigma_j}\,\big|\imath k_0-e(\mathbf{k})\big|$$

for all k in the $j^{\rm th}$ neighbourhood, all $K\in\mathfrak{K}_j$ and all K'.

ii)

$$|u(K)|_{1,\Sigma_j}\le const\Big[\tfrac{\lambda_0^{1-\upsilon}}{M^{j-1}}+\|K\|_{1,\Sigma_j}\Big]\mathfrak{e}_j(\|K\|_{1,\Sigma_j}),$$
$$\Big|\tfrac{d}{ds}u(K+sK')\big|_{s=0}\Big|_{1,\Sigma_j}\le const\,\mathfrak{e}_j(\|K\|_{1,\Sigma_j})\,\|K'\|_{1,\Sigma_j}.$$

iii) Let $\mathfrak{j}\in(j,j+2]$. Then

$$\Big|u(K)*\big(\nu^{(\ge j)}-\nu^{(\ge\mathfrak{j})}\big)\hat{}\,\Big|_{1,\Sigma_j}\le \mathrm{const}\,\Big[\tfrac{\lambda_0^{1-\upsilon}}{M^{j-1}}+\|K\|_{1,\Sigma_j}\Big]\mathfrak{e}_j(\|K\|_{1,\Sigma_j}).$$

Proof. i) By Remark VIII.2.iii,

$$\begin{aligned}&\Big|\check u(k;K)+\check K(\mathbf{k})\nu^{(\ge j+2)}(k)\Big|\\&=\Big|\sum_{i=2}^{j-1}\check p^{(i)}(k)+\delta\check u(k;K)-\check K(\mathbf{k})\big[1-\nu^{(\ge j+2)}(k)\big]\Big|\\&\le\sum_{i=2}^{j-1}\big|\check p^{(i)}(k)\big|+\big|\delta\check u(k;K)\big|+\big|\check K(\mathbf{k})\big|\big[1-\nu^{(\ge j+2)}(k)\big]\end{aligned}$$

and

$$\begin{aligned}\Big|\tfrac{d}{ds}\check u(k;K+sK')\big|_{s=0}+\check K'(\mathbf{k})\nu^{(\ge j+2)}(k)\Big|&\le\Big|\tfrac{d}{ds}\delta\check u(k;K+sK')\big|_{s=0}\Big|\\&\quad+\big|\check K'(\mathbf{k})\big|\big[1-\nu^{(\ge j+2)}(k)\big]\end{aligned}$$

for all k in the $j^{\rm th}$ neighbourhood. Because $\check p^{(i)}$ vanishes at $k_0=0$, Lemma XII.12 of [FKTo3] and (VIII.1) imply that

$$\big|\check p^{(i)}(k)\big|\le 2|k_0|\,\tfrac{\partial}{\partial t_{(1,0,0)}}|p^{(i)}|_{1,\Sigma_i}\Big|_{t=0}\le 2|k_0|\,\lambda_0^{1-\upsilon}\tfrac{\mathfrak{l}_i}{M^i}M^i. \tag{VIII.7}$$

Similarly, by Lemma XII.12 of [FKTo3], (VIII.3) and Definition VI.9,

$$\begin{aligned}\big|\delta\check u(k;K)\big|&\le 2|k_0|\,\big|\mathcal{D}^{(1,0,0)}_{1,2}\delta u(K)\big|_{1,\Sigma_{j-1}}\Big|_{t=0}\\&\le 2|k_0|\,\lambda_0^{1-\upsilon}M^{j-3}\frac{1}{1-M^j\lambda_0^{1-\upsilon}\frac{\mathfrak{l}_{j+1}}{M^{j+1}}}\lambda_0^{1-\upsilon}\tfrac{\mathfrak{l}_{j+1}}{M^{j+1}}\\&\le 4|k_0|\,\lambda_0^{2-2\upsilon}\mathfrak{l}_j \qquad\qquad\text{(VIII.8)}\\ \Big|\tfrac{d}{ds}\delta\check u(k;K+sK')\big|_{s=0}\Big|&\le 2|k_0|\,\Big|\mathcal{D}^{(1,0,0)}_{1,2}\tfrac{d}{ds}\delta u(K+sK')\big|_{s=0}\Big|_{1,\Sigma_{j-1}}\Big|_{t=0}\\&\le 2|k_0|\,\lambda_0^{1-\upsilon}M^{j-3}\frac{1}{1-M^j\lambda_0^{1-\upsilon}\frac{\mathfrak{l}_{j+1}}{M^{j+1}}}\|K'\|_{1,\Sigma_j}\\&\le 4|k_0|\,\lambda_0^{1-\upsilon}M^{j-3}\|K'\|_{1,\Sigma_j}\end{aligned}$$

and, by Lemma XII.12 of [FKTo3], Definition VI.9 and Definition VIII.1.i of [FKTo2],

$$\begin{aligned}
\big|\check K(\mathbf{k})\big|\big[1-\nu^{(\ge j+2)}(k)\big] &\le 2\|K\|_{1,\Sigma_j}\big|_{t=0}\,\big[1-\nu^{(\ge j+2)}(k)\big]\\
&\le 2\lambda_0^{1-\upsilon}\tfrac{\mathfrak{l}_{j+1}}{M^{j+1}}\big[1-\nu^{(\ge j+2)}(k)\big]\\
&\le 2\lambda_0^{1-\upsilon}\tfrac{\mathfrak{l}_{j+1}}{M^{j+1}}\frac{|\imath k_0-e(\mathbf{k})|}{\sqrt{1/M}\,\frac{1}{M^{j+1}}}\\
&\le 2\sqrt{M}\lambda_0^{1-\upsilon}\mathfrak{l}_{j+1}|\imath k_0-e(\mathbf{k})|\\
&\le \tfrac{1}{10}\lambda_0^{1-\upsilon}\mathfrak{l}_j|\imath k_0-e(\mathbf{k})| \qquad\qquad \text{(VIII.9)}\\
\big|\check K'(\mathbf{k})\big|\big[1-\nu^{(\ge j+2)}(k)\big] &\le 2\|K'\|_{1,\Sigma_j}\big|_{t=0}\,\big[1-\nu^{(\ge j+2)}(k)\big]\\
&\le 2\|K'\|_{1,\Sigma_j}\,\frac{|\imath k_0-e(\mathbf{k})|}{\sqrt{1/M}\,\frac{1}{M^{j+1}}}\\
&\le 2M^{j+\frac{3}{2}}\|K'\|_{1,\Sigma_j}\,|\imath k_0-e(\mathbf{k})|
\end{aligned}$$

if M is large enough. In the last inequality of the bound on $\big|\check K(\mathbf{k})\big|\big[1-\nu^{(\ge j+2)}(k)\big]$, we used that $\frac{\mathfrak{l}_{j+1}}{\mathfrak{l}_j}=\frac{1}{M^{\aleph}}$ with $\aleph>\frac{1}{2}$. Combining (VIII.7), (VIII.8) and (VIII.9),

$$\begin{aligned}
&\big|\check u(k;K)+\check K(\mathbf{k})\nu^{(\ge j+2)}(k)\big|\\
&\quad\le\Big\{\sum_{i=2}^{j-1}2\mathfrak{l}_i+4\lambda_0^{1-\upsilon}\mathfrak{l}_j+\tfrac{1}{10}\mathfrak{l}_j\Big\}\lambda_0^{1-\upsilon}|\imath k_0-e(\mathbf{k})|\le\lambda_0^{1-\upsilon}|\imath k_0-e(\mathbf{k})|\\
&\Big|\tfrac{d}{ds}\check u(k;K+sK')\big|_{s=0}+\check K'(\mathbf{k})\nu^{(\ge j+2)}(k)\Big|\\
&\quad\le\Big\{4\lambda_0^{1-\upsilon}M^{-3}+2M^{\frac{3}{2}}\Big\}M^j\|K'\|_{1,\Sigma_j}\,|\imath k_0-e(\mathbf{k})|\\
&\quad\le 4M^{j+\frac{3}{2}}\|K'\|_{1,\Sigma_j}\,\big|\imath k_0-e(\mathbf{k})\big|.
\end{aligned}$$

ii) By Corollary XIX.13, Remark XIX.5 and Proposition E.10.i of [FKTo4],

$$\begin{aligned}
&|u(K)|_{1,\Sigma_j}\\
&\quad=\Big|\sum_{i=2}^{j-1}p_{\Sigma_j}^{(i)}+\big[\delta u(K)-K_{\rm ext}\big]_{\Sigma_j}\Big|_{1,\Sigma_j}\\
&\quad\le const\;\mathfrak{c}_{j-1}\Big[\sum_{i=2}^{j-1}\lambda_0^{1-\upsilon}M\tfrac{\mathfrak{l}_i}{M^j}+\lambda_0^{1-\upsilon}\mathfrak{e}_j(\|K\|_{1,\Sigma_j})\big\|K\big\|_{1,\Sigma_j}+\big\|K\big\|_{1,\Sigma_j}\Big]\\
&\quad\le const\;\Big[\tfrac{\lambda_0^{1-\upsilon}}{M^{j-1}}\mathfrak{c}_j+\lambda_0^{1-\upsilon}\mathfrak{e}_j(\|K\|_{1,\Sigma_j})\mathfrak{c}_j\big\|K\big\|_{1,\Sigma_j}+\mathfrak{c}_j\big\|K\big\|_{1,\Sigma_j}\Big]\\
&\quad\le const\;\Big[\tfrac{\lambda_0^{1-\upsilon}}{M^{j-1}}+\big(1+\lambda_0^{1-\upsilon}\big)\big\|K\big\|_{1,\Sigma_j}\Big]\mathfrak{e}_j(\|K\|_{1,\Sigma_j})\\
&\Big|\tfrac{d}{ds}u(K+sK')\big|_{s=0}\Big|_{1,\Sigma_j}\\
&\quad\le const\;\mathfrak{c}_{j-1}\Big[\lambda_0^{1-\upsilon}\mathfrak{e}_j(\|K\|_{1,\Sigma_j})\|K'\|_{1,\Sigma_j}+\|K'\|_{1,\Sigma_j}\Big]\\
&\quad\le const\;\mathfrak{e}_j(\|K\|_{1,\Sigma_j})\,\|K'\|_{1,\Sigma_j}.
\end{aligned}$$

iii) By Lemma XIII.7 of [FKTo3] with $\mu(t)=\varphi(t/M)-\varphi(M^{2(j-j)}t/M)$, where φ is the function used in Definition I.2, and $\Lambda=M^j$ we have

$$\begin{aligned}\left|u(K) * \left(\nu^{(\geq j)} - \nu^{(\geq j)}\right)^{\widehat{}}\right|_{1,\Sigma_j} &\le \text{const }\mathfrak{c}_j\, |u(K)|_{1,\Sigma_j}\\ &\le \text{const }\mathfrak{c}_j \left[\tfrac{\lambda_0^{1-\upsilon}}{M^{j-1}} + \|K\|_{1,\Sigma_j}\right]\mathfrak{e}_j(\|K\|_{1,\Sigma_j})\\ &\le \text{const } \left[\tfrac{\lambda_0^{1-\upsilon}}{M^{j-1}} + \|K\|_{1,\Sigma_j}\right]\mathfrak{e}_j(\|K\|_{1,\Sigma_j}). \quad \square\end{aligned}$$

IX. One Recursion Step

The data of Theorem VIII.5 are constructed recursively. In this section, we implement one recursion step, analogous to the map $\Omega_j \circ \mathcal{O}_j$ of §III.

1. Input and Output Data. We now impose the actual conditions on the input and output data, analogous to Definitions III.8 and III.9.

Definition IX.1 (Input Data). *The input data just before integrating out the $j^{\rm th}$ scale is the set $\mathcal{D}_{\rm in}^{(j)}$ of interaction quadruples, in the sense of Definition VIII.1, $(\mathcal{W}, \mathcal{G}, u, \vec{p})$ that fulfill*

(I1) *$\mathcal{W}(K)$ has a Σ_j–sectorized representative*

$$\begin{aligned}w(\phi,\psi;K) = \sum_{m,n}\ \sum_{s_1,\cdots,s_n\in\Sigma_j} \int d\eta_1\cdots d\eta_m\, d\xi_1\cdots d\xi_n\ w_{m,n}(\eta_1,\cdots,\eta_m\,(\xi_1,s_1),\cdots,(\xi_n,s_n);\,K)\\ \times\,\phi(\eta_1)\cdots\phi(\eta_m)\ \psi((\xi_1,s_1))\cdots\psi((\xi_n,s_n))\end{aligned}$$

with $w_{m,n}$ antisymmetric separately in the η and in the ξ variables, $w_{0,2} = 0$, $w_{m,0} = 0$ for all $m \ge 0$ and

$$\begin{aligned}N_j\big(w(K), 64\alpha, \|K\|_{1,\Sigma_j}\big) &\le \mathfrak{e}_j\big(\|K\|_{1,\Sigma_j}\big),\\ N_j\big(\tfrac{d}{ds} w(K+sK')\big|_{s=0}, 64\alpha, \|K\|_{1,\Sigma_j}\big) &\le M^j\ \mathfrak{e}_j\big(\|K\|_{1,\Sigma_j}\big)\,\|K'\|_{1,\Sigma_j},\end{aligned}$$

for all $K \in \mathfrak{K}_j$ and all K'.

(I2) *There is a family $\vec{F}$ of antisymmetric kernels*

$$F^{(i)}((\xi_1,s_1),\cdots,(\xi_4,s_4)) \in \mathcal{F}_0(4,\Sigma_i), \qquad 2 \le i \le j-1$$

(independent of K) and an antisymmetric kernel $\delta F^{(j)}((\xi_1,s_1),\cdots,(\xi_4,s_4);K) \in \mathcal{F}_0(4,\Sigma_j)$ such that

$$w_{0,4}(K) = \delta F^{(j)}(K) + \sum_{i=2}^{j-1} F^{(i)}_{\Sigma_j} + \tfrac{1}{8}\mathrm{Ant}\Big(V_{\rm ph}\big(\mathcal{L}^{(j)}(\vec{p},\vec{F})\big)\Big)_{\Sigma_j},$$

where the particle–hole value $V_{\rm ph}$ was defined in Definition VII.4. Furthermore, for all $2 \le i \le j-1$,

$$\big|F^{(i)}\big|_{3,\Sigma_i} \le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^7}\,\mathfrak{l}_i^{1/n_0}\,\mathfrak{c}_i$$

and

$$\big|\delta F^{(j)}(K)\big|_{3,\Sigma_j} \le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^4}\Big\{\tfrac{\mathfrak{l}_j^{1/n_0}}{\alpha^3} + \tfrac{1}{\mathrm{B}^2}M^j\|K\|_{1,\Sigma_j}\Big\}\mathfrak{e}_j(\|K\|_{1,\Sigma_j}) \qquad \textit{for all } K \in \mathfrak{K}_j.$$

(I3) *For each* $K \in \mathfrak{K}_j$,

$$N\big(\mathcal{G}(K) - \tfrac{1}{2}\phi J C^{(<j)} J\phi\big) \le 4\sum_{i=2}^{j-1} \tfrac{1}{\sqrt[4]{\mathfrak{l}_i M^i}}.$$

Let $\mathcal{G}_2(K) = \int d\eta_1 d\eta_2\ G_2(\eta_1,\eta_2,K)\ \phi(\eta_1)\phi(\eta_2)$ *be the part of* $\mathcal{G}(K)$ *that is homogeneous of degree two. Then*

$$\begin{aligned} N\big(\tfrac{d}{ds}\big[\mathcal{G}(K+sK') - \mathcal{G}_2(K+sK')\big]_{s=0}\big) &\le \tfrac{1}{2} M^j \|K'\|_{1,\Sigma_j} \\ \big|\big|\big|\tfrac{d}{ds} G_2(K+sK')\big|_{s=0}\big|\big|\big|_\infty &\le \tfrac{1}{2} M^j \|K'\|_{1,\Sigma_j} \end{aligned}$$

for all $K \in \mathfrak{K}_j$ *and all* K'.

The input data is said to be **real** *if*

- ○ *the interaction quadruple* $(\mathcal{W},\mathcal{G},u,\vec{p})$ *is real*
- ○ *for real* K, *the* Σ_j*–sectorized representative* $w(\phi,\psi;K)$ *of* $\mathcal{W}(K)$ *is* k_0*–reversal real, in the sense of Definition B.1.R of [FKTo2] and*
- ○ *each* $F^{(i)}$ *is* k_0*–reversal real.*

Definition IX.2 (Output Data). *The output data just after integrating out the* $j^{\rm th}$ *scale is the set* $\mathcal{D}^{(j)}_{\rm out}$ *of interaction quadruples* $(\mathcal{W},\mathcal{G},u,\vec{p})$ *that fulfill*

(O1) $\mathcal{W}(K)$ *has a* Σ_j*–sectorized representative*

$$\begin{aligned} w(\phi,\psi;K) = \sum_{m,n} \sum_{s_1,\cdots,s_n\in\Sigma_j} \int & d\eta_1\cdots d\eta_m\, d\xi_1\cdots d\xi_n\ w_{m,n}(\eta_1,\cdots,\eta_m\,(\xi_1,s_1),\cdots,(\xi_n,s_n);K) \\ & \times \phi(\eta_1)\cdots\phi(\eta_m)\ \psi((\xi_1,s_1))\cdots\psi((\xi_n,s_n)) \end{aligned}$$

with $w_{m,n}$ *antisymmetric separately in the* η *and in the* ξ *variables,* $w_{m,0}=0$ *for all* $m \ge 0$ *and*

$$\begin{aligned} \big|w_{0,2}(K)\big|_{1,\Sigma_j} &\le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^7}\ \tfrac{\mathfrak{l}_j}{M^j}\ \mathfrak{e}_j\big(\|K\|_{1,\Sigma_j}\big), \\ N_j\big(w(K),\alpha,\|K\|_{1,\Sigma_j}\big) &\le \mathfrak{e}_j\big(\|K\|_{1,\Sigma_j}\big), \\ N_j\big(\tfrac{d}{ds} w(K+sK')\big|_{s=0},\alpha,\ \|K\|_{1,\Sigma_j}\big), &\le M^j\ \mathfrak{e}_j\big(\|K\|_{1,\Sigma_j}\big)\,\|K'\|_{1,\Sigma_j} \end{aligned}$$

for all $K \in \mathfrak{K}_j$ *and all* K'.

(O2) *There is a family* $\vec{F}$ *of antisymmetric kernels*

$$F^{(i)}((\xi_1,s_1),\cdots,(\xi_4,s_4)) \in \mathcal{F}_0(4,\Sigma_i), \qquad 2 \le i \le j$$

(independent of K*) and an antisymmetric kernel* $\delta F^{(j+1)}((\xi_1,s_1),\cdots,(\xi_4,s_4);K) \in \mathcal{F}_0(4,\Sigma_j)$ *such that*

$$w_{0,4}(K) = \delta F^{(j+1)}(K) + \sum_{i=2}^{j} F^{(i)}_{\Sigma_j} + \tfrac{1}{8}{\rm Ant}\Big(V_{\rm ph}\big(\mathcal{L}^{(j+1)}(\vec{p},\vec{F})\big)\Big).$$

Furthermore,

$$\big|F^{(i)}\big|_{3,\Sigma_i} \le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^7}\ \mathfrak{l}_i^{1/n_0}\ \mathfrak{c}_i \quad \textit{for all } 2 \le i \le j$$

and

$$\big|\delta F^{(j+1)}(K)\big|_{3,\Sigma_j} \le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^4}\Big\{\tfrac{\mathfrak{l}_{j+1}^{1/n_0}}{\alpha^4} + \tfrac{1}{{\rm B}^2} M^j \|K\|_{1,\Sigma_j}\Big\}\mathfrak{e}_j(\|K\|_{1,\Sigma_j}) \quad \textit{for all } K \in \mathfrak{K}_j$$

(O3) *For each* $K \in \mathfrak{K}_j$,

$$N\big(\mathcal{G}(K) - \tfrac{1}{2}\phi J C^{(\le j)} J\phi\big) \le 4\sum_{i=2}^{j-1} \tfrac{1}{\sqrt[4]{\mathfrak{l}_i M^i}} + \tfrac{2}{\sqrt[4]{\mathfrak{l}_j M^j}} \quad \textit{for all } K \in \mathfrak{K}_j$$

Let $\mathcal{G}_2(K) = \int d\eta_1 d\eta_2\ G_2(\eta_1, \eta_2, K)\ \phi(\eta_1)\phi(\eta_2)$ *be the part of* $\mathcal{G}(K)$ *that is homogeneous of degree two. Then*

$$\begin{aligned} N\big(\tfrac{d}{ds}\big[\mathcal{G}(K + sK') - \mathcal{G}_2(K + sK')\big]_{s=0}\big) &\le M^j \|K'\|_{1,\Sigma_j} \\ \big|\big|\big|\tfrac{d}{ds} G_2(K + sK')\big|_{s=0}\big|\big|\big|_\infty &\le M^j \|K'\|_{1,\Sigma_j} \end{aligned}$$

for all $K \in \mathfrak{K}_j$ *and all* K'.

The output data is said to be **real** *if*

- *the interaction quadruple* $(\mathcal{W}, \mathcal{G}, u, \vec{p})$ *is real,*
- *for real* K, *the* Σ_j*–sectorized representative* $w(\phi, \psi; K)$ *of* $\mathcal{W}(K)$ *is* k_0*–reversal real,*
- *each* $F^{(i)}$ *is* k_0*–reversal real.*

Remark IX.3. Conditions (O1) and (O2) coincide with Conditions (R1) and (R2) of Theorem VIII.5, while Condition (O3) implies Condition (R3).

2. Integrating Out a Scale. In this subsection we implement the map $\Omega_j : \mathcal{D}_{\rm in}^{(j)} \to \mathcal{D}_{\rm out}^{(j)}$, analogous to that of Definition III.6, that integrates out fields of scale j. We use the covariances

$$\begin{aligned} C_u^{(j)}(k) &= \tfrac{\nu^{(j)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{u}(k;K)}, \\ C_u^{(\ge j)}(k) &= \tfrac{\nu^{(\ge j)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{u}(k;K)}, \\ D_j(u;K)(k) &= \tfrac{\nu^{(\ge j+1)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{u}(k;K) - \check{K}(\mathbf{k})\nu^{(\ge j+2)}(k)}, \\ C_j(u;K)(k) &= C_u^{(j)}(k) + D_j(u;K)(k) \end{aligned}$$

from Definition III.5 and Remark III.7.

Definition IX.4. *Integrating out the fields of scale* j *is implemented by the map* Ω_j, *which maps an interaction quadruple* $(\mathcal{W}, \mathcal{G}, u, \vec{p}) \in \mathcal{D}_{\rm in}^{(j)}$ *to the quadruple* $(\mathcal{W}', \mathcal{G}', u, \vec{p})$ *determined by*

$$\begin{aligned} {:}\mathcal{W}'(\phi,\psi;K){:}_{\psi, D_j(u;K)} &= \log \tfrac{1}{Z(\phi)} \int e^{\phi J\zeta}\, e^{:\mathcal{W}(\phi,\psi+\zeta;K):_{\psi, C_j(u;K)}}\, d\mu_{C^{(j)}_{u(K)}}(\zeta), \\ \mathcal{G}'(\phi) &= \mathcal{G}(\phi) + \log \tfrac{Z(\phi)}{Z(0)}, \end{aligned}$$

where

$$\log Z(\phi) = \int \Big[\log \int e^{\phi J\zeta}\, e^{:\mathcal{W}(\phi,\psi+\zeta;K):_{\psi, C_j(u;K)}}\, d\mu_{C^{(j)}_{u(K)}}(\zeta)\Big] d\mu_{D_j(u;K)}(\psi).$$

Theorem IX.5. *If* $(\mathcal{W}, \mathcal{G}, u, \vec{p}) \in \mathcal{D}_{\rm in}^{(j)}$ *then* $\Omega_j(\mathcal{W}, \mathcal{G}, u, \vec{p}) \in \mathcal{D}_{\rm out}^{(j)}$.

Proof. Let $(\mathcal{W}', \mathcal{G}', u, \vec{p}) = \Omega_j(\mathcal{W}, \mathcal{G}, u, \vec{p})$. As in Definition III.6, one easily verifies that $(\mathcal{W}', \mathcal{G}', u, \vec{p})$ is an interaction quadruple of scale j.

Define $\mathcal{W}''$ by

$$
\begin{aligned}
{:}\mathcal{W}''(\phi, \psi; K){:}_{\psi, D_j(u;K)} &= \tilde{\Omega}_{C_u^{(j)}}\big({:}\mathcal{W}(\phi, \psi; K){:}_{\psi, C_j(u;K)}\big) \\
&= \log \tfrac{1}{Z(0)} \int e^{\phi J \zeta}\, e^{{:}\mathcal{W}(\phi,\psi+\zeta;K){:}_{\psi, C_j(u;K)}}\, d\mu_{C_u^{(j)}}(\zeta).
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathcal{W}'(\phi, \psi; K) &= \mathcal{W}''(\phi, \psi; K) - \mathcal{W}''(\phi, 0; K), \\
\mathcal{G}'(\phi) &= \mathcal{G}(\phi) + \mathcal{W}''(\phi, 0; K).
\end{aligned}
\tag{IX.1}
$$

We now apply Theorems XV.3 and XV.7 of [FKTo3] to bound $\mathcal{W}''$. By (I1) and parts (i) and (ii) of Lemma VIII.7, the hypotheses of Theorem XV.3 of [FKTo3], with $\mu = \mathit{const}$, $\Lambda = \frac{\lambda_0^{1-\upsilon}}{M^{j-1}}$ and $X = \|K\|_{1,\Sigma_j}$, are fulfilled. Therefore $\mathcal{W}''$ has a sectorized representative w'' obeying

$$
\begin{aligned}
& N_j(w'' - \tfrac{1}{2}\phi J C_u^{(j)} J\phi - w, \alpha, \ \|K\|_{1,\Sigma_j}) \\
&\quad \le \tfrac{\mathrm{const}}{\alpha}\ \tfrac{N_j(w, 64\alpha,\ \|K\|_{1,\Sigma_j})}{1 - \frac{\mathrm{const}}{\alpha} N_j(w, 64\alpha,\ \|K\|_{1,\Sigma_j})}, \\
& |w''_{0,2}|_{1,\Sigma_j} \\
&\quad \le \tfrac{\mathrm{const}}{\alpha^8 \rho_{0;2}}\, \tfrac{\mathfrak{l}_j}{M^j}\ \tfrac{N_j(w, 64\alpha,\ \|K\|_{1,\Sigma_j})^2}{1 - \frac{\mathrm{const}}{\alpha} N_j(w, 64\alpha,\ \|K\|_{1,\Sigma_j})}, \\
& \Big| w''_{0,4} - w_{0,4} - \tfrac{1}{4}\sum_{\ell=1}^{\infty} (-1)^\ell (12)^{\ell+1} L_\ell(w_{0,4}; C_u^{(j)}, D_j(u;K)) \Big|_{3,\Sigma_j} \\
&\quad \le \tfrac{\mathrm{const}}{\alpha^{10} \rho_{0;4}}\, \mathfrak{l}_j\ \tfrac{N_j(w, 64\alpha,\ \|K\|_{1,\Sigma_j})^2}{1 - \frac{\mathrm{const}}{\alpha} N_j(w, 64\alpha,\ \|K\|_{1,\Sigma_j})}.
\end{aligned}
\tag{IX.2}
$$

The hypotheses of Theorem XV.7 of [FKTo3], with, in addition, $Y = \mathit{const}\, \|K'\|_{1,\Sigma_j}$ and $\varepsilon = \mathrm{const}\, M^j\, \|K'\|_{1,\Sigma_j}\big|_{t=0}$, are fulfilled. Hence

$$
\begin{aligned}
& N_j\big(\tfrac{d}{ds}\big[w''(K + sK') - \tfrac{1}{2}\phi J C^{(j)}_{u(K+sK')} J\phi\big]_{s=0}, \alpha,\ \|K\|_{1,\Sigma_j}\big) \\
&\quad \le N_j\big(\tfrac{d}{ds} w(K + sK')\big|_{s=0}, \alpha,\ \|K\|_{1,\Sigma_j}\big) \\
&\qquad + \tfrac{\mathrm{const}}{\alpha} N_j\big(\tfrac{d}{ds} w(K + sK')\big|_{s=0}, 16\alpha,\ \|K\|_{1,\Sigma_j}\big) \\
&\qquad + \tfrac{\mathrm{const}}{\alpha}\ \tfrac{N_j(w, 64\alpha,\ \|K\|_{1,\Sigma_j})}{1 - \frac{\mathrm{const}}{\alpha^2} N_j(w, 64\alpha,\ \|K\|_{1,\Sigma_j})} \\
&\qquad \times \Big\{ N_j\big(\tfrac{d}{ds} w(K + sK')\big|_{s=0}, 16\alpha,\ \|K\|_{1,\Sigma_j}\big) + M^j Y \Big\} \\
&\qquad + \tfrac{\mathrm{const}}{\alpha^2}\ \tfrac{N_j(w, 64\alpha,\ \|K\|_{1,\Sigma_j})^2}{1 - \frac{\mathrm{const}}{\alpha^2} N_j(w, 63\alpha,\ \|K\|_{1,\Sigma_j})} \Big\{ M^j Y + \varepsilon \Big\}.
\end{aligned}
\tag{IX.3}
$$

By (I1) and Corollary A.5.ii of [FKTo1],

$$
\frac{N_j(w, 64\alpha,\ \|K\|_{1,\Sigma_j})}{1 - \frac{\mathrm{const}}{\alpha} N_j(w, 64\alpha,\ \|K\|_{1,\Sigma_j})} \le \frac{\mathfrak{e}_j(\|K\|_{1,\Sigma_j})}{1 - \frac{\mathrm{const}}{\alpha} \mathfrak{e}_j(\|K\|_{1,\Sigma_j})} \le \mathrm{const}\, \mathfrak{e}_j(\|K\|_{1,\Sigma_j})
$$

and

$$
\frac{N_j(w, 64\alpha,\ \|K\|_{1,\Sigma_j})^2}{1 - \frac{\mathrm{const}}{\alpha} N_j(w, 64\alpha,\ \|K\|_{1,\Sigma_j})} \le \mathrm{const}\, \mathfrak{e}_j(\|K\|_{1,\Sigma_j})^2 \le \mathrm{const}\, \mathfrak{e}_j(\|K\|_{1,\Sigma_j}).
$$

Therefore, by (IX.2) and (I1), recalling that w vanishes when $\psi = 0$,

$$\begin{aligned} N_j(w'' - \tfrac{1}{2}\phi J C_u^{(j)} J\phi, \alpha, \ \|K\|_{1,\Sigma_j}) &\le N_j(w, \alpha, \ \|K\|_{1,\Sigma_j}) + \tfrac{\mathrm{const}}{\alpha}\, \mathfrak{e}_j(\|K\|_{1,\Sigma_j}) \\ &\le \tfrac{1}{64} N_j(w, 64\alpha, \ \|K\|_{1,\Sigma_j}) + \tfrac{\mathrm{const}}{\alpha}\, \mathfrak{e}_j(\|K\|_{1,\Sigma_j}) \\ &\le \mathfrak{e}_j(\|K\|_{1,\Sigma_j}) \end{aligned} \tag{IX.4}$$

and

$$|w''_{0,2}|_{1,\Sigma_j} \le \mathrm{const}\, \tfrac{\lambda_0^{1-\upsilon}}{\alpha^8} \tfrac{\mathfrak{l}_j}{M^j}\, \mathfrak{e}_j(\|K\|_{1,\Sigma_j}) \le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^7} \tfrac{\mathfrak{l}_j}{M^j}\, \mathfrak{e}_j(\|K\|_{1,\Sigma_j}) \tag{IX.5}$$

and, by (IX.3) and (I1),

$$\begin{aligned} &N_j\big(\tfrac{d}{ds}\big[w''(K+sK') - \tfrac{1}{2}\phi J C_{u(K+sK')}^{(j)} J\phi\big]_{s=0}, \alpha, \ \|K\|_{1,\Sigma_j}\big) \\ &\quad\le N_j\big(\tfrac{d}{ds} w(K+sK')\big|_{s=0}, \alpha, \ \|K\|_{1,\Sigma_j}\big) + \tfrac{\mathrm{const}}{\alpha}\, \mathfrak{e}_j(\|K\|_{1,\Sigma_j})\, M^j \|K'\|_{1,\Sigma_j} \\ &\quad\le \tfrac{1}{64} N_j\big(\tfrac{d}{ds} w(K+sK')\big|_{s=0}, 64\alpha, \ \|K\|_{1,\Sigma_j}\big) + \tfrac{\mathrm{const}}{\alpha}\, \mathfrak{e}_j(\|K\|_{1,\Sigma_j})\, M^j \|K'\|_{1,\Sigma_j} \\ &\quad\le M^j \mathfrak{e}_j(\|K\|_{1,\Sigma_j})\, \|K'\|_{1,\Sigma_j} \end{aligned} \tag{IX.6}$$

By (IX.1), $w'(\phi,\psi;K) = w''(\phi,\psi;K) - w''(\phi,0;K)$ is a Σ_j–sectorized representative for $\mathcal{W}'$. (O1) now follows from (IX.5), (IX.4) and (IX.6).

In preparation for the verification of (O2), set

$$\begin{aligned} \delta F_1'(K) &= w''_{0,4}(K) - w_{0,4}(K) \\ &\qquad - \tfrac{1}{4} \sum_{\ell=1}^{\infty} (-1)^\ell (12)^{\ell+1} \mathrm{Ant}\, L_\ell\big(w_{0,4}(K); C_{u(K)}^{(j)}, D_j(u(K);K)\big). \end{aligned}$$

By (IX.2),

$$\big|\delta F_1'(K)\big|_{3,\Sigma_j} \le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^9}\, \mathfrak{l}_j\, \mathfrak{e}_j(\|K\|_{1,\Sigma_j}).$$

In particular

$$\big|\delta F_1'(0)\big|_{3,\Sigma_j} \le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^9}\, \mathfrak{l}_j\, \mathfrak{c}_j. \tag{IX.7}$$

Define

$$\begin{aligned} \delta F_2'(K) &= w''_{0,4}(K) - w''_{0,4}(0) \\ \delta F_3' &= \tfrac{1}{4} \sum_{\ell=1}^{\infty} (-1)^\ell (12)^{\ell+1} \mathrm{Ant}\, V_{\mathrm{pp}}\Big(L_\ell\Big(w_{0,4}(0); C_{u(0)}^{(j)}, C_{u(0)}^{(\ge j+1)}\Big)^{\mathrm{pp}}\Big) \end{aligned}$$

and

$$\delta F'^{(j+1)}(K) = \delta F_1'(0) + \delta F_2'(K) + \delta F_3'.$$

Observe that $D_j(u(0);0) = C_{u(0)}^{(\ge j+1)}$ and, by Lemma VII.5.ii,

$$\begin{aligned} L_\ell\big(w_{0,4}(0); C_{u(0)}^{(j)}, C_{u(0)}^{(\ge j+1)}\big) &= V_{\mathrm{pp}}\Big(L_\ell\Big(w_{0,4}(0); C_{u(0)}^{(j)}, C_{u(0)}^{(\ge j+1)}\Big)^{\mathrm{pp}}\Big) \\ &\qquad + V_{\mathrm{ph}}\Big(L_\ell\Big(w_{0,4}(0); C_{u(0)}^{(j)}, C_{u(0)}^{(\ge j+1)}\Big)^{\mathrm{ph}}\Big). \end{aligned}$$

Therefore

$$w''_{0,4}(K) = w_{0,4}(0) + \delta F'^{(j+1)}(K) + \tfrac{1}{4}\sum_{\ell=1}^{\infty} (-1)^{\ell}(12)^{\ell+1}\mathrm{Ant}\, V_{\rm ph}\Big(L_\ell\Big(w_{0,4}(0);\, C^{(j)}_{u(0)}, C^{(\ge j+1)}_{u(0)}\Big)^{\rm ph}\Big). \tag{IX.8}$$

We now estimate $\big|\delta F'^{(j+1)}(K)\big|_{3,\Sigma_j}$. By (IX.6),

$$\begin{aligned}\big|\delta F'_2\big|_{3,\Sigma_j} &\le \Big|\int_0^1 \tfrac{d}{ds'} w''_{0,4}(sK+s'K)\Big|_{s'=0}ds\Big|_{3,\Sigma_j}\\ &\le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^4 \mathrm{B}^2}\,\mathfrak{e}_j(\|K\|_{1,\Sigma_j})\, M^j \|K\|_{1,\Sigma_j}.\end{aligned} \tag{IX.9}$$

By Proposition VII.6, for $\ell \ge 1$,

$$\Big|V_{\rm pp}\Big(L_\ell\Big(w_{0,4}(0);\, C^{(j)}_{u(0)}, C^{(\ge j+1)}_{u(0)}\Big)^{\rm pp}\Big)\Big|_{3,\Sigma_j} \le \big(\mathrm{const}\, \mathfrak{l}_j^{1/n_0}\mathfrak{c}_j\big)^{\ell}\, |w_{0,4}(0)|^{\ell+1}_{3,\Sigma_j}.$$

The hypotheses of this proposition are fulfilled by parts (i) and (ii) of Lemma VIII.7. Observe that, by (I1),

$$\tfrac{M^{2j}}{\mathfrak{l}_j}\mathfrak{e}_j(\|K\|_{1,\Sigma_j})(64\alpha)^4\big(\tfrac{\mathfrak{l}_j \mathrm{B}}{M^j}\big)^2\, \tfrac{1}{\lambda_0^{1-\upsilon}}\tfrac{1}{\mathfrak{l}_j}\, |w_{0,4}(K)|_{3,\Sigma_j} \le \mathfrak{e}_j(\|K\|_{1,\Sigma_j})$$

so that

$$|w_{0,4}(K)|_{3,\Sigma_j} \le \tfrac{\lambda_0^{1-\upsilon}}{(64\alpha)^4\mathrm{B}^2}\mathfrak{e}_j(\|K\|_{1,\Sigma_j}).$$

In particular,

$$|w_{0,4}(0)|_{3,\Sigma_j} \le \tfrac{\lambda_0^{1-\upsilon}}{(64\alpha)^4\mathrm{B}^2}\mathfrak{c}_j.$$

Therefore, by Corollary A.5.ii of [FKTo1],

$$\begin{aligned}\big|\delta F'_3\big|_{3,\Sigma_j} &\le \sum_{\ell=1}^{\infty}\big(\mathrm{const}\, \mathfrak{l}_j^{1/n_0}\mathfrak{c}_j\big)^{\ell}\, |w_{0,4}(0)|^{\ell+1}_{3,\Sigma_j}\\ &\le \sum_{\ell=1}^{\infty}\big(\mathrm{const}\, \mathfrak{l}_j^{1/n_0}\mathfrak{c}_j\big)^{\ell}\big(\tfrac{\lambda_0^{1-\upsilon}}{\alpha^4}\mathfrak{c}_j\big)^{\ell+1}\\ &\le \mathrm{const}\big(\tfrac{\lambda_0^{1-\upsilon}}{\alpha^4}\big)^2 \mathfrak{l}_j^{1/n_0}\,\mathfrak{c}_j^3\,\frac{1}{1-\mathrm{const}\,\frac{\lambda_0^{1-\upsilon}}{\alpha^4}\mathfrak{l}_j^{1/n_0}\,\mathfrak{c}_j^2}\\ &\le \mathrm{const}\,\tfrac{\lambda_0^{2-2\upsilon}}{\alpha^8}\mathfrak{l}_j^{1/n_0}\,\mathfrak{c}_j.\end{aligned} \tag{IX.10}$$

Hence, by (IX.7), (IX.9) and (IX.10),

$$\begin{aligned}\big|\delta F'^{(j+1)}(K)\big|_{3,\Sigma_j} &\le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^4}\Big\{\tfrac{\mathfrak{l}_j}{\alpha^5} + \tfrac{1}{\mathrm{B}^2}M^j\|K\|_{1,\Sigma_j} + \tfrac{\mathrm{const}\,\lambda_0^{1-\upsilon}}{\alpha^4}\mathfrak{l}_j^{1/n_0}\Big\}\,\mathfrak{e}_j(\|K\|_{1,\Sigma_j})\\ &\le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^4}\Big\{\tfrac{\mathfrak{l}_{j+1}^{1/n_0}}{\alpha^4} + \tfrac{1}{\mathrm{B}^2}M^j\|K\|_{1,\Sigma_j}\Big\}\mathfrak{e}_j(\|K\|_{1,\Sigma_j}).\end{aligned} \tag{IX.11}$$

To verify (O2), set $F'^{(i)} = F^{(i)}$ for all $0 \le i \le j-1$, $F'^{(j)} = \delta F^{(j)}(0)$. By (I2),

$$w_{0,4}(0) = \sum_{i=2}^{j} F'^{(i)}_{\Sigma_j} + \tfrac{1}{8}\mathrm{Ant}\Big(V_{\rm ph}\big(\mathcal{L}^{(j)}(\vec{p}, \vec{F}')\big)\Big)_{\Sigma_j}.$$

Therefore, by (IX.8) and the Definition VII.7 of iterated particle–hole ladders

$$\begin{aligned}w'_{0,4}(K) &= \sum_{i=2}^{j} F'^{(i)}_{\Sigma_j} + \delta F'^{(j+1)}(K) + \tfrac{1}{8}\mathrm{Ant}\Big(V_{\rm ph}\big(\mathcal{L}^{(j)}(\vec p, \vec F')\big)\Big)_{\Sigma_j} \\ &\quad + \tfrac{1}{4}\sum_{\ell=1}^{\infty}(-1)^\ell (12)^{\ell+1}\mathrm{Ant}\, V_{\rm ph}\bigg(L_\ell\Big(w_{0,4}(0); C^{(j)}_{u(0)}, C^{(\ge j+1)}_{u(0)}\Big)^{\rm ph}\bigg) \\ &= \sum_{i=2}^{j} F'^{(i)}_{\Sigma_j} + \delta F'^{(j+1)}(K) + \tfrac{1}{8}\mathrm{Ant}\Big(V_{\rm ph}\big(\mathcal{L}^{(j+1)}(\vec p, \vec F')\big)\Big).\end{aligned}$$

The estimate on $\big|\delta F'^{(j+1)}(K)\big|_{3,\Sigma_j}$ required for (O2) was verified in (IX.11). By (I2),

$$\big|F'^{(j)}\big|_{3,\Sigma_j} = \big|\delta F^{(j)}(0)\big|_{3,\Sigma_j} \le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^7}\mathfrak{l}_j^{1/n_0}\mathfrak{c}_j$$

and $\big|F'^{(i)}\big|_{3,\Sigma_i} = \big|F^{(i)}\big|_{3,\Sigma_i} \le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^7}\mathfrak{l}_i^{1/n_0}\mathfrak{c}_i$ for $2 \le i \le j-1$.

This leaves only the verification of (O3). By (IX.1),

$$\begin{aligned}\mathcal{G}'(\phi; K) - \tfrac{1}{2}\phi J C^{(\le j)} J\phi &= \Big(\mathcal{G}(\phi; K) - \tfrac{1}{2}\phi J C^{(<j)} J\phi\Big) \\ &\quad + \Big(w''(\phi, 0; K) - \tfrac{1}{2}\phi J C^{(j)}_{u(K)} J\phi\Big) \\ &\quad - \tfrac{1}{2}\phi J\big(C^{(j)} - \phi C^{(j)}_{u(K)}\big)J\phi.\end{aligned}$$

Now

$$C^{(j)}(k) - C^{(j)}_{u(K)}(k) = -\nu^{(j)}(k)\tfrac{\check u(k;K)}{[\imath k_0 - e(\mathbf{k})][\imath k_0 - e(\mathbf{k}) - \check u(k;K)]}.$$

By Lemma VIII.7.i,

$$\big|\check u(k; K) + \check K(k)\nu^{(\ge j+2)}(k)\big| \le \lambda_0^{1-\upsilon}\,|\imath k_0 - e(\mathbf{k})|.$$

On the support of $\nu^{(j)}$, $\nu^{(\ge j+2)}(k)$ vanishes and $\big|\check u(k; K)\big| \le \lambda_0^{1-\upsilon}\,|\imath k_0 - e(\mathbf{k})|$ so that

$$\big|C^{(j)}(k) - C^{(j)}_{u(K)}(k)\big| \le \tfrac{2\lambda_0^{1-\upsilon}}{|\imath k_0 - e(\mathbf{k})|}\nu^{(j)}(k).$$

On the support of $\nu^{(j)}$, $|\imath k_0 - e(\mathbf{k})| \le \frac{\sqrt{2M}}{M^j}$, so that

$$\begin{aligned}\big|\!\big|\!\big|C^{(j)} - C^{(j)}_{u(K)}\big|\!\big|\!\big|_\infty &\le \int \tfrac{d^3k}{(2\pi)^3}\tfrac{2\lambda_0^{1-\upsilon}}{|\imath k_0 - e(\mathbf{k})|}\nu^{(j)}(k) \le const\,\lambda_0^{1-\upsilon}\int_{|\imath x - y| \le \frac{\sqrt{2M}}{M^j}} dx\,dy\,\tfrac{1}{|\imath x - y|} \\ &\le const\,\tfrac{\sqrt{M}}{M^j}\lambda_0^{1-\upsilon}\end{aligned}$$

and

$$\begin{aligned}N\big(\phi J(C^{(j)} - C^{(j)}_{u(K)})J\phi\big) &= \tfrac{1}{\lambda_0^{1-\upsilon}}\big|\!\big|\!\big|J\big(C^{(j)} - C^{(j)}_{u(K)}\big)J\big|\!\big|\!\big|_\infty \le \tfrac{1}{\lambda_0^{1-\upsilon}}\big|\!\big|\!\big|C^{(j)} - C^{(j)}_{u(K)}\big|\!\big|\!\big|_\infty \\ &\le const\,\tfrac{\sqrt{M}}{M^j} = const\,\tfrac{\sqrt{M}}{M^{j(3+\aleph)/4}}\tfrac{1}{\sqrt[4]{\mathfrak{l}_j M^j}} \\ &\le \tfrac{1/2}{\sqrt[4]{\mathfrak{l}_j M^j}}\end{aligned}$$

if $j \geq 1$ and M is big enough. Therefore, by (IX.4) and Remark VI.8.ii,

$$\begin{aligned}
&N\big(\mathcal{G}'(\phi;K) - \tfrac{1}{2}\phi J C^{(j)} J\phi - \mathcal{G}(\phi;K)\big)\\
&\quad\leq N\Big(w''(\phi,0;K) - \tfrac{1}{2}\phi J C^{(j)}_{u(K)} J\phi\Big) + \tfrac{1}{2} N\big(\phi J (C^{(j)} - C^{(j)}_{u(K)}) J\phi\big)\\
&\quad\leq N_j\big(w''(K) - \tfrac{1}{2}\phi J C^{(j)}_{u(K)} J\phi\,,\ \alpha\,,\ \|K\|_{1,\Sigma_j}\big)_{\mathbf{0}} + \tfrac{1/2}{\sqrt[4]{\mathfrak{l}_j M^j}}\\
&\quad\leq \tfrac{2}{\sqrt[4]{\mathfrak{l}_j M^j}}.
\end{aligned} \tag{IX.12}$$

Consequently, by (I3),

$$\begin{aligned}
&N\big(\mathcal{G}'(\phi;K) - \tfrac{1}{2}\phi J C^{(\leq j)} J\phi\big)\\
&\quad\leq N\Big(\mathcal{G}(\phi;K) - \tfrac{1}{2}\phi J C^{(<j)} J\phi\Big) + N\big(\mathcal{G}'(\phi;K) - \tfrac{1}{2}\phi J C^{(j)} J\phi - \mathcal{G}(\phi;K)\big)\\
&\quad\leq 4\sum_{i=2}^{j-1} \tfrac{1}{\sqrt[4]{\mathfrak{l}_i M^i}} + \tfrac{2}{\sqrt[4]{\mathfrak{l}_j M^j}}.
\end{aligned}$$

By (IX.1), (IX.6) and Remark VI.8.ii,

$$\begin{aligned}
&N\Big(\tfrac{d}{ds}\big[\mathcal{G}'(\phi;\,{\scriptstyle K+sK'}) - \mathcal{G}(\phi;\,{\scriptstyle K+sK'}) - \tfrac{1}{2}\phi J C^{(j)}_{u(K+sK')} J\phi\big]_{s=0}\Big)\\
&\quad= N\Big(\tfrac{d}{ds}\big[w''(\phi,0;\,{\scriptstyle K+sK'}) - \tfrac{1}{2}\phi J C^{(j)}_{u(K+sK')} J\phi\big]_{s=0}\Big)\\
&\quad\leq \tfrac{1}{\sqrt[4]{\mathfrak{l}_j M^j}}\, N_j\Big(\tfrac{d}{ds}\big[w''({\scriptstyle K+sK'}) - \tfrac{1}{2}\phi J C^{(j)}_{u(K+sK')} J\phi\big]_{s=0}\,,\ \alpha\,,\ \|K\|_{1,\Sigma_j}\Big)_{\mathbf{0}}\\
&\quad\leq \tfrac{1}{\sqrt[4]{\mathfrak{l}_j M^j}}\, M^j\, \mathfrak{e}_j(\|K\|_{1,\Sigma_j})\|K'\|_{1,\Sigma_j}\big|_{t=0}\\
&\quad\leq \tfrac{1}{4}\, M^j \|K'\|_{1,\Sigma_j}.
\end{aligned} \tag{IX.13}$$

It follows directly from this bound and (I3) that

$$N\big(\tfrac{d}{ds}\big[\mathcal{G}(K+sK') - \mathcal{G}_2(K+sK')\big]_{s=0}\big). \leq M^j\|K'\|_{1,\Sigma_j}.$$

To prove the last inequality of (O3), observe that, by parts (i) and (ii) of Lemma VIII.7 and Lemma XII.12 of [FKTo3],

$$\begin{aligned}
\Big|\tfrac{d}{ds} C^{(j)}_{u(K+sK')}\big|_{s=0}\Big| &= \Big|\tfrac{\frac{d}{ds}\check{u}(k;K+sK')|_{s=0}}{[\imath k_0 - e(\mathbf{k}) - \check{u}(k;K)]^2}\Big|\nu^{(j)}(k)\\
&\leq const\, \tfrac{\|K'\|_{1,\Sigma_j}}{|\imath k_0 - e(\mathbf{k})|^2}\nu^{(j)}(k).
\end{aligned}$$

Hence

$$\begin{aligned}
\Big|\!\Big|\!\Big|\tfrac{d}{ds} C^{(j)}_{u(K+sK')}\big|_{s=0}\Big|\!\Big|\!\Big|_\infty &\leq const\, \|K'\|_{1,\Sigma_j}\int d^3k\ \tfrac{\nu^{(j)}(k)}{|\imath k_0 - e(\mathbf{k})|^2}\\
&\leq const\, \|K'\|_{1,\Sigma_j}\int_{\frac{1}{\sqrt{M}}\frac{1}{M^j}\leq r\leq\sqrt{2M}\frac{1}{M^j}} dr\ \tfrac{1}{r}\\
&= const\, \|K'\|_{1,\Sigma_j}\ln\big(\sqrt{2}\,M\big)\\
&\leq \tfrac{1}{4}M^j\|K'\|_{1,\Sigma_j}
\end{aligned} \tag{IX.14}$$

Combining (I3), (IX.37) and (IX.14),

$$
\begin{aligned}
&\big|\!\big|\!\big| \tfrac{d}{ds} G_2'(K+sK')\big|_{s=0} \big|\!\big|\!\big|_\infty \\
&\quad \le \big|\!\big|\!\big| \tfrac{d}{ds} G_2(K+sK')\big|_{s=0} \big|\!\big|\!\big|_\infty \\
&\qquad +\lambda_0^{1-\upsilon} N\Big(\tfrac{d}{ds}\big[\mathcal{G}'(\phi;\, {\scriptstyle K+sK'}) - \mathcal{G}(\phi;\, {\scriptstyle K+sK'}) - \tfrac12 \phi J C^{(j)}_{u(K+sK')} J\phi\big]_{s=0}\Big) \\
&\qquad +\tfrac12 \big|\!\big|\!\big| \tfrac{d}{ds} C^{(j)}_{u(K+sK')}\big|_{s=0} \big|\!\big|\!\big|_\infty \\
&\quad \le M^j \|K'\|_{1,\Sigma_j}. \qquad \square
\end{aligned}
$$

Remark IX.6. Let $(\mathcal{W}, \mathcal{G}, u, \vec{p}) \in \mathcal{D}^{(j)}_{\rm in}$ and $(\mathcal{W}', \mathcal{G}', u, \vec{p}) = \Omega_j(\mathcal{W}, \mathcal{G}, u, \vec{p})$.

(i) The data $F'^{(2)}, \cdots, F'^{(j-1)}$ of (O2) coincides with the data $F^{(2)}, \cdots, F^{(j-1)}$ of (I2).

(ii) By (IX.12), $N\big(\mathcal{G}'(\phi; K) - \frac12 \phi J C^{(j)} J\phi - \mathcal{G}(\phi; K)\big) \le \frac{2}{\sqrt[4]{\mathfrak{l}_j M^j}}$.

(iii) By Remark XV.11 of [FKTo3], the sectorized representative w' of $\mathcal{W}'$ and the function $\mathcal{G}'(\phi)$ may be obtained from the sectorized representative w of $\mathcal{W}$ and the function $\mathcal{G}(\phi)$ by

$$
\begin{aligned}
{:}w''(\phi,\psi;K){:}_{\psi, D_j(u;K)_{\Sigma_j}} &= \tfrac12 \phi J C^{(j)}_u J\phi + \Omega_{C^{(j)}_{u,\Sigma_j}}\big({:}w{:}_{\psi, C_j(u;K)_{\Sigma_j}}\big) \\
&\qquad \times (\phi, \psi + C^{(j)}_u J\phi), \\
w'(\phi,\psi;K) &= w''(\phi,\psi;K) - w''(\phi,0;K), \\
\mathcal{G}'(\phi) &= \mathcal{G}(\phi) + w''(\phi,0;K).
\end{aligned}
$$

The covariances $C^{(j)}_{u,\Sigma_j}$, $C_j(u;K)_{\Sigma_j}$ and $D_j(u;K)_{\Sigma_j}$ are defined as follows. Let $C(k)$ be one of $C^{(j)}_u(k)$, $C_j(u;K)(k)$, $D_j(u;K)(k)$, as specified just before Definition IX.4 and let $c\big((\cdot,s),(\cdot,s')\big)$ be the Fourier transform of $\chi_s(k)\, C(k)\, \chi_{s'}(k)$ as in (III.1) and (III.2). Then

$$
C_\Sigma\big((\xi,s),(\xi',s')\big) = \sum_{\substack{t\cap s\ne\emptyset \\ t'\cap s'\ne\emptyset}} c\big((\xi,t),(\xi',t')\big).
$$

(iv) If the input data are analytic functions of K, then, by Remark XV.11 of [FKTo3], the output data are analytic functions of the input data and K.

(v) If the input data is real, then, by Remark B.5 of [FKTo2], the output data is real.

3. Sector Refinement, ReWick Ordering and Renormalization. We now implement an analog with estimates of the formal power series map

$$
\mathcal{O}_j : \mathcal{D}^{(j)}_{\rm out} \to \mathcal{D}^{(j+1)}_{\rm in}
$$

that reWick orders. For each $(\mathcal{W}, \mathcal{G}, u, \vec{p}) \in \mathcal{D}^{(j)}_{\rm out}$ we choose a sectorized representative

$$
\begin{aligned}
w(\phi,\psi;K) = \sum_{m,n} \sum_{s_1,\cdots,s_n\in\Sigma_j} \int & d\eta_1\cdots d\eta_m\, d\xi_1\cdots d\xi_n\; w_{m,n}\big(\eta_1,\cdots,\eta_m\,(\xi_1,s_1),\cdots,(\xi_n,s_n);\,K\big) \\
&\times \phi(\eta_1)\cdots\phi(\eta_m)\; \psi\big((\xi_1,s_1)\big)\cdots\psi\big((\xi_n,s_n)\big)
\end{aligned}
$$

for $\mathcal{W}$ satisfying (O1) and (O2).

We first solve the re-Wick ordering equation, that – in the context of formal power series – was discussed in Lemma III.11. As in (III.11), we set, for any function $q : \mathfrak{K}_{j+1} \to \mathcal{F}_0(2, \Sigma_j)$,

$$\begin{aligned} \delta K\big((\mathbf{x}, s), (\mathbf{x}', s'); K'; q\big) &= \int dx_0 \,\big(q(K') * \hat{\nu}^{(\geq j+1)}\big)\big((x_0, \mathbf{x}, s), (0, \mathbf{x}', s')\big) \\ K(K'; q) &= K'_{\Sigma_j} + \delta K(K'; q) \\ u'(K'; q) &= u(K(K'; q))_{\Sigma_{j+1}} + q(K')_{\Sigma_{j+1}} * \hat{\nu}^{(\geq j+1)} \\ E(K'; q) &= C_{j+1}\big(u'(\,\cdot\,; q); K'\big) - D_j(u; K(K'; q)). \end{aligned} \tag{IX.15}$$

Let $e\big((\cdot, s), (\cdot, s')\big)$ be the Fourier transform of $\chi_s(k)\, \check{E}(k; K'; q)\, \chi_{s'}(k)$ in the sense of Definition IX.3 of [FKTo2]. As in Proposition XII.8 of [FKTo3], e defines a covariance on the vector space V_{Σ_j}, generated by the fields $\psi(\xi, s)$, by

$$E_{\Sigma_j}\big(\psi(\xi,s), \psi(\xi',s'); K'; q\big) = \sum_{\substack{t \cap s \neq \emptyset \\ t' \cap s' \neq \emptyset}} e\big((\xi, t), (\xi', t')\big).$$

Set

$$\tilde{w}(\phi, \psi; K'; q) = \int w(\phi, \psi + \psi'; K(K'; q))\, d\mu_{E_{\Sigma_j}(K';q)}(\psi') \tag{IX.16}$$

and expand

$$\tilde{w}(0, \psi; K'; q) = \sum_{n \geq 0} \sum_{s_1, \cdots, s_n \in \Sigma_j} \int d\xi_1 \cdots d\xi_n\, \tilde{w}_{0,n}\big((\xi_1, s_1), \cdots, (\xi_n, s_n); K'; q\big) \times \psi\big((\xi_1, s_1)\big) \cdots \psi\big((\xi_n, s_n)\big).$$

Lemma IX.7. *For each $K' \in \mathfrak{K}_{j+1}$ there exists an antisymmetric, spin independent, particle number conserving function $q_0(K') \in \mathcal{F}_0(2, \Sigma_j)$ that solves the equation*

$$\tfrac{1}{2} q(K') = \tilde{w}_{0,2}(K'; q(K'))$$

and fulfills

$$\begin{aligned} |q_0(K')|_{1,\Sigma_j} &\leq \tfrac{\lambda_0^{1-\upsilon}}{\alpha^6} \tfrac{\mathfrak{l}_j}{M^j}\, \mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}}), \\ \Big|\tfrac{d}{ds} q_0(K' + sK'')\big|_{s=0}\Big|_{1,\Sigma_j} &\leq \tfrac{\lambda_0^{1-\upsilon}}{\alpha}\, \mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})\, \|K''\|_{1,\Sigma_{j+1}}. \end{aligned}$$

If w and u are analytic in K, then q_0 is jointly analytic in K', w and u.

Proof. The proof is an application the implicit function theorem and is given following Lemma B.3 in Appendix B. □

Define, for $K' \in \mathfrak{K}_{j+1}$,

$$\begin{aligned} \delta K(K') &= \delta K\big(K'; q_0(K')\big), \\ \mathrm{ren}_{j,j+1}(K', \mathcal{W}, u) &= K(K') = K(K'; q_0), \end{aligned} \tag{IX.17}$$

If w and u are analytic in K, then δK and $\mathrm{ren}_{j,j+1}$ are analytic in K', w and u. If the output data is real and K' is real, then $\mathrm{ren}_{j,j+1}(K', \mathcal{W}, u)$ is real.

Lemma IX.8. *There is a constant* const, *independent of* M *and* j, *such that if* $K' \in \mathfrak{K}_{j+1}$, *then*

(i)

$$\begin{aligned}\|\delta K(K')\|_{1,\Sigma_j} &\le \frac{\lambda_0^{1-\upsilon}}{\alpha^6}\frac{\mathfrak{l}_j}{M^j}\,\mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}}),\\ \Big\|\tfrac{d}{ds}\delta K(K'+sK'')\big|_{s=0}\Big\|_{1,\Sigma_j} &\le \frac{\lambda_0^{1-\upsilon}}{\alpha}\,\mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})\ \|K''\|_{1,\Sigma_{j+1}},\end{aligned}$$

(ii)

$$\begin{aligned}\|K(K')\|_{1,\Sigma_j} &\le \mathrm{const}\,\frac{\mathfrak{l}_j}{\mathfrak{l}_{j+1}}\mathfrak{c}_{j-1}\|K'\|_{1,\Sigma_{j+1}} + \frac{\lambda_0^{1-\upsilon}}{\alpha^6}\frac{\mathfrak{l}_j}{M^j}\mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}},)\\ \Big\|\tfrac{d}{ds}K(K'+sK'')\big|_{s=0}\Big\|_{1,\Sigma_j} &\le \mathrm{const}\,M^{\aleph}\mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})\|K''\|_{1,\Sigma_{j+1}},\end{aligned}$$

(iii)

$$\mathfrak{e}_j(\|K(K')\|_{1,\Sigma_j}) \le \mathrm{const}\ \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}},)$$

Proof. The proof is given following Lemma B.3 in Appendix B. □

Proposition IX.9. *Let* $(\mathcal{W},\mathcal{G},u,\vec{p}) \in \mathcal{D}_{\mathrm{out}}^{(j)}$ *and* $K' \in \mathfrak{K}_{j+1}$. *Then*

$$\mathrm{ren}_{j,j+1}(K',\mathcal{W},u) \in \mathfrak{K}_j$$

Proof. As $\mathrm{ren}_{j,j+1}(K',\mathcal{W},u) = K(K';q_0)$, the proposition follows from Lemma B.1.i and Lemma IX.7. □

We define, for each $K' \in \mathfrak{K}_{j+1}$,

$$\begin{aligned}\tilde{w}(\phi,\psi;K') &= \tilde{w}(\phi,\psi;K';q_0(K'))\\ w''(\phi,\psi;K') &= \tilde{w}\big(\phi,\psi;K'\big) - \tilde{w}\big(\phi,0;K'\big) - \tfrac{1}{2}\sum_{s_1,s_1\in\Sigma_j}\int d\xi_1 d\xi_2\\ &\quad\times q_0\big((\xi_1,s_1),(\xi_2,s_2);K'\big)\psi\big((\xi_1,s_1)\big)\psi\big((\xi_2,s_2)\big),\end{aligned} \tag{IX.18}$$

and let $\mathcal{W}'$ be the Grassmann function having sectorized representative w''. Also set

$$\begin{aligned}\mathcal{G}'(\phi;K') &= \mathcal{G}(\phi;K(K')) + \tilde{w}(\phi,0;K') - \tilde{w}(0,0;K'),\\ u'(K') &= u'(K';q_0),\\ {p'}^{(i)} &= p^{(i)} \qquad \text{for all } 2\le i\le j-1,\\ {p'}^{(j)} &= \delta u(\delta K(0))_{\Sigma_j} * \hat{\nu}^{(\ge j)} + q_0(0) * \hat{\nu}^{(\ge j+1)}\\ &\quad -\big[\delta K(0)_{\mathrm{ext}}\big]_{\Sigma_j} * \hat{\nu}^{(\ge j)}.\end{aligned} \tag{IX.19}$$

Observe that

$$\begin{aligned}\check{u}'(k;K') &= \check{u}(k;K(K'))\,\tilde{\nu}^{(\ge j+1)}(k) + \check{q}_0(k;K')\,\nu^{(\ge j+1)}(k),\\ \check{p'}^{(j)}(k) &= \delta\check{u}(k;\delta K(0))\,\nu^{(\ge j)}(k)\\ &\quad + \check{q}_0(k;0)\,\nu^{(\ge j+1)}(k) - \delta\check{K}(\mathbf{k};0)\,\nu^{(\ge j)}(k).\end{aligned} \tag{IX.20}$$

We now define

$$\mathcal{O}_j(\mathcal{W},\mathcal{G},u,\vec{p}) = (\mathcal{W}',\mathcal{G}',u',\vec{p}').$$

Theorem IX.10. *Let* $(\mathcal{W}, \mathcal{G}, u, \vec{p}) \in \mathcal{D}^{(j)}_{\rm out}$. *Then* $\mathcal{O}_j(\mathcal{W}, \mathcal{G}, u, \vec{p}) \in \mathcal{D}^{(j+1)}_{\rm in}$.

Proof. Set $(\mathcal{W}', \mathcal{G}', u', \vec{p}\,') = \mathcal{O}_j(\mathcal{W}, \mathcal{G}, u, \vec{p})$. Let w be the sectorized representative of $\mathcal{W}$ and

$$u(K) = \sum_{i=2}^{j-1} p^{(i)}_{\Sigma_j} + \big[\delta u(K) - K_{\rm ext}\big]_{\Sigma_j}$$

be the decomposition of u specified in (VIII.2). Let $\tilde{w},\ w''$ be as in (IX.18), q_0 be the function of Lemma IX.7 and let $\delta K(K'),\ K(K')$ be as in (IX.17).

Verification that $(\mathcal{W}', \mathcal{G}', u', \vec{p}\,')$ *is an interaction quadruple of scale* $j+1$ We first check the properties of $\vec{p}\,'$. As $p'^{(i)} = p^{(i)}$, for all $2 \le i \le j-1$, we need only discuss $p'^{(j)}$. By (IX.20), $\check{p}'^{(j)}(k)$ is supported on the $j^{\rm th}$ neighbourhood and, since $\delta\check{K}(\mathbf{k}; 0) = \check{q}_0\big((0,\mathbf{k}); 0\big)\,\nu^{(\ge j+1)}((0,\mathbf{k}))$, vanishes at $k_0 = 0$. By Lemma XIII.7 of [FKTo3], Remark XIX.5 of [FKTo4], Lemma E.5 of [FKTo4], (VIII.3), Lemma IX.7, Lemma IX.8.i and Corollary A.5.i of [FKTo1],

$$\begin{aligned}
\big|p'^{(j)}\big|_{1,\Sigma_j} &\le \big|\delta u(\delta K(0))_{\Sigma_j} * \hat{\nu}^{(\ge j)}(k)\big|_{1,\Sigma_j} + \big|q_0(0) * \hat{\nu}^{(\ge j+1)}\big|_{1,\Sigma_j} \\
&\quad + \big|\big[\delta K(0)_{\rm ext}\big]_{\Sigma_j} * \hat{\nu}^{(\ge j)}\big|_{1,\Sigma_j} \\
&\le {\rm const}\, \mathfrak{c}_j \Big[\big|\delta u(\delta K(0))\big|_{1,\Sigma_{j-1}} + \big|q_0(0)\big|_{1,\Sigma_j} + \|\delta K(0)\|_{1,\Sigma_j}\Big] \\
&\le {\rm const}\Big[\lambda_0^{1-\upsilon}\, \mathfrak{e}_j\big(\|\delta K(0)\|_{1,\Sigma_j}\big)\, \|\delta K(0)\|_{1,\Sigma_j} + \tfrac{\lambda_0^{1-\upsilon}}{\alpha^6}\, \tfrac{\mathfrak{l}_j}{M^j}\, \mathfrak{c}_j\Big] \\
&\le {\rm const}\, \tfrac{\lambda_0^{1-\upsilon}}{\alpha^6}\, \tfrac{\mathfrak{l}_j}{M^j}\, \mathfrak{c}_j \Big[\lambda_0^{1-\upsilon}\, \mathfrak{e}_j\Big(\tfrac{\lambda_0^{1-\upsilon}}{\alpha^6}\, \tfrac{\mathfrak{l}_j}{M^j}\, \mathfrak{c}_j\Big) + 1\Big] \\
&\le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^5}\, \tfrac{\mathfrak{l}_j}{M^j}\, \mathfrak{c}_j.
\end{aligned} \tag{IX.21}$$

Thus, (VIII.1) holds for $\vec{p}\,'$.
Next, we construct the decomposition (VIII.2) for u'. Set

$$\delta u'(K') = \delta u(K(L))_{\Sigma_j} + q_0(L) * \hat{\nu}^{(\ge j+1)} - \delta K(L)_{\rm ext} * \hat{\nu}^{(\ge j)}\Big|_{L=0}^{L=K'}.$$

By construction, $\delta u' \in \mathcal{F}_0(2, \Sigma_j)$ and $\delta u'(0) = 0$. Since $\delta\check{u}\big((0,\mathbf{k}); K'\big)$ vanishes,

$$\delta\check{u}'\big((0,\mathbf{k}); K'\big) = \check{q}_0\big((0,\mathbf{k}); L\big)\,\nu^{(\ge j+1)}\big((0,\mathbf{k})\big) - \delta\check{K}(\mathbf{k}; L)\,\nu^{(\ge j)}\big((0,\mathbf{k})\big)\Big|_{L=0}^{L=K'} = 0$$

by (IX.15) and (IX.17). Observe that, by Remark XIX.3.iii of [FKTo4],

$$\begin{aligned}
\delta u'(K')_{\Sigma_{j+1}} &= \delta u(K(L))_{\Sigma_{j+1}} + q_0(L)_{\Sigma_{j+1}} * \hat{\nu}^{(\ge j+1)} - \big[\delta K(L)_{\rm ext}\big]_{\Sigma_{j+1}}\Big|_{L=0}^{L=K'}, \\
p'^{(j)}_{\Sigma_{j+1}} &= \delta u(K(L))_{\Sigma_{j+1}} + q_0(L)_{\Sigma_{j+1}} * \hat{\nu}^{(\ge j+1)} - \big[\delta K(L)_{\rm ext}\big]_{\Sigma_{j+1}}\Big|_{L=0}.
\end{aligned}$$

Since $K(K') = K'_{\Sigma_j} + \delta K(K')$,

$$\begin{aligned}
u'(K') &= u(K(K'))_{\Sigma_{j+1}} + q_0(K')_{\Sigma_{j+1}} * \hat{\nu}^{(\ge j+1)} \\
&= \sum_{i=2}^{j-1} p^{(i)}_{\Sigma_{j+1}} + \big[\delta u(K(K')) - K(K')_{\rm ext}\big]_{\Sigma_{j+1}} + q_0(K')_{\Sigma_{j+1}} * \hat{\nu}^{(\ge j+1)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=2}^{j-1} p^{(i)}_{\Sigma_{j+1}} + \Big[\delta u(K(L))_{\Sigma_{j+1}} + q_0(L)_{\Sigma_{j+1}} * \hat\nu^{(\ge j+1)} - [\delta K(L)_{\rm ext}]_{\Sigma_{j+1}}\Big]_{L=K'} \\
&\quad -[K'_{\rm ext}]_{\Sigma_{j+1}} \\
&= \sum_{i=2}^{j-1} p'^{(i)}_{\Sigma_{j+1}} + p'^{(j)}_{\Sigma_{j+1}} + \big[\delta u'(K') - K'_{\rm ext}\big]_{\Sigma_{j+1}}
\end{aligned}
$$

and $u'(K')$ has the desired form.
We next bound $\big|\mathcal{D}^{(1,0,0)}_{1,2} \frac{d}{ds}\delta u'(K' + sK'')\big|_{s=0}\big|_{1,\Sigma_j}$, for $K', K'' \in \mathfrak{K}_{j+1}$. By definition

$$
\begin{aligned}
&\big|\mathcal{D}^{(1,0,0)}_{1,2} \tfrac{d}{ds}\delta u'(K' + sK'')\big|_{s=0}\big|_{1,\Sigma_j} \\
&\quad \le \big|\mathcal{D}^{(1,0,0)}_{1,2} \tfrac{d}{ds}\delta u(K'_{\Sigma_j} + sK''_{\Sigma_j} + \delta K(K' + sK''))_{\Sigma_j}\big|_{s=0}\big|_{1,\Sigma_j} \\
&\qquad + \big|\mathcal{D}^{(1,0,0)}_{1,2} \tfrac{d}{ds} q_0(K' + sK'')\big|_{s=0} * \hat\nu^{(\ge j+1)}\big|_{1,\Sigma_j} \\
&\qquad + \big|\mathcal{D}^{(1,0,0)}_{1,2} \tfrac{d}{ds}\delta K(K' + sK'')_{\rm ext} * \hat\nu^{(\ge j)}\big|_{s=0}\big|_{1,\Sigma_j}. \qquad\qquad \text{(IX.22)}
\end{aligned}
$$

By Remark XII.11 of [FKTo3] and Lemma E.5 of [FKTo4], Lemma IX.8.i and Remark XV.2.iv of [FKTo3], the third term is bounded by

$$
\begin{aligned}
&\big|\mathcal{D}^{(1,0,0)}_{1,2} \tfrac{d}{ds}\delta K(K' + sK'')_{\rm ext} * \hat\nu^{(\ge j)}\big|_{s=0}\big|_{1,\Sigma_j} \\
&\quad \le \tfrac{\partial}{\partial t_0}\big|\tfrac{d}{ds}\delta K(K' + sK'')_{\rm ext} * \hat\nu^{(\ge j)}\big|_{s=0}\big|_{1,\Sigma_j} \\
&\quad \le const\, \tfrac{\partial}{\partial t_0}\Big(\mathfrak{c}_j \,\big\|\tfrac{d}{ds}\delta K(K' + sK'')\big|_{s=0}\big\|_{1,\Sigma_j}\Big) \\
&\quad \le const\, \tfrac{\partial}{\partial t_0}\Big(\mathfrak{c}_j \tfrac{\lambda_0^{1-\upsilon}}{\alpha}\, \mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})\ \|K''\|_{1,\Sigma_{j+1}}\Big) \\
&\quad \le const\, \tfrac{\lambda_0^{1-\upsilon}}{\alpha} \|K''\|_{1,\Sigma_{j+1}} \tfrac{\partial}{\partial t_0}\Big(\mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})\Big) \\
&\quad \le const\, \tfrac{\lambda_0^{1-\upsilon}}{\alpha} M^j\, \mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})\, \|K''\|_{1,\Sigma_{j+1}} + \sum_{\delta_0 = r_0} \infty\, t^\delta \\
&\quad \le \Big(const\, \tfrac{M^2}{\alpha}\Big)\lambda_0^{1-\upsilon} M^{(j+1)-3}\, \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\, \|K''\|_{1,\Sigma_{j+1}} \\
&\qquad + \sum_{\delta_0 = r_0} \infty\, t^\delta. \qquad\qquad \text{(IX.23)}
\end{aligned}
$$

Similarly, by Remark XII.11 and Lemma XIII.7 of [FKTo3], and Lemma IX.7, the second term is bounded by

$$
\begin{aligned}
&\big|\mathcal{D}^{(1,0,0)}_{1,2} \tfrac{d}{ds} q_0(K' + sK'')\big|_{s=0} * \hat\nu^{(\ge j+1)}\big|_{1,\Sigma_j} \\
&\quad \le \tfrac{\partial}{\partial t_0}\big|\tfrac{d}{ds} q_0(K' + sK'')\big|_{s=0} * \hat\nu^{(\ge j+1)}\big|_{1,\Sigma_j} \\
&\quad \le const\, \tfrac{\partial}{\partial t_0}\Big(\mathfrak{c}_j \big|\tfrac{d}{ds} q_0(K' + sK'')\big|_{s=0}\big|_{1,\Sigma_j}\Big) \\
&\quad \le const\, \tfrac{\partial}{\partial t_0}\Big(\mathfrak{c}_j \tfrac{\lambda_0^{1-\upsilon}}{\alpha}\, \mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})\ \|K''\|_{1,\Sigma_{j+1}}\Big) \\
&\quad \le \Big(const\, \tfrac{M^2}{\alpha}\Big)\lambda_0^{1-\upsilon} M^{(j+1)-3}\, \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\, \|K''\|_{1,\Sigma_{j+1}} \\
&\qquad + \sum_{\delta_0 = r_0} \infty\, t^\delta. \qquad\qquad \text{(IX.24)}
\end{aligned}
$$

By the chain rule,

$$\begin{aligned}
&\tfrac{d}{ds}\delta u(K'_{\Sigma_j} + sK''_{\Sigma_j} + \delta K(K' + sK''))\big|_{s=0} \\
&\quad= \tfrac{d}{ds}\delta u(K'_{\Sigma_j} + \delta K(K') + sK''_{\Sigma_j})\big|_{s=0} \\
&\qquad+ \tfrac{d}{ds}\delta u(K'_{\Sigma_j} + \delta K(K') + s\tfrac{d\delta K(K'+xK'')}{dt}\big|_{x=0})\big|_{s=0} \\
&\quad= \tfrac{d}{ds}\delta u(K(K') + sK''_{\Sigma_j})\big|_{s=0} \\
&\qquad+ \tfrac{d}{ds}\delta u(K(K') + s\tfrac{d}{dx}\delta K(K' + xK'')\big|_{x=0})\big|_{s=0}.
\end{aligned} \tag{IX.25}$$

We bound these two contributions to the first term of the right hand side of (IX.22) separately. By Remark XIX.5 of [FKTo4], (VIII.3), with K replaced by $K(K')$, Proposition E.10.ii of [FKTo4] and Lemma IX.8.iii,

$$\begin{aligned}
&\big|\mathcal{D}_{1,2}^{(1,0,0)} \tfrac{d}{ds}\delta u(K(K') + sK''_{\Sigma_j})_{\Sigma_j}\big|_{s=0}\big|_{1,\Sigma_j} \\
&\quad\le const\, \mathfrak{c}_{j-1}\big|\mathcal{D}_{1,2}^{(1,0,0)} \tfrac{d}{ds}\delta u(K(K') + sK''_{\Sigma_j})\big|_{s=0}\big|_{1,\Sigma_{j-1}} \\
&\quad\le const\, \lambda_0^{1-\upsilon} M^{j-3}\, \mathfrak{e}_j\big(\|K(K')\|_{1,\Sigma_j}\big)\, \|K''_{\Sigma_j}\|_{1,\Sigma_j} + \sum_{\delta_0=r_0} \infty\, t^\delta \\
&\quad\le const\, \lambda_0^{1-\upsilon} M^{j-3}\, \mathfrak{e}_{j+1}\big(\|K'\|_{1,\Sigma_{j+1}}\big)\, \tfrac{\mathfrak{l}_j}{\mathfrak{l}_{j+1}}\mathfrak{c}_{j-1}\|K''\|_{1,\Sigma_{j+1}} + \sum_{\delta_0=r_0} \infty\, t^\delta \\
&\quad\le \Big(const\,\tfrac{1}{M}\tfrac{\mathfrak{l}_j}{\mathfrak{l}_{j+1}}\Big)\lambda_0^{1-\upsilon} M^{(j+1)-3}\, \mathfrak{e}_{j+1}\big(\|K'\|_{1,\Sigma_{j+1}}\big)\|K''\|_{1,\Sigma_{j+1}} \\
&\qquad+ \sum_{\delta_0=r_0} \infty\, t^\delta.
\end{aligned} \tag{IX.26}$$

Similarly, by Remark XIX.5 of [FKTo4], (VIII.3), parts (i) and (iii) of Lemma IX.8 and Corollary A.5.ii of [FKTo1],

$$\begin{aligned}
&\big|\mathcal{D}_{1,2}^{(1,0,0)} \tfrac{d}{ds}\delta u(K(K') + s\tfrac{d}{dx}\delta K(K' + xK'')\big|_{x=0})_{\Sigma_j}\big|_{s=0}\big|_{1,\Sigma_j} \\
&\quad\le const\, \mathfrak{c}_{j-1}\big|\mathcal{D}_{1,2}^{(1,0,0)} \tfrac{d}{ds}\delta u(K(K') + s\tfrac{d}{dx}\delta K(K' + xK'')\big|_{x=0})\big|_{s=0}\big|_{1,\Sigma_{j-1}} \\
&\quad\le const\, \lambda_0^{1-\upsilon} M^{j-3}\, \mathfrak{e}_j\big(\|K(K')\|_{1,\Sigma_j}\big)\, \big\|\tfrac{d}{dx}\delta K(K' + xK'')\big|_{x=0}\big\|_{1,\Sigma_j} + \sum_{\delta_0=r_0} \infty\, t^\delta \\
&\quad\le const\, \lambda_0^{1-\upsilon} M^{j-3}\, \mathfrak{e}_{j+1}\big(\|K'\|_{1,\Sigma_{j+1}}\big)\, \tfrac{\lambda_0^{1-\upsilon}}{\alpha}\, \mathfrak{e}_j\big(\|K'\|_{1,\Sigma_{j+1}}\big)\, \|K''\|_{1,\Sigma_{j+1}} + \sum_{\delta_0=r_0} \infty\, t^\delta \\
&\quad\le \Big(const\,\tfrac{\lambda_0^{1-\upsilon}}{\alpha}\Big)\lambda_0^{1-\upsilon} M^{(j+1)-3}\, \mathfrak{e}_{j+1}\big(\|K'\|_{1,\Sigma_{j+1}}\big)\|K''\|_{1,\Sigma_{j+1}} + \sum_{\delta_0=r_0} \infty\, t^\delta.
\end{aligned} \tag{IX.27}$$

Substituting (IX.25) into (IX.22) and applying the bounds (IX.23), (IX.24), (IX.26) and (IX.27), we have

$$\begin{aligned}
&\big|\mathcal{D}_{1,2}^{(1,0,0)} \tfrac{d}{ds}\delta u'(K' + sK'')\big|_{s=0}\big|_{1,\Sigma_j} \\
&\quad\le const_0\Big(\tfrac{M^2}{\alpha} + \tfrac{1}{M}\tfrac{\mathfrak{l}_j}{\mathfrak{l}_{j+1}} + \tfrac{\lambda_0^{1-\upsilon}}{\alpha}\Big)\lambda_0^{1-\upsilon} M^{(j+1)-3}\, \mathfrak{e}_{j+1}\big(\|K'\|_{1,\Sigma_{j+1}}\big)\, \|K''\|_{1,\Sigma_{j+1}} \\
&\qquad+ \sum_{\delta_0=r_0} \infty\, t^\delta
\end{aligned}$$

with an M–independent constant $const_0$. If M is sufficiently large

$$const_0\,\tfrac{1}{M}\tfrac{\mathfrak{l}_j}{\mathfrak{l}_{j+1}} = const_0\,\tfrac{1}{M^{1-\aleph}} \le \tfrac{1}{2}.$$

If α is sufficiently large and λ_0 is sufficiently small, depending on M,

$$const_0\Big(\frac{M^2}{\alpha}+\frac{\lambda_0^{1-\upsilon}}{\alpha}\Big)\le\frac{1}{2}$$

and

$$\Big|\mathcal{D}_{1,2}^{(1,0,0)}\tfrac{d}{ds}\delta u'(K'+sK'')\big|_{s=0}\Big|_{1,\Sigma_j}\le\lambda_0^{1-\upsilon}M^{(j+1)-3}\,\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\|K''\|_{1,\Sigma_{j+1}}+\sum_{\delta_0=r_0}\infty\,t^\delta.$$

The remaining bound required in (VIII.3), namely,

$$\Big|\tfrac{d}{ds}\delta u'(K'+sK'')\big|_{s=0}\Big|_{1,\Sigma_j}\le\lambda_0^{1-\upsilon}\,\mathfrak{e}_{j+1}\big(\|K'\|_{1,\Sigma_{j+1}}\big)\,\|K''\|_{1,\Sigma_{j+1}}$$

is now a consequence of Corollary XIX.12 of [FKTo4].
The remaining requirements of Definition VIII.1 are easily verified.

Preparation for the verification of (I1), (I2) and (I3) Clearly

$$w'(K')=w''_{\Sigma_{j+1}}(K') \tag{IX.28}$$

is a Σ_{j+1}–sectorized representative of $\mathcal{W}'$. Write

$$\begin{aligned}\tilde w(\phi,\psi;K')&=\sum_{m,n}\sum_{s_1,\cdots,s_n\in\Sigma_j}\int d\eta_1\cdots d\eta_m\,d\xi_1\cdots d\xi_n\;\tilde w_{m,n}(\eta_1,\cdots,\eta_m\,(\xi_1,s_1),\cdots,(\xi_n,s_n);K')\\&\qquad\times\phi(\eta_1)\cdots\phi(\eta_m)\;\psi((\xi_1,s_1))\cdots\psi((\xi_n,s_n)),\\w''(\phi,\psi;K')&=\sum_{m,n}\sum_{s_1,\cdots,s_n\in\Sigma_j}\int d\eta_1\cdots d\eta_m\,d\xi_1\cdots d\xi_n\;w''_{m,n}(\eta_1,\cdots,\eta_m\,(\xi_1,s_1),\cdots,(\xi_n,s_n);K')\\&\qquad\times\phi(\eta_1)\cdots\phi(\eta_m)\;\psi((\xi_1,s_1))\cdots\psi((\xi_n,s_n)),\\w'(\phi,\psi;K')&=\sum_{m,n}\sum_{s_1,\cdots,s_n\in\Sigma_j}\int d\eta_1\cdots d\eta_m\,d\xi_1\cdots d\xi_n\;w'_{m,n}(\eta_1,\cdots,\eta_m\,(\xi_1,s_1),\cdots,(\xi_n,s_n);K')\\&\qquad\times\phi(\eta_1)\cdots\phi(\eta_m)\;\psi((\xi_1,s_1))\cdots\psi((\xi_n,s_n)).\end{aligned}$$

By Lemma IX.7, $\tilde w_{0,2}(K')=\frac{1}{2}q_0(K')$ and hence, by (IX.18), $w''_{0,2}=0$. Consequently, by (IX.28), $w'_{0,2}=0$.

By (IX.16) and Proposition A.2 of [FKTr1],

$$\tilde w(\phi,\psi;K')=\,:w(\phi,\psi;K(K')):_{\psi,-E_{\Sigma_j}(K';q_0)}. \tag{IX.29}$$

By Lemma B.1.iii, $\sqrt{\lambda_0^{1-\upsilon}\mathfrak{l}_j}\sqrt{\frac{\mathfrak{l}_j\mathrm{B}}{M^j}}$ is an integral bound for E_{Σ_j}. Hence by Corollary II.32.ii of [FKTr1], with $f(\psi)=w(\phi,\psi;K(K'))$ and $f'(\psi)=\,:w(\phi,\psi;K(K')):_{\psi,-E_{\Sigma_j}}$,

$$N_j\Big(\tilde w(K')-w\big(K(K')\big),\tfrac{\alpha}{2},\,X\Big)\le\frac{8\lambda_0^{1-\upsilon}}{\alpha^2}\mathfrak{l}_j\;N_j(w(K(K')),\alpha,\,X) \tag{IX.30}$$

for all $X\in\mathfrak{N}_{d+1}$. In particular

$$N_j\big(\tilde w(K'),\tfrac{\alpha}{2},\,X\big)\le\tfrac{3}{2}\,N_j\big(w(K(K')),\alpha,\,X\big). \tag{IX.31}$$

To get a similar bound on $\frac{d}{ds}\tilde{w}(K'+sK'')\big|_{s=0}$, observe that, by (IX.29)

$$\frac{d}{ds}\tilde{w}(K'+sK'')\big|_{s=0} =: \frac{d}{ds}w\big(K(K'+sK'')\big)\!:_{\psi,-E_{\Sigma_j}(K';q_0)}\big|_{s=0} \\ +\frac{d}{ds}\!:\!w\big(K(K')\big)\!:_{\psi,-E_{\Sigma_j}(K'+sK'';q_0)}\big|_{s=0}. \tag{IX.32}$$

By Corollary II.32 of [FKTr1], (O1), Lemma IX.8, parts (ii) and (iii), and Corollary A.5.ii of [FKTo1], the first term is bounded by

$$\begin{aligned} &N_j\Big(:\tfrac{d}{ds}w\big(K(K'+sK'')\big):_{\psi,-E_{\Sigma_j}(K';q_0)}\big|_{s=0}\,,\ \tfrac{\alpha}{2}\,,\ \|K'\|_{1,\Sigma_{j+1}}\Big)\\ &\le N_j\Big(\tfrac{d}{ds}w\big(K(K'+sK'')\big)\big|_{s=0}\,,\ \alpha\,,\ \|K'\|_{1,\Sigma_{j+1}}\Big)\\ &\le \mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})N_j\Big(\tfrac{d}{ds}w\big(K(K'+sK'')\big)\big|_{s=0}\,,\ \alpha\,,\ \|K(K')\|_{1,\Sigma_j}\Big)\\ &\le M^j\mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})\mathfrak{e}_j(\|K(K')\|_{1,\Sigma_j})\big\|\tfrac{d}{dx}K(K'+xK'')\big\|_{1,\Sigma_j}\big|_{x=0}\\ &\le const\ M^{j+\aleph}\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})^3\|K''\|_{1,\Sigma_{j+1}}\\ &\le const\ M^{j+\aleph}\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\|K''\|_{1,\Sigma_{j+1}}. \end{aligned} \tag{IX.33}$$

In preparation for bounding the second term of (IX.32), observe that, by Remark II.30 of [FKTr1] and Lemma B.1,

$$\frac{d}{ds}E_{\Sigma_j}(K'+sK'';q_0)\big|_{s=0} = \frac{d}{ds}E_{\Sigma_j}(K'+sK'';q_0(K'))\big|_{s=0} \\ +\frac{d}{ds}E_{\Sigma_j}(K';q_0(K')+s\tfrac{d}{dx}q_0(K'+xK'')|_{x=0})\big|_{s=0}$$

has integral bound

$$\text{const}\sqrt{\mathfrak{l}_j\|K''\|_{1,\Sigma_{j+1}}\big|_{t=0}} + \text{const}\sqrt{\mathfrak{l}_j\tfrac{\lambda_0^{1-\upsilon}}{\alpha}\|K''\|_{1,\Sigma_{j+1}}\big|_{t=0}} \\ \le \text{const}\sqrt{\mathfrak{l}_j\|K''\|_{1,\Sigma_{j+1}}\big|_{t=0}}$$

since, by Lemma IX.7,

$$\big|\tfrac{d}{dx}q_0(K'+xK'')\big|_{x=0}\big|_{1,\Sigma_j} \le \tfrac{\lambda_0^{1-\upsilon}}{\alpha}\,\mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})\ \|K''\|_{1,\Sigma_{j+1}}.$$

Consequently, by Corollary II.32.iii of [FKTr1], (O1), Lemma IX.8, parts (ii) and (iii), and Corollary A.5.ii of [FKTo1],

$$\begin{aligned} &N_j\Big(\tfrac{d}{ds}:w\big(K(K')\big):_{\psi,-E_{\Sigma_j}(K'+sK'';q_0)}\big|_{s=0}\,,\ \tfrac{\alpha}{2}\,,\ \|K'\|_{1,\Sigma_{j+1}}\Big)\\ &\le \tfrac{\rm const}{(\alpha-1)^2}\,N_j\Big(w\big(K(K')\big)\,,\ \alpha\,,\ \|K'\|_{1,\Sigma_{j+1}}\Big)M^j\|K''\|_{1,\Sigma_{j+1}}\\ &\le \tfrac{\rm const}{(\alpha-1)^2}\mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})\,N_j\Big(w\big(K(K')\big)\,,\ \alpha\,,\ \|K(K')\|_{1,\Sigma_j}\Big)M^j\|K''\|_{1,\Sigma_{j+1}}\\ &\le \tfrac{\rm const}{(\alpha-1)^2}\mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})\mathfrak{e}_j(\|K(K')\|_{1,\Sigma_j})\ M^j\|K''\|_{1,\Sigma_{j+1}}\\ &\le \tfrac{\rm const}{(\alpha-1)^2}\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\ M^j\|K''\|_{1,\Sigma_{j+1}}. \end{aligned} \tag{IX.34}$$

Combining (IX.32), (IX.33) and (IX.34),

$$N_j\Big(\tfrac{d}{ds}\tilde{w}(K'+sK'')\big|_{s=0}\,,\ \tfrac{\alpha}{2}\,,\ \|K'\|_{1,\Sigma_{j+1}}\Big) \\ \le const\ M^{j+\aleph}\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\|K''\|_{1,\Sigma_{j+1}} \tag{IX.35}$$

Verification of (I3) By the definition, (IX.19), of $\mathcal{G}'$

$$\begin{aligned} N\big(\mathcal{G}'(\phi; K') - \mathcal{G}(\phi; K(K'))\big) &= N\big(\tilde{w}(\phi, 0; K') - \tilde{w}(0, 0; K')\big) \\ &\le \tfrac{1}{\sqrt[4]{\mathfrak{l}_j M^j}}\, N_j\big(\tilde{w}(\phi, 0; K')\,,\ \tfrac{\alpha}{2}\,,\ 0\big)_{\mathbf{0}} \\ &\le \tfrac{3/2}{\sqrt[4]{\mathfrak{l}_j M^j}}\, N_j\big(w(K(K'))\,,\ \alpha\,,\ 0\big)_{\mathbf{0}} \\ &\le \tfrac{3/2}{\sqrt[4]{\mathfrak{l}_j M^j}}\, N_j\big(w(K(K'))\,,\ \alpha\,,\ \|K(K')\|_{1,\Sigma_j}\big)_{\mathbf{0}} \\ &\le \tfrac{2}{\sqrt[4]{\mathfrak{l}_j M^j}} \end{aligned} \tag{IX.36}$$

by Remark VI.8.ii, (IX.31) and (O1). Therefore, by (O3),

$$\begin{aligned} &N\big(\mathcal{G}'(\phi; K') - \tfrac{1}{2}\phi J C^{(<j+1)} J\phi\big) \\ &\quad \le N\Big(\mathcal{G}(\phi; K(K')) - \tfrac{1}{2}\phi J C^{(\le j)} J\phi\Big) + N\big(\mathcal{G}'(\phi; K') - \mathcal{G}(\phi; K(K'))\big) \\ &\quad \le 4 \sum_{i=0}^{j} \tfrac{1}{\sqrt[4]{\mathfrak{l}_i M^i}}. \end{aligned}$$

From (IX.19),

$$\begin{aligned} &N\Big(\tfrac{d}{ds}\big[\mathcal{G}'(\phi; {\scriptstyle K'+sK''}) - \mathcal{G}_2'(\phi; {\scriptstyle K'+sK''})\big]_{s=0}\Big) \\ &\quad \le N\Big(\tfrac{d}{ds}\big[\mathcal{G}(\phi; {\scriptstyle K(K'+sK'')}) - \mathcal{G}_2(\phi; {\scriptstyle K(K'+sK'')})\big]_{s=0}\Big) + N\Big(\tfrac{d}{ds}\tilde{w}(K' + sK'')\big|_{\substack{s=0\\ \psi=0}}\Big) \\ &\quad \le N\Big(\tfrac{d}{ds}\big[\mathcal{G}\big(\phi; {\scriptstyle K(K')+sL}\big) - \mathcal{G}_2\big(\phi; {\scriptstyle K(K')+sL}\big)\big]_{s=0}\Big) \\ &\qquad + \tfrac{1}{\sqrt[4]{\mathfrak{l}_j M^j}} N_j\Big(\tfrac{d}{ds}\tilde{w}(K' + sK'')\big|_{s=0}\,,\ \tfrac{\alpha}{2}\,,\ {\scriptstyle \|K'\|_{1,\Sigma_{j+1}}}\Big)_{\mathbf{0}}, \end{aligned}$$

where $L = \frac{d}{dx} K(K' + xK'')|_{x=0}$ obeys $\|L\|_{1,\Sigma_j} \le \mathit{const}\, M^{\aleph}\, \mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})\ \|K''\|_{1,\Sigma_{j+1}}$, by Lemma IX.8.ii. Hence, by (O3) and (IX.35)

$$\begin{aligned} &N\Big(\tfrac{d}{ds}\big[\mathcal{G}'(\phi; {\scriptstyle K'+sK''}) - \mathcal{G}_2'(\phi; {\scriptstyle K'+sK''})\big]_{s=0}\Big) \\ &\quad \le M^j \|L\|_{1,\Sigma_j} + \mathit{const}\, M^{j+\aleph} \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}}) \|K''\|_{1,\Sigma_{j+1}} \\ &\quad \le \mathit{const}\, M^{j+\aleph} \mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}}) \|K''\|_{1,\Sigma_{j+1}} \\ &\quad \le \tfrac{1}{2}\, M^{j+1} \|K''\|_{1,\Sigma_{j+1}} + \sum_{\delta \ne 0} \infty\, t^{\delta}. \end{aligned} \tag{IX.37}$$

Similarly,

$$\begin{aligned} &\Big\|\!\Big|\tfrac{d}{ds} G_2'(\phi; K' + sK'')\big|_{s=0}\Big|\!\Big\|_\infty \\ &\quad \le \Big\|\!\Big|\tfrac{d}{ds} G_2(\phi; {\scriptstyle K(K'+sK'')})\big|_{s=0}\Big|\!\Big\|_\infty + \lambda_0^{1-\upsilon} N\Big(\tfrac{d}{ds}\tilde{w}(K' + sK'')\big|_{\substack{s=0\\ \psi=0}}\Big) \\ &\quad \le M^j \|L\|_{1,\Sigma_j} + \mathit{const}\, \lambda_0^{1-\upsilon}\, \tfrac{1}{\sqrt[4]{\mathfrak{l}_j M^j}}\, M^{j+\aleph} \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}}) \|K''\|_{1,\Sigma_{j+1}} \\ &\quad \le \tfrac{1}{2}\, M^{j+1} \|K''\|_{1,\Sigma_{j+1}} + \sum_{\delta \ne 0} \infty\, t^{\delta}. \end{aligned}$$

Verification of (I2) Observe that by (IX.30), (O1) and Lemma IX.8.iii

$$\begin{aligned}
&\tfrac{M^{2j}}{\mathfrak{l}_j}\,\tfrac{\alpha^4}{16}\left(\tfrac{\mathfrak{l}_j \mathrm{B}}{M^j}\right)^2 \tfrac{1}{\lambda_0^{1-\upsilon}\mathfrak{l}_j}\,\left|\tilde w_{0,4}(K') - w_{0,4}\big(K(K')\big)\right|_{3,\Sigma_j}\\
&\quad\le \tfrac{8\lambda_0^{1-\upsilon}}{\alpha^2}\mathfrak{l}_j\; N_j\big(w(K(K')),\alpha,\ \|K(K')\|_{1,\Sigma_j}\big)\\
&\quad\le \tfrac{8\lambda_0^{1-\upsilon}}{\alpha^2}\mathfrak{l}_j\;\mathfrak{e}_j\big(\|K(K')\|_{1,\Sigma_j}\big)\\
&\quad\le const\,\tfrac{\lambda_0^{1-\upsilon}}{\alpha^2}\mathfrak{l}_j\;\mathfrak{e}_{j+1}\big(\|K'\|_{1,\Sigma_{j+1}}\big)
\end{aligned}$$

so that, by Remark XIX.5 of [FKTo4] and Corollary A.5.ii of [FKTo1]

$$\left|\Big(\tilde w_{0,4}(K') - w_{0,4}\big(K(K')\big)\Big)_{\Sigma_{j+1}}\right|_{3,\Sigma_{j+1}} \le const\,\tfrac{\lambda_0^{2-2\upsilon}}{\alpha^6}\mathfrak{l}_j\;\mathfrak{e}_{j+1}\big(\|K'\|_{1,\Sigma_{j+1}}\big). \tag{IX.38}$$

Set $F'^{(i)} = F^{(i)}$ for all $2 \le i \le j$ and

$$\delta F'^{(j+1)}(K') = \delta F^{(j+1)}(K(K'))_{\Sigma_{j+1}} + \big(\tilde w_{0,4}(K') - w_{0,4}(K(K'))\big)_{\Sigma_{j+1}}.$$

By Definition VII.7, $\mathcal{L}^{(j+1)}(\vec p, \vec F)$ depends only on $p^{(2)},\cdots,p^{(j-1)}$ and $F^{(2)},\cdots,F^{(j)}$. In particular $\mathcal{L}^{(j+1)}(\vec p\,', \vec F') = \mathcal{L}^{(j+1)}(\vec p, \vec F)$. Therefore

$$\begin{aligned}
w'_{0,4}(K') &= \tilde w_{0,4}(K')_{\Sigma_{j+1}}\\
&= \delta F'^{(j+1)}(K') + \sum_{i=2}^{j} F'^{(i)}_{\Sigma_{j+1}} + \tfrac{1}{8}\mathrm{Ant}\Big(V_{\rm ph}\big(\mathcal{L}^{(j+1)}(\vec p\,', \vec F')\big)\Big)_{\Sigma_{j+1}}
\end{aligned}$$

and

$$\left|F'^{(i)}\right|_{3,\Sigma_i} \le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^7}\mathfrak{l}_i^{1/n_0}\mathfrak{c}_i \qquad \text{for all } 2 \le i \le j.$$

Furthermore, by Remark XIX.5 of [FKTo4], (IX.38), (O2), Lemma IX.8 and Corollary A.5.ii of [FKTo1]

$$\begin{aligned}
&\left|\delta F'^{(j+1)}(K')\right|_{3,\Sigma_{j+1}}\\
&\quad\le const\,\mathfrak{c}_j\left|\delta F^{(j+1)}(K(K'))\right|_{3,\Sigma_j} + const\,\tfrac{\lambda_0^{2-2\upsilon}}{\alpha^6}\mathfrak{l}_j\;\mathfrak{e}_{j+1}\big(\|K'\|_{1,\Sigma_{j+1}}\big)\\
&\quad\le const\,\tfrac{\lambda_0^{1-\upsilon}}{\alpha^4}\left\{\tfrac{\mathfrak{l}_{j+1}^{1/n_0}}{\alpha^4} + \tfrac{1}{\mathrm{B}^2}M^j\|K(K')\|_{1,\Sigma_j}\right\}\mathfrak{e}_j\big(\|K(K')\|_{1,\Sigma_j}\big)\\
&\qquad + const\,\tfrac{\lambda_0^{2-2\upsilon}}{\alpha^6}\mathfrak{l}_j\;\mathfrak{e}_{j+1}\big(\|K'\|_{1,\Sigma_{j+1}}\big)\\
&\quad\le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^4}\left\{const\,\tfrac{\mathfrak{l}_{j+1}^{1/n_0}}{\alpha^4} + const\,\tfrac{1}{\mathrm{B}^2}\tfrac{\mathfrak{l}_j}{\mathfrak{l}_{j+1}}M^j\|K'\|_{1,\Sigma_{j+1}} + \mathrm{const}\,\tfrac{\lambda_0^{1-\upsilon}}{\alpha^2}\mathfrak{l}_j\right\}\mathfrak{e}_{j+1}\big(\|K'\|_{1,\Sigma_{j+1}}\big)\\
&\quad\le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^4}\left\{\tfrac{\mathfrak{l}_{j+1}^{1/n_0}}{\alpha^3} + const\,\tfrac{1}{M^{1-\aleph}}\tfrac{1}{\mathrm{B}^2}M^{j+1}\|K'\|_{1,\Sigma_{j+1}}\right\}\mathfrak{e}_{j+1}\big(\|K'\|_{1,\Sigma_{j+1}}\big),
\end{aligned} \tag{IX.39}$$

since $\mathrm{const}\,\lambda_0^{1-\upsilon}\mathfrak{l}_j \le \frac{\mathfrak{l}_{j+1}^{1/n_0}}{2\alpha}$, by the hypothesis $\alpha < \frac{1}{\lambda_0^{\upsilon/10}}$ of Theorem VIII.5 and the requirement of Definition V.6 that $0 < \upsilon < \frac{1}{4}$. In particular

$$\left|\delta F'^{(j+1)}(K')\right|_{3,\Sigma_{j+1}} \le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^4}\left\{\tfrac{\mathfrak{l}_{j+1}^{1/n_0}}{\alpha^3} + \tfrac{1}{\mathrm{B}^2}M^{j+1}\|K'\|_{1,\Sigma_{j+1}}\right\}\mathfrak{e}_{j+1}\big(\|K'\|_{1,\Sigma_{j+1}}\big).$$

Verification of (I1) Set

$$\begin{aligned}\tilde{\omega}_{m,n} &= \sum_{s_1,\cdots,s_n\in\Sigma_j} \int d\eta_1\cdots d\eta_m\, d\xi_1\cdots d\xi_n\ \tilde{w}_{m,n}(\eta_1,\cdots,\eta_m\,(\xi_1,s_1),\cdots,(\xi_n,s_n)) \\ &\quad\times \phi(\eta_1)\cdots\phi(\eta_m)\ \psi((\xi_1,s_1))\cdots\psi((\xi_n,s_n)), \\ \omega'_{m,n} &= \sum_{s_1,\cdots,s_n\in\Sigma_{j+1}} \int d\eta_1\cdots d\eta_m\, d\xi_1\cdots d\xi_n\ w'_{m,n}(\eta_1,\cdots,\eta_m\,(\xi_1,s_1),\cdots,(\xi_n,s_n)) \\ &\quad\times\phi(\eta_1)\cdots\phi(\eta_m)\ \psi((\xi_1,s_1))\cdots\psi((\xi_n,s_n)).\end{aligned}$$

Then

$$\begin{aligned}w'' &= \sum_{m,n\geq 1}\tilde{\omega}_{m,n} + \sum_{n\geq 4}\tilde{\omega}_{0,n}, \\ w' &= \sum_{m,n\geq 1}\omega'_{m,n} + \sum_{n\geq 4}\omega'_{0,n}.\end{aligned}$$

By (O2), (IX.39), Remark XIX.5 of [FKTo4] and Theorem VII.8 (with $\rho = \lambda_0^{1-\upsilon}$, $\varepsilon = \frac{\aleph}{n_0}$)

$$\begin{aligned}\big|w'_{0,4}(K')\big|_{3,\Sigma_{j+1}} &\leq \big|\delta F'^{(j+1)}(K')\big|_{3,\Sigma_{j+1}} + \sum_{i=2}^{j}\big|F'^{(i)}_{\Sigma_{j+1}}\big|_{3,\Sigma_{j+1}} \\ &\quad+\tfrac{1}{8}\Big|\mathrm{Ant}\Big(V_{\mathrm{ph}}\big(\mathcal{L}^{(j+1)}(\vec{p}\,',\vec{F}')\big)\Big)_{\Sigma_{j+1}}\Big|_{3,\Sigma_{j+1}} \\ &\leq const\,\tfrac{\lambda_0^{1-\upsilon}}{\alpha^4}\Big\{\sum_{i=2}^{j+1}\tfrac{\mathfrak{l}_i^{1/n_0}}{\alpha^3} + \tfrac{1}{M^{1-\aleph}}\tfrac{1}{\mathrm{B}^2}M^{j+1}\|K'\|_{1,\Sigma_{j+1}}\Big\}\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}}) \\ &\quad+\mathrm{const}\,\lambda_0^{2-2\upsilon}\mathfrak{e}_{j+1} \\ &\leq const\,\tfrac{\lambda_0^{1-\upsilon}}{\alpha^4}\Big\{\tfrac{1}{\alpha^3} + \tfrac{1}{M^{1-\aleph}}\tfrac{1}{\mathrm{B}^2}M^{j+1}\|K'\|_{1,\Sigma_{j+1}}\Big\}\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}}).\end{aligned}$$

Consequently, by Proposition XIX.1 of [FKTo4],

$$\big|w'_{0,4}(K')\big|_{1,\Sigma_{j+1}} \leq const\,\tfrac{\lambda_0^{1-\upsilon}}{\alpha^4\,\mathfrak{l}_{j+1}}\Big\{\tfrac{1}{\alpha^3} + \tfrac{1}{M^{1-\aleph}}\tfrac{1}{\mathrm{B}^2}M^{j+1}\|K'\|_{1,\Sigma_{j+1}}\Big\}\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}}).$$

Therefore, by Corollary A.5 of [FKTo1] ,

$$\begin{aligned}&N_{j+1}(\omega'_{0,4}(K'),64\alpha,\ \|K'\|_{1,\Sigma_{j+1}}) \\ &\quad= 2^{24}\,\alpha^4\,\mathrm{B}^2\ \tfrac{\mathfrak{l}_{j+1}}{\lambda_0^{1-\upsilon}}\,\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\Big(\big|w'_{0,4}(K')\big|_{1,\Sigma_{j+1}} + \tfrac{1}{\mathfrak{l}_{j+1}}\big|w'_{0,4}(K')\big|_{3,\Sigma_{j+1}}\Big) \\ &\quad\leq const\Big\{\tfrac{\mathrm{B}^2}{\alpha^3} + \tfrac{1}{M^{1-\aleph}}M^{j+1}\|K'\|_{1,\Sigma_{j+1}}\Big\}\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}}) \\ &\quad\leq \tfrac{1}{2}\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}}).\end{aligned}\tag{IX.40}$$

By (IX.31), (O1) and Lemma IX.8.iii

$$\begin{aligned}N_j(w''(K'),\tfrac{\alpha}{2},0) &\leq \tfrac{3}{2}\,N_j(w(K(K')),\alpha,0) \\ &\leq \tfrac{3}{2}\,N_j(w(K(K')),\alpha,\ \|K(K')\|_{1,\Sigma_j}) \\ &\leq \tfrac{3}{2}\,\mathfrak{e}_j(\|K(K')\|_{1,\Sigma_j}) \\ &\leq const\ \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\end{aligned}$$

so that, by Corollary XIX.8 of [FKTo4] and Corollary A.5 of [FKTo1],

$$\begin{aligned} N_{j+1}(w'(K') - \omega'_{0,4}(K'), 64\alpha, \|K'\|_{1,\Sigma_{j+1}}) &\le \tfrac{1}{M^{(1-\aleph)/8}}\, \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\, N_j\big(w''(K') - \tilde\omega_{0,4}(K'), \tfrac{\alpha}{2}, \|K'\|_{1,\Sigma_{j+1}}\big) \\ &\le \tfrac{1}{M^{(1-\aleph)/8}}\, \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})^2\, N_j\big(w''(K'), \tfrac{\alpha}{2}, 0\big) \\ &\le const\, \tfrac{1}{M^{(1-\aleph)/8}}\, \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})^3 \\ &\le \tfrac{1}{2}\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}}). \end{aligned} \tag{IX.41}$$

Combining (IX.40) and (IX.41), we get

$$N_{j+1}(w'(K'), 64\alpha, \|K'\|_{1,\Sigma_{j+1}}) \le \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}}).$$

By (IX.18) and (IX.35),

$$N_j\Big(\tfrac{d}{ds} w''(K' + sK'')\big|_{s=0}\,,\ \tfrac{\alpha}{2}\,,\ \|K'\|_{1,\Sigma_{j+1}}\Big) \le const\, M^{j+\aleph}\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\|K''\|_{1,\Sigma_{j+1}}.$$

Therefore by Corollary XIX.8 of [FKTo4] and Corollary A.5 of [FKTo1],

$$\begin{aligned} N_{j+1}\Big(\tfrac{d}{ds} w'(K' + sK'')\big|_{s=0}, 64\alpha, \|K'\|_{1,\Sigma_{j+1}}\Big) &\le const\, \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\, N_j\Big(\tfrac{d}{ds} w''(K' + sK'')\big|_{s=0}, \tfrac{\alpha}{2}, \|K'\|_{1,\Sigma_{j+1}}\Big) \\ &\le const\, M^{j+\aleph}\, \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})^2 \|K''\|_{1,\Sigma_{j+1}} \\ &\le M^{j+1}\, \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}}) \|K''\|_{1,\Sigma_{j+1}}. \quad \square \end{aligned}$$

Remark IX.11. Let $(\mathcal{W}, \mathcal{G}, u, \vec{p}) \in \mathcal{D}^{(j)}_{\rm out}$ and $(\mathcal{W}', \mathcal{G}', u', \vec{p}') = \mathcal{O}_j(\mathcal{W}, \mathcal{G}, u, \vec{p}) \in \mathcal{D}^{(j+1)}_{\rm in}$.

(i) The data ${F'}^{(2)}, \cdots, {F'}^{(j)}$ of (I2) coincides with the data $F^{(2)}, \cdots, F^{(j)}$ of (O2). Also, by (IX.19), ${p'}^{(i)} = p^{(i)}$ for all $2 \le i \le j-1$.

(ii) By (IX.36),

$$N\Big(\mathcal{G}'(\phi; K') - \mathcal{G}\big(\phi; {\rm ren}_{j,j+1}(K', \mathcal{W}, u)\big)\Big) \le \tfrac{2}{\sqrt[4]{\mathfrak{l}_j M^j}}.$$

(iii) If the output data are analytic functions of K, then, by Lemma IX.7, the input data are analytic functions of the output data and K.

(iv) If the output data is real, then the input data is real.

X. The Recursive Construction of the Green's Functions

In this section, we construct the data of Theorem VIII.5, recursively in j.

1. Initialization at $j = j_0$. We set

- $\delta e_{j_0}(K)(\mathbf{k}) = \check{K}(\mathbf{k})$ for $K \in \mathfrak{K}_{j_0}$,
- $p^{(2)} = p^{(3)} = \cdots = p^{(j_0-1)} = 0$,
- $F^{(2)} = \cdots = F^{(j_0)} = 0$,

and define $\mathcal{W}_{j_0}$, $\mathcal{G}^{\rm rg}_{j_0}$ and u_{j_0} as follows.

$$\begin{aligned} u_{j_0}(K) &= -\big[K_{\rm ext}\big]_{\Sigma_{j_0}}, \\ \mathcal{W}_{j_0}(K) &= \tilde{\Omega}_{C^{(\le j_0)}_{u_{j_0}(K)}}(\mathcal{V}(\psi))(\phi,\psi) - \tilde{\Omega}_{C^{(\le j_0)}_{u_{j_0}(K)}}(\mathcal{V}(\psi))(\phi,0), \\ \mathcal{G}^{\rm rg}_{j_0}(K) &= \tilde{\Omega}_{C^{(\le j_0)}_{u_{j_0}(K)}}(\mathcal{V}(\psi))(\phi,0). \end{aligned}$$

Clearly, $\big(\mathcal{W}_{j_0}, \mathcal{G}^{\rm rg}_{j_0}, u_{j_0}, (p^{(2)},\cdots,p^{(j_0-1)})\big)$ is an interaction quadruple at scale j_0. Next, we verify that it is in $\mathcal{D}^{(j_0)}_{\rm out}$. Let $\tilde{w}(\phi,\psi;K)$ be the Σ_{j_0}–sectorized representative for

$$\tilde{\Omega}_{C^{(\le j_0)}_{u(K)}}\big(\mathcal{V}(\psi)\big)(\phi,\psi) - \tfrac12 \phi J C^{(\le j_0)}_{u_{j_0}(K)} J\phi = \mathcal{W}_{j_0}(K)(\phi,\psi) + \mathcal{G}^{\rm rg}_{j_0}(K)(\phi) - \tfrac12 \phi J C^{(\le j_0)}_{u_{j_0}(K)} J\phi$$

chosen in Theorem VI.12 and set

$$\begin{aligned} w(\phi,\psi;K) &= \tilde{w}(\phi,\psi;K) - \tilde{w}(\phi,0;K) \\ &= \sum_{m,n} \sum_{s_1,\cdots,s_n\in\Sigma_{j_0}} \int d\eta_1\cdots d\eta_m\, d\xi_1\cdots d\xi_n\ w_{m,n}(\eta_1,\cdots,\eta_m\,(\xi_1,s_1),\cdots,(\xi_n,s_n);K) \\ &\quad \times \phi(\eta_1)\cdots\phi(\eta_m)\ \psi((\xi_1,s_1))\cdots\psi((\xi_n,s_n)). \end{aligned}$$

By Theorem VI.12,

$$N_{j_0}\big(w(K),\alpha,\|K\|_{1,\Sigma_{j_0}}\big) \le N_{j_0}\big(\tilde{w}(K),\alpha,\|K\|_{1,\Sigma_{j_0}}\big) \le {\rm const}\,\alpha^4\lambda_0^{\upsilon}\mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big)$$

and

$$\begin{aligned} N_{j_0}\big(\tfrac{d}{ds}w(K+sK')\big|_{s=0},\alpha,\ \|K\|_{1,\Sigma_{j_0}}\big) &\le N_{j_0}\big(\tfrac{d}{ds}\tilde{w}(K+sK')\big|_{s=0},\alpha,\ \|K\|_{1,\Sigma_{j_0}}\big) \\ &\le M^{j_0}\ \mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big)\ \|K'\|_{1,\Sigma_{j_0}}. \end{aligned}$$

In particular,

$$\begin{aligned} N_{j_0}\big(w(K),\alpha,\|K\|_{1,\Sigma_{j_0}}\big) &\le \mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big) \\ \big|w_{0,2}(K)\big|_{1,\Sigma_{j_0}} &\le \tfrac{1}{M^{j_0}{\rm B}\alpha^2\,\rho_{0;2}} N_{j_0}\big(w(K),\alpha,\|K\|_{1,\Sigma_{j_0}}\big) \\ &\le {\rm const}\ \alpha^2\lambda_0\ \mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big) \\ &\le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^7}\ \tfrac{\mathfrak{l}_{j_0}}{M^{j_0}}\ \mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big) \end{aligned}$$

so that (O1) is satisfied. Similarly,

$$\begin{aligned} \big|w_{0,4}(K)\big|_{3,\Sigma_{j_0}} &\le \tfrac{1}{{\rm B}^2\alpha^4\,\rho_{0;4}(\lambda_0)} N_{j_0}\big(w(K),\alpha,\|K\|_{1,\Sigma_{j_0}}\big) \\ &\le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^9}\mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big). \end{aligned}$$

Setting $\delta F^{(j_0+1)}(K) = w_{0,4}(K)$, (O2) is satisfied. Observe that

$$\mathcal{G}^{\rm rg}_{j_0}(K) - \tfrac12\phi J C^{(\le j_0)} J\phi = \tilde{w}(\phi,0;K) - \tfrac12\phi J\big[C^{(\le j_0)} - C^{(\le j_0)}_{u_{j_0}(K)}\big]J\phi.$$

Now

$$C^{(\le j_0)}(k) - C^{(\le j_0)}_{u_{j_0}(K)}(k) = \tfrac{\check K(\mathbf{k})[U(\mathbf{k})-\nu^{(>j_0)}(k)]}{[\imath k_0 - e(\mathbf{k})][\imath k_0 - e(\mathbf{k}) + \check K(\mathbf{k})]}.$$

On the support of $U(\mathbf{k})-\nu^{(>j_0)}(k)$, $|\imath k_0 - e(\mathbf{k})| \ge \frac{1}{\sqrt{M}\,M^{j_0}}$ and $|\check K(\mathbf{k})| \le const\,\lambda_0^{1-\upsilon}\,\frac{\mathfrak{l}_{j_0+1}}{M^{j_0+1}}$ so that

$$\begin{aligned}
N\big(\phi J\big[C^{(\le j_0)} - C^{(\le j_0)}_{u_{j_0}(K)}\big]J\phi\big) &= \tfrac{1}{\lambda_0^{1-\upsilon}}\big|\big|\big|J\big(C^{(\le j_0)} - C^{(\le j_0)}_{u_{j_0}(K)}\big)J\big|\big|\big|_\infty \\
&\le const\ \tfrac{\mathfrak{l}_{j_0+1}}{M^{j_0+1}}\int \tfrac{d^3k}{(2\pi)^3}\,\tfrac{U(\mathbf{k})-\nu^{(>j_0)}(k)}{|\imath k_0 - e(\mathbf{k})|^2} \\
&\le const\ \tfrac{\mathfrak{l}_{j_0+1}}{M^{j_0+1}}\int_{\substack{|\imath x - y| \ge \frac{1}{\sqrt{M}M^{j_0}} \\ |y|\le const}} dx\,dy\ \tfrac{1}{|\imath x - y|^2} \\
&\le const\ \tfrac{\mathfrak{l}_{j_0+1}}{M^{j_0+1}}\ln\big(\sqrt{M}\,M^{j_0}\big) \\
&\le \tfrac{1}{\sqrt[4]{\mathfrak{l}_{j_0}M^{j_0}}}
\end{aligned}$$

if M is big enough. Therefore, by Remark VI.8.ii and Theorem VI.12,

$$\begin{aligned}
N\big(\mathcal{G}^{\rm rg}_{j_0}(K) - \tfrac12\phi J C^{(\le j_0)}J\phi\big) &\le \tfrac{1}{\sqrt[4]{\mathfrak{l}_{j_0}M^{j_0}}} N_{j_0}\big(\tilde w(K), \alpha, \|K\|_{1,\Sigma_{j_0}}\big)_{\mathbf{0}} \\
&\quad + N\big(\phi J\big[C^{(\le j_0)} - C^{(\le j_0)}_{u_{j_0}(K)}\big]J\phi\big) \\
&\le \tfrac{2}{\sqrt[4]{\mathfrak{l}_{j_0}M^{j_0}}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&N\big(\tfrac{d}{ds}\big[\mathcal{G}^{\rm rg}_{j_0}(K+sK') - \mathcal{G}_{j_0,2}(K+sK')\big]_{s=0}\big) \\
&\quad = N\big(\tfrac{d}{ds}\tilde w(\phi, 0; K+sK')\big|_{s=0}\big) \\
&\quad \le \tfrac{1}{\sqrt[4]{\mathfrak{l}_{j_0}M^{j_0}}} N_{j_0}\big(\tfrac{d}{ds}\tilde w(K+sK')\big|_{s=0}, \alpha, \|K\|_{1,\Sigma_{j_0}}\big)_{\mathbf{0}} \\
&\quad \le M^{j_0}\|K'\|_{1,\Sigma_{j_0}}
\end{aligned}$$

and, since $\frac{d}{ds}C^{(\le j_0)}_{u_{j_0}(K+sK')}(k)\big|_{s=0} = -\frac{\check K'(\mathbf{k})[U(\mathbf{k})-\nu^{(>j_0)}(k)]}{[\imath k_0 - e(\mathbf{k}) + \check K(\mathbf{k})]^2}$,

$$\begin{aligned}
&\big|\big|\big|\tfrac{d}{ds}G_{j_0,2}(K+sK')\big|_{s=0}\big|\big|\big|_\infty \\
&\quad \le \big|\big|\big|\tfrac{d}{ds}\tilde w_{2,0}(K+sK'))\big|_{s=0}\big|\big|\big|_\infty + \tfrac12\big|\big|\big|J\tfrac{d}{ds}C^{(\le j_0)}_{u_{j_0}(K+sK')}\big|_{s=0}J\big|\big|\big|_\infty \\
&\quad \le \lambda_0^{1-\upsilon}N\big(\tfrac{d}{ds}\tilde w(\phi, 0; K+sK')\big|_{s=0}\big) + const\ \sup_{\mathbf{k}}\big|\check K'(\mathbf{k})\big|\int \tfrac{d^3k}{(2\pi)^3}\,\tfrac{U(\mathbf{k})-\nu^{(>j_0)}(k)}{|\imath k_0 - e(\mathbf{k})|^2} \\
&\quad \le \lambda_0^{1-\upsilon}M^{j_0}\|K'\|_{1,\Sigma_{j_0}} + const\ \sup_{\mathbf{k}}\big|\check K'(\mathbf{k})\big|\int_{\substack{|\imath x - y| \ge \frac{1}{\sqrt{M}M^{j_0}} \\ |y|\le const}} dx\,dy\ \tfrac{1}{|\imath x - y|^2} \\
&\quad \le \lambda_0^{1-\upsilon}M^{j_0}\|K'\|_{1,\Sigma_{j_0}} + const\ \|K'\|_{1,\Sigma_{j_0}}\ln\big(\sqrt{M}\,M^{j_0}\big) \\
&\quad \le M^{j_0}\|K'\|_{1,\Sigma_{j_0}}
\end{aligned}$$

if M is big enough, for all $K \in \mathfrak{K}_{j_0}$ and all K'. This completes the verification that (O3) is satisfied.

As pointed out in Remark IX.3, conditions (O1–3) imply conditions (R1–3) of Theorem VIII.5. For $j = j_0$, (R4) is vacuous. Condition (R5) was verified formally as (III.10). The analyticity and reality conditions of Theorem VIII.5 follow from Theorem VI.12.

2. Recursive Step $j \to j+1$. Fix $j \ge j_0$ and assume that

- $\circ$ maps $\delta e_{j'}$, $\mathrm{ren}_{i,j'}$, $j_0 \le i \le j' \le j$,
- $\circ$ $p^{(2)}, \cdots, p^{(j-1)}$,
- $\circ$ $F^{(2)}, \cdots, F^{(j)}$,

and output data $\big(\mathcal{W}_j, \mathcal{G}_j^{\mathrm{rg}}, u_j, (p^{(2)},\cdots,p^{(j-1)})\big)$ at scale j have been constructed and fulfill the conclusions of Theorem VIII.5.

Define $\mathcal{W}_{j+1}, \mathcal{G}_{j+1}^{\mathrm{rg}}, u_{j+1}$ and $p^{(j)}$ by

$$\big(\mathcal{W}_{j+1}, \mathcal{G}_{j+1}^{\mathrm{rg}}, u_{j+1}, (p^{(2)},\cdots,p^{(j)})\big) = \Omega_{j+1} \circ \mathcal{O}_j\big(\mathcal{W}_j, \mathcal{G}_j^{\mathrm{rg}}, u_j, (p^{(2)},\cdots,p^{(j-1)})\big).$$

By Theorems IX.10 and IX.5, the left hand side is an output datum of scale $j+1$ and, by Remark IX.3, satisfy conditions (R1–3). By Remarks IX.11.i and IX.6.i, the $F^{(j+1)}$ of (O2) may be appended to $F^{(2)}, \cdots, F^{(j)}$ so that (R2) is satisfied. The analyticity and reality conditions of Theorem VIII.5, follow from Remarks IX.6.iv,v and IX.11.iii,iv.

Define $\mathrm{ren}_{j,j+1}$ to be the map $\mathrm{ren}_{j,j+1}(\,\cdot\,, \mathcal{W}_j, u_j)$ of (IX.17). By Remarks IX.11.ii and IX.6.ii, (R4) is satisfied. Define, for $j_0 \le i \le j$,

$$\mathrm{ren}_{i,j+1} = \mathrm{ren}_{i,j} \circ \mathrm{ren}_{j,j+1}$$

and

$$\delta e_{j+1}(K)(\mathbf{k}) = \big[\mathrm{ren}_{j_0,j+1}(K)\big]\check{}(\mathbf{k}).$$

Then the algebraic conditions of Definition VIII.3 are fulfilled. The analyticity and reality of $\mathrm{ren}_{j,j+1}$ was observed following (IX.17). That the estimates are also fulfilled is proven in

Lemma X.1. *i) For all $K \in \mathfrak{K}_{j+1}$,*

$$\big\|\mathrm{ren}_{i,j+1}(K)\big\|_{1,\Sigma_i} \le \lambda_0^{1-\upsilon} \tfrac{\mathfrak{l}_i}{M^i} + \sum_{\delta\ne 0} \infty\, t^{\delta},$$
$$|||\delta\hat{e}_{j+1}(K)|||_{1,\infty} \le \lambda_0^{1-\upsilon}.$$

ii) There is a universal constant const such that, for all $j_0 \le i \le j+1$ and $K, K' \in \mathfrak{K}_{j+1}$,

$$\big\|\tfrac{d}{ds}\mathrm{ren}_{i,j+1}(K+sK')\big|_{s=0}\big\|_{1,\Sigma_i} \le \mathit{const}^{\,j+1-i}\ \tfrac{\mathfrak{l}_i}{\mathfrak{l}_{j+1}} \|K'\|_{1,\Sigma_{j+1}} + \sum_{\delta\ne 0} \infty\, t^{\delta}.$$

iii) For all $j_0 \le j' \le j+1$,

$$\big\|\mathrm{ren}_{i,j+1}(0) - \mathrm{ren}_{i,j'}(0)\big\|_{1,\Sigma_i} \le \lambda_0^{1-\upsilon} \tfrac{1}{2^{j'}} + \sum_{\delta\ne 0} \infty\, t^{\delta},$$
$$|||\delta e_{j+1}(0) - \delta e_{j'}(0)|||_{1,\infty} \le \lambda_0^{1-\upsilon} \tfrac{1}{2^{j'}}.$$

Proof. i) We prove, by induction on $j-i$ that

$$\mathrm{ren}_{i,j+1}(K) = \sum_{\ell=i}^{j+1} P_{\ell,j+1}(K)_{\Sigma_i} \tag{X.1}$$

with

$$P_{\ell,j+1}(K) \in \mathfrak{K}_\ell \qquad P_{j+1,j+1}(K) = K$$

and

$$\|P_{\ell,j+1}(K)\|_{1,\Sigma_\ell} \le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^5}\ \tfrac{\mathfrak{l}_\ell}{M^\ell}\,\mathfrak{e}_j\big(\|K\|_{1,\Sigma_{j+1}}\big) \tag{X.2}$$

for $\ell < j+1$. This will then imply

$$\begin{aligned}
&\big\|\mathrm{ren}_{i,j+1}(K)\big\|_{1,\Sigma_i}\\
&\le \sum_{\ell=i}^{j+1}\big\|P_{\ell,j+1}(K)_{\Sigma_i}\big\|_{1,\Sigma_i}\\
&\le const \sum_{\ell=i}^{j+1}\tfrac{\mathfrak{l}_i}{\mathfrak{l}_\ell}\,\mathfrak{c}_{i-1}\,\big\|P_{\ell,j+1}(K)\big\|_{1,\Sigma_\ell}\text{ by Proposition E.10.ii of [FKTo4]}\\
&\le const\ \tfrac{\mathfrak{l}_i}{\mathfrak{l}_{j+1}}\,\mathfrak{c}_{i-1}\,\big\|K\big\|_{1,\Sigma_{j+1}} + const\ \sum_{\ell=i}^{j}\tfrac{\lambda_0^{1-\upsilon}}{\alpha^5}\ \tfrac{\mathfrak{l}_i}{M^\ell}\,\mathfrak{c}_{i-1}\,\mathfrak{e}_j\big(\|K\|_{1,\Sigma_{j+1}}\big)\\
&\le const\ \tfrac{\mathfrak{l}_i}{\mathfrak{l}_{j+1}}\,\mathfrak{c}_{i-1}\,\big\|K\big\|_{1,\Sigma_{j+1}} + const\ \sum_{\ell=i}^{j}\tfrac{\lambda_0^{1-\upsilon}}{\alpha^5}\ \tfrac{\mathfrak{l}_i}{M^\ell}\,\mathfrak{e}_j\big(\|K\|_{1,\Sigma_{j+1}}\big)\\
&\qquad\text{by Example A.3 of [FKTo1]}\\
&\le const\ \tfrac{\mathfrak{l}_i}{\mathfrak{l}_{j+1}}\,\mathfrak{c}_{i-1}\,\big\|K\big\|_{1,\Sigma_{j+1}} + const\ \tfrac{\lambda_0^{1-\upsilon}}{\alpha^5}\ \tfrac{\mathfrak{l}_i}{M^i}\,\mathfrak{e}_j\big(\|K\|_{1,\Sigma_{j+1}}\big)\\
&\le const\ \tfrac{\mathfrak{l}_i}{\mathfrak{l}_{j+1}}\,\mathfrak{c}_{i-1}\,\big\|K\big\|_{1,\Sigma_{j+1}} + \tfrac{\lambda_0^{1-\upsilon}}{\alpha^4}\ \tfrac{\mathfrak{l}_i}{M^i}\,\mathfrak{e}_j\big(\|K\|_{1,\Sigma_{j+1}}\big).
\end{aligned}\tag{X.3}$$

In particular, setting $t=0$,

$$\begin{aligned}
\big\|\mathrm{ren}_{i,j+1}(K)\big\|_{1,\Sigma_i}\Big|_{t=0} &\le const\ \tfrac{\mathfrak{l}_i}{\mathfrak{l}_{j+1}}\,\lambda_0^{1-\upsilon}\tfrac{\mathfrak{l}_{j+2}}{M^{j+2}} + \tfrac{\lambda_0^{1-\upsilon}}{\alpha^4}\ \tfrac{\mathfrak{l}_i}{M^i}\,\frac{1}{1-M^j\lambda_0^{1-\upsilon}\tfrac{\mathfrak{l}_{j+2}}{M^{j+2}}}\\
&\le \lambda_0^{1-\upsilon}\tfrac{\mathfrak{l}_i}{M^{j+2}} + 2\tfrac{\lambda_0^{1-\upsilon}}{\alpha^4}\ \tfrac{\mathfrak{l}_i}{M^i}\\
&\le \lambda_0^{1-\upsilon}\tfrac{\mathfrak{l}_i}{M^i}
\end{aligned}\tag{X.4}$$

by Definitions VI.9 and V.2.iii, if M is big enough. Substituting $i=j_0$ and using

$$\Big|\!\Big|\!\Big|\sum_{s,s'\in\Sigma_i}\varphi\big((\,\cdot\,,s),(\,\cdot\,,s')\big)\Big|\!\Big|\!\Big|_{1,\infty} \le \tfrac{const}{\mathfrak{l}_i}\|\varphi\|_{1,\Sigma_i}$$

which applies to any translation invariant sectorized function on $\big(\mathbb{R}^2\times\Sigma_i\big)^2$, also gives the desired bound on δe_{j+1}.

Now suppose that (X.1) and (X.2) hold for $\mathrm{ren}_{i+1,j+1}(K)$. Then, defining $\delta K^{(i+1)}(K') = \mathrm{ren}_{i,i+1}(K') - K'_{\Sigma_i}$,

$$\begin{aligned}
\mathrm{ren}_{i,j+1}(K) &= \mathrm{ren}_{i,i+1}\Big(\mathrm{ren}_{i+1,j+1}(K)\Big)\\
&= \Big(\mathrm{ren}_{i+1,j+1}(K)\Big)_{\Sigma_i} + \delta K^{(i+1)}\Big(\mathrm{ren}_{i+1,j+1}(K)\Big)\\
&= \sum_{\ell=i+1}^{j+1}P_{\ell,j+1}(K)_{\Sigma_i} + \delta K^{(i+1)}\Big(\mathrm{ren}_{i+1,j+1}(K)\Big)\\
&= \sum_{\ell=i}^{j+1}P_{\ell,j+1}(K)_{\Sigma_i}
\end{aligned}$$

if we choose

$$P_{i,j+1}(K) = \delta K^{(i+1)}\Big(\mathrm{ren}_{i+1,j+1}(K)\Big).$$

By Lemma IX.8.i,

$$\|\delta K^{(i+1)}(K')\|_{1,\Sigma_i} \le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^6}\,\tfrac{\mathfrak{l}_i}{M^i}\,\mathfrak{e}_i\big(\|K'\|_{1,\Sigma_{i+1}}\big).$$

By the inductive hypothesis, (X.3) applies when i is replaced by $i+1$, so

$$\begin{aligned}
&\|P_{i,j+1}(K)\|_{1,\Sigma_i}\\
&\le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^6}\,\tfrac{\mathfrak{l}_i}{M^i}\,\mathfrak{e}_i\big(\|\mathrm{ren}_{i+1,j+1}(K)\|_{1,\Sigma_{i+1}}\big)\\
&\le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^6}\,\tfrac{\mathfrak{l}_i}{M^i}\,\mathfrak{e}_i\Big(const\,\tfrac{\mathfrak{l}_{i+1}}{\mathfrak{l}_{j+1}}\,\mathfrak{c}_i\,\|K\|_{1,\Sigma_{j+1}}+\tfrac{\lambda_0^{1-\upsilon}}{\alpha^4}\,\tfrac{\mathfrak{l}_{i+1}}{M^{i+1}}\,\mathfrak{e}_j(\|K\|_{1,\Sigma_{j+1}})\Big)\\
&\le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^6}\,\tfrac{\mathfrak{l}_i}{M^i}\,\frac{\mathfrak{c}_i}{1-const\,\frac{\mathfrak{l}_{i+1}}{\mathfrak{l}_{j+1}}M^i\,\mathfrak{c}_i\,\|K\|_{1,\Sigma_{j+1}}-\frac{\lambda_0^{1-\upsilon}}{\alpha^4}\,\frac{\mathfrak{l}_{i+1}}{M}\,\mathfrak{e}_j\big(\|K\|_{1,\Sigma_{j+1}}\big)}\\
&\le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^6}\,\tfrac{\mathfrak{l}_i}{M^i}\,\tfrac{\mathfrak{c}_i}{1-const\,M^j\,\mathfrak{c}_j\,\|K\|_{1,\Sigma_{j+1}}-\lambda_0^{1-\upsilon}\,\mathfrak{e}_j(\|K\|_{1,\Sigma_{j+1}})}.
\end{aligned}$$

By Lemma A.4.ii and Corollary A.5.ii of [FKTo1],

$$\begin{aligned}
\|P_{i,j+1}(K)\|_{1,\Sigma_i} &\le const\,\tfrac{\lambda_0^{1-\upsilon}}{\alpha^6}\,\tfrac{\mathfrak{l}_i}{M^i}\,\tfrac{\mathfrak{c}_i}{1-const\,M^j\,\mathfrak{c}_j\,\|K\|_{1,\Sigma_{j+1}}}\,\tfrac{1}{1-\lambda_0^{1-\upsilon}\mathfrak{e}_j(\|K\|_{1,\Sigma_{j+1}})}\\
&\le const\,\tfrac{\lambda_0^{1-\upsilon}}{\alpha^6}\,\tfrac{\mathfrak{l}_i}{M^i}\,\tfrac{\mathfrak{c}_j}{1-const M^j\mathfrak{c}_j\|K\|_{1,\Sigma_{j+1}}}\,\mathfrak{e}_j\big(\|K\|_{1,\Sigma_{j+1}}\big).
\end{aligned}$$

Corollary A.5.i of [FKTo1], with $\mu = const$ and $X = M^j\|K\|_{1,\Sigma_{j+1}}$, yields

$$\tfrac{\mathfrak{c}_j}{1-const M^j\mathfrak{c}_j\|K\|_{1,\Sigma_{j+1}}} \le const\;\mathfrak{e}_j\big(\|K\|_{1,\Sigma_{j+1}}\big)$$

which, by Corollary A.5.ii of [FKTo1], implies

$$\|P_{i,j+1}(K)\|_{1,\Sigma_i} \le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^5}\,\tfrac{\mathfrak{l}_i}{M^i}\,\mathfrak{e}_j\big(\|K\|_{1,\Sigma_{j+1}}\big)$$

as desired.

ii) We use induction on $j-i$. Introduce the local notation

$$|||K|||_{j+1} = \|K\|_{1,\Sigma_{j+1}}\big|_{t=0}$$

As $\mathrm{ren}_{i,j+1}$ is the identity map for $i=j+1$, the case $i=j+1$ is trivial. As

$$\begin{aligned}
\tfrac{d}{ds}\mathrm{ren}_{i,j+1}(K+sK')\big|_{s=0} = \tfrac{d}{ds}\mathrm{ren}_{i,i+1}\Big(&\mathrm{ren}_{i+1,j+1}(K)\\
&+s\tfrac{d}{ds'}\mathrm{ren}_{i+1,j+1}(K+s'K')\big|_{s'=0}\Big)\Big|_{s=0}
\end{aligned}$$

we have, by Lemma IX.8.ii,

$$\begin{aligned}
\Big\|\tfrac{d}{ds}\mathrm{ren}_{i,j+1}(K+sK')\big|_{s=0}\Big\|_{1,\Sigma_i} \le\; & const\;M^{\aleph}\mathfrak{e}_i\big(\|\mathrm{ren}_{i+1,j+1}(K)\|_{1,\Sigma_{i+1}}\big)\\
&\times\Big\|\tfrac{d}{ds'}\mathrm{ren}_{i+1,j+1}(K+s'K')\big|_{s'=0}\Big\|_{1,\Sigma_{i+1}}
\end{aligned}$$

Setting $t = 0$,

$$\begin{aligned}
&\left|\!\left|\!\left| \tfrac{d}{ds}\mathrm{ren}_{i,j+1}(K+sK')\big|_{s=0} \right|\!\right|\!\right|_i \\
&\quad \le const\, M^{\aleph} \frac{1}{1-M^i |||\mathrm{ren}_{i+1,j+1}(K)|||_{i+1}} \left|\!\left|\!\left| \tfrac{d}{ds}\mathrm{ren}_{i+1,j+1}(K+sK')\big|_{s=0} \right|\!\right|\!\right|_{i+1} \\
&\quad \le const\, M^{\aleph} \frac{1}{1-M^i \lambda_0^{1-\upsilon}\frac{\mathfrak{l}_{i+1}}{M^{i+1}}} \left|\!\left|\!\left| \tfrac{d}{ds}\mathrm{ren}_{i+1,j+1}(K+sK')\big|_{s=0} \right|\!\right|\!\right|_{i+1} \qquad \text{by (X.4)} \\
&\quad \le const\, M^{\aleph} \left|\!\left|\!\left| \tfrac{d}{ds}\mathrm{ren}_{i+1,j+1}(K+sK')\big|_{s=0} \right|\!\right|\!\right|_{i+1}.
\end{aligned}$$

By induction

$$\left|\!\left|\!\left| \tfrac{d}{ds}\mathrm{ren}_{i,j+1}(K+sK')\big|_{s=0} \right|\!\right|\!\right|_i \le \left(const\, M^{\aleph}\right)^{j+1-i} |||K'|||_{j+1}$$

as desired.

iii)

$$\mathrm{ren}_{i,j+1}(0) - \mathrm{ren}_{i,j'}(0) = \mathrm{ren}_{i,j'}\big(\mathrm{ren}_{j',j+1}(0)\big) - \mathrm{ren}_{i,j'}(0).$$

Hence, by part (ii),

$$\left\|\mathrm{ren}_{i,j+1}(0) - \mathrm{ren}_{i,j'}(0)\right\|_{1,\Sigma_i} \le const^{j'-j_0}\, \frac{\mathfrak{l}_{j_0}}{\mathfrak{l}_{j'}} \|\mathrm{ren}_{j',j+1}(0)\|_{1,\Sigma_{j'}} + \sum_{\delta\neq 0} \infty\, t^{\delta}.$$

By (X.4)

$$\begin{aligned}
\left\|\mathrm{ren}_{i,j+1}(0) - \mathrm{ren}_{i,j'}(0)\right\|_{1,\Sigma_i} &\le const^{j'-j_0}\, \frac{\mathfrak{l}_{j_0}}{\mathfrak{l}_{j'}} \lambda_0^{1-\upsilon} \frac{\mathfrak{l}_{j'}}{M^{j'}} + \sum_{\delta\neq 0} \infty\, t^{\delta} \\
&\le \lambda_0^{1-\upsilon} \left(\tfrac{const}{M}\right)^{j'} + \sum_{\delta\neq 0} \infty\, t^{\delta} \\
&\le \lambda_0^{1-\upsilon} \frac{1}{2^{j'}} + \sum_{\delta\neq 0} \infty\, t^{\delta},
\end{aligned}$$

and, setting $i = j_0$,

$$\begin{aligned}
|||\delta e_{j+1}(0) - \delta e_{j'}(0)|||_{1,\infty} &\le \frac{const}{\mathfrak{l}_{j_0}} \left\|\mathrm{ren}_{j_0,j+1}(0) - \mathrm{ren}_{j_0,j'}(0)\right\|_{1,\Sigma_{j_0}}\Big|_{t=0} \\
&\le const\; const^{j'-j_0}\, \lambda_0^{1-\upsilon} \frac{1}{M^{j'}} \\
&\le \lambda_0^{1-\upsilon} \frac{1}{2^{j'}}. \quad \square
\end{aligned}$$

This completes the proof of Theorem VIII.5.

Appendix B. Self–Consistent ReWick Ordering

In this appendix, we prove Lemma IX.7 and parts (i) and (ii) of Lemma IX.8. We view any fixed $\varrho \in \mathcal{F}_0(2, \Sigma_j)$ as the constant function $K' \mapsto \varrho$ on $\mathfrak{K}_{j+1}$. In this sense, the definitions of (IX.15) apply. For example,

$$\delta K\big((\mathbf{x}, s), (\mathbf{x}', s'); \varrho\big) = \int dx_0\, \big(\varrho * \hat{\nu}^{(\ge j+1)}\big)\big((x_0, \mathbf{x}, s), (0, \mathbf{x}', s')\big).$$

Lemma B.1. *Assume that $K' \in \mathfrak{K}_{j+1}$ and $\varrho \in \mathcal{F}_0(2, \Sigma_j)$ obeys*

$$|\varrho|_{1,\Sigma_j} \le \frac{\lambda_0^{1-\upsilon}}{\alpha} \frac{\mathfrak{l}_j}{M^j} \mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}}).$$

Then

i) $K(K';\,Q) \in \mathfrak{K}_j$.
ii) There are constants const *, independent of j but possibly depending on M, and* $_{const}$, *independent of M and j, such that*

$$\mathfrak{e}_j\big(\|K(K';Q)\|_{1,\Sigma_j}\big) \le \text{const}\ \mathfrak{e}_j\big(\|K'\|_{1,\Sigma_{j+1}}\big),$$
$$\mathfrak{e}_j\big(\|K(K';Q)\|_{1,\Sigma_j}\big) \le {\scriptstyle const}\ \mathfrak{e}_{j+1}\big(\|K'\|_{1,\Sigma_{j+1}}\big).$$

iii)

- $\sqrt{\lambda_0^{1-\upsilon}\mathfrak{l}_j}\sqrt{\frac{\mathfrak{l}_j \mathrm{B}}{M^j}}$ *is an integral bound for* $E_{\Sigma_j}(K';\,Q)$,
- $\text{const}\sqrt{\frac{\mathfrak{l}_j}{M^j}\Big(M^j|Q'|_{1,\Sigma_j} + \frac{\partial\ \mathfrak{c}_j|Q'|_{1,\Sigma_j}}{\partial\ t_0}\Big)_{t=0}}$ *is an integral bound for* $\frac{d}{ds}E_{\Sigma_j}(K';\,Q+sQ')\big|_{s=0}$. *In particular, if* $\mathfrak{d} \in \mathfrak{N}_{d+1}$ *is independent of* t_0 *and* $|Q'|_{1,\Sigma_j} \le \mathfrak{d}\mathfrak{c}_j$, *then* $\text{const}\sqrt{\mathfrak{l}_j\mathfrak{d}_{\mathbf{0}}}$ *is an integral bound for* $\frac{d}{ds}E_{\Sigma_j}(K';\,Q+sQ')\big|_{s=0}$.
- $\text{const}\sqrt{\mathfrak{l}_j\|K''\|_{1,\Sigma_{j+1}}\big|_{t=0}}$ *is an integral bound for* $\frac{d}{ds}E_{\Sigma_j}(K'+sK'';\,Q)\big|_{s=0}$.

Proof. i) Observe that

$$\delta\check{K}'(\mathbf{k};\,Q) = \check{Q}\big((0,\mathbf{k})\big)\,\nu^{(\ge j+1)}((0,\mathbf{k})).$$

By Proposition E.10.ii of [FKTo4] and Lemma XIII.7 of [FKTo3],

$$\begin{aligned}
\|K'_{\Sigma_j} + \delta K(Q)\|_{1,\Sigma_j} &\le \|K'_{\Sigma_j}\|_{1,\Sigma_j} + \big|Q * \hat{\nu}^{(\ge j+1)}\big|_{1,\Sigma_j}\\
&\le {\scriptstyle const}\,\frac{\mathfrak{l}_j}{\mathfrak{l}_{j+1}}\mathfrak{c}_{j-1}\|K'\|_{1,\Sigma_{j+1}} + {\scriptstyle const}\ \mathfrak{c}_{j+1}\big|Q\big|_{1,\Sigma_j}\\
&\le {\scriptstyle const}\ M^{\aleph}\mathfrak{c}_{j-1}\|K'\|_{1,\Sigma_{j+1}} + {\scriptstyle const}\ \mathfrak{c}_{j+1}\frac{\lambda_0^{1-\upsilon}}{\alpha}\frac{\mathfrak{l}_j}{M^j}\mathfrak{e}_j\\
&\quad\times\big(\|K'\|_{1,\Sigma_{j+1}}\big)\\
&\le {\scriptstyle const}\ M^{\aleph}\lambda_0^{1-\upsilon}\frac{\mathfrak{l}_{j+2}}{M^{j+2}} + {\scriptstyle const}\ \frac{\lambda_0^{1-\upsilon}}{\alpha}\frac{\mathfrak{l}_j}{M^j} + \sum_{\delta\ne 0}\infty\, t^{\delta}\\
&\le \lambda_0^{1-\upsilon}\frac{\mathfrak{l}_{j+1}}{M^{j+1}} + \sum_{\delta\ne 0}\infty\, t^{\delta} \qquad \text{(B.1)}
\end{aligned}$$

if M is large enough and α is large enough, depending on M. By definition, $\operatorname{supp}\check{K}' \subset \operatorname{supp}\nu^{(\ge j+2)}(0,\mathbf{k}) \subset \operatorname{supp}\nu^{(\ge j+1)}(0,\mathbf{k})$, and by construction $\operatorname{supp}\check{\delta}K \subset \operatorname{supp}\nu^{(\ge j+1)}(0,\mathbf{k})$, so $\check{K}(K';\,Q)$ fulfills the required support property.

ii) By (B.1)

$$\begin{aligned}
&\mathfrak{e}_j\big(\|K(K';Q)\|_{1,\Sigma_j}\big)\\
&\quad= \frac{\mathfrak{c}_j}{1 - M^j\|K'_{\Sigma_j} + \delta K(Q)\|_{1,\Sigma_j}}\\
&\quad\le \frac{\mathfrak{c}_j}{1 - M^j\Big[{\scriptstyle const}\ M^{\aleph}\mathfrak{c}_{j-1}\|K'\|_{1,\Sigma_{j+1}} + {\scriptstyle const}\ \mathfrak{c}_{j+1}\frac{\lambda_0^{1-\upsilon}}{\alpha}\frac{\mathfrak{l}_j}{M^j}\mathfrak{e}_j\big(\|K'\|_{1,\Sigma_{j+1}}\big)\Big]}\\
&\quad\le \frac{\mathfrak{c}_j}{1 - M^{j+1}\mathfrak{c}_j\|K'\|_{1,\Sigma_{j+1}} - {\scriptstyle const}\frac{\lambda_0^{1-\upsilon}}{\alpha}\ \mathfrak{c}_{j+1}\ \mathfrak{e}_j\big(\|K'\|_{1,\Sigma_{j+1}}\big)}\\
&\quad\le \frac{\mathfrak{c}_j}{1 - M^{j+1}\mathfrak{c}_j\|K'\|_{1,\Sigma_{j+1}} - \lambda_0^{1-\upsilon}\ \mathfrak{e}_j\big(\|K'\|_{1,\Sigma_{j+1}}\big)}
\end{aligned}$$

if α is large enough, since

$$\mathfrak{c}_{j+1}\mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}}) \le M^{r+r_0}\mathfrak{c}_j\mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}}) \le \text{const}\,\mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}}). \tag{B.2}$$

By Lemma A.4.ii and Corollary A.5.ii of [FKTo1],

$$\begin{aligned}\mathfrak{e}_j(\|K(K';Q)\|_{1,\Sigma_j}) &\le const\ \frac{\mathfrak{c}_j}{1-M^{j+1}\mathfrak{c}_j\|K'\|_{1,\Sigma_{j+1}}}\ \frac{1}{1-\lambda_0^{1-\upsilon}\,\mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})}\\ &\le const\ \frac{\mathfrak{c}_j}{1-M^{j+1}\mathfrak{c}_j\|K'\|_{1,\Sigma_{j+1}}}\ \mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}}).\end{aligned}$$

Corollary A.5.i of [FKTo1], with $\mu = M$ and $X = M^j\|K'\|_{1,\Sigma_{j+1}}$, yields

$$\frac{\mathfrak{c}_j}{1-M^{j+1}\mathfrak{c}_j\|K'\|_{1,\Sigma_{j+1}}} \le \text{const}\,\mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})$$

which, by Corollary A.5.ii of [FKTo1], implies the first bound. On the other hand, Corollary A.5.i of [FKTo1], with $\mu = 1$ and $X = M^{j+1}\|K'\|_{1,\Sigma_{j+1}}$, yields

$$\frac{\mathfrak{c}_j}{1-M^{j+1}\mathfrak{c}_j\|K'\|_{1,\Sigma_{j+1}}} \le \frac{\mathfrak{c}_{j+1}}{1-M^{j+1}\mathfrak{c}_{j+1}\|K'\|_{1,\Sigma_{j+1}}} \le const\ \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})$$

which, by Corollary A.5.ii of [FKTo1], implies the second bound.

iii) Set

$$\begin{aligned}V(K';Q) &= u'(K';Q) + K'_{\rm ext} * \hat\nu^{(\ge j+3)},\\ v(K';Q) &= u(K(K';Q)) + K(K';Q)_{\rm ext} * \hat\nu^{(\ge j+2)}.\end{aligned}$$

Then

$$\begin{aligned}E(K';Q) &= C_{j+1}\big(u'(\,\cdot\,;Q);K'\big) - D_j\big(u;K(K';Q)\big)\\ &= \frac{\nu^{(\ge j+1)}(k)}{\imath k_0 - e(\mathbf{k}) - \check u'(k;K';Q) - \check K'(\mathbf{k})\nu^{(\ge j+3)}(k)}\\ &\quad - \frac{\nu^{(\ge j+1)}(k)}{\imath k_0 - e(\mathbf{k}) - \check u(k;K(K';Q)) - \check K(\mathbf{k};K';Q)\nu^{(\ge j+2)}(k)}\\ &= \frac{\nu^{(\ge j+1)}(k)}{\imath k_0 - e(\mathbf{k}) - \check V(k;K';Q)} - \frac{\nu^{(\ge j+1)}(k)}{\imath k_0 - e(\mathbf{k}) - \check v(k;K';Q)}.\end{aligned} \tag{B.3}$$

Also

$$\begin{aligned}&\check V(k;K';Q) - \check v(k;K';Q)\\ &\quad= \Big[\check u'(K';Q) + \check K'\nu^{(\ge j+3)}\Big] - \Big[\check u(K(K';Q)) + \big(\check K'_{\Sigma_j} + \delta\check K'(Q)\big)\nu^{(\ge j+2)}\Big]\\ &\quad= \Big[\check u(K(K';Q)) + \check Q\,\nu^{(\ge j+1)} + \check K'\nu^{(\ge j+3)}\Big]\\ &\qquad - \Big[\check u(K(K';Q)) + \big(\check K'_{\Sigma_j} + \delta\check K'(Q)\big)\nu^{(\ge j+2)}\Big]\\ &\quad= -\check K'(\mathbf{k})\nu^{(j+2)}(k) + \check Q(k)\,\nu^{(\ge j+1)}(k)\\ &\qquad -\check Q\big((0,\mathbf{k})\big)\,\nu^{(\ge j+1)}((0,\mathbf{k}))\nu^{(\ge j+2)}(k).\end{aligned} \tag{B.4}$$

For the last equality, we used that $\check K'(\mathbf{k}) = \check K'_{\Sigma_j}(\mathbf{k})$, by Definitions E.7 of [FKTo4] and XII.4 of [FKTo3], since $\check K'(\mathbf{k})$ vanishes outside the support of $\nu^{(\ge j+2)}((0,\mathbf{k}))$. By Definition VI.9, Lemma XII.12 of [FKTo3] and Definition VIII.1 of [FKTo2],

$$
\begin{aligned}
\left|\check{K}'(\mathbf{k})\nu^{(j+2)}(k)\right| &\le 2\lambda_0^{1-\upsilon}\tfrac{\mathfrak{l}_{j+2}}{M^{j+2}}\nu^{(j+2)}(k) \le 2\lambda_0^{1-\upsilon}\tfrac{\mathfrak{l}_{j+2}}{M^{j+2}}\tfrac{|\imath k_0-e(\mathbf{k})|}{\frac{1}{\sqrt{M}}\frac{1}{M^{j+2}}}\\
&= 2\sqrt{M}\lambda_0^{1-\upsilon}\mathfrak{l}_{j+2}|\imath k_0-e(\mathbf{k})|\\
&\le \tfrac{1}{10}\lambda_0^{1-\upsilon}\mathfrak{l}_j|\imath k_0-e(\mathbf{k})|. \qquad \text{(B.5)}
\end{aligned}
$$

Similarly, using Lemma XIII.7 of [FKTo3],

$$
\begin{aligned}
&\left|\check{Q}(k)\,\nu^{(\ge j+1)}(k)-\check{Q}\big((0,\mathbf{k})\big)\,\nu^{(\ge j+1)}((0,\mathbf{k}))\right|\nu^{(j+2)}(k)\\
&\quad\le 2|k_0|\,\tfrac{\partial}{\partial t_0}\left|Q*\hat{\nu}^{(\ge j+1)}\right|_{1,\Sigma_j}\Big|_{t=0}\\
&\quad\le const\,|k_0|\,\tfrac{\partial}{\partial t_0}\big(\mathfrak{c}_{j+1}\left|Q\right|_{1,\Sigma_j}\big)\Big|_{t=0} \qquad \text{(B.6)}
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\check{Q}(k)\big(1-\nu^{(\ge j+2)}(k)\big)\,\nu^{(\ge j+1)}(k)\right| &\le 2\left|Q\right|_{1,\Sigma_j}\nu^{(j+1)}(k)\le 2\left|Q\right|_{1,\Sigma_j}\tfrac{|\imath k_0-e(\mathbf{k})|}{\frac{1}{\sqrt{M}}\frac{1}{M^{j+1}}}\\
&= 2M^{j+\frac{3}{2}}\left|Q\right|_{1,\Sigma_j}|\imath k_0-e(\mathbf{k})|. \qquad \text{(B.7)}
\end{aligned}
$$

Combining (B.4)–(B.7)

$$
\begin{aligned}
&|\check{V}(k;K';Q)-\check{v}(k;K';Q)|\\
&\quad\le\Big[\tfrac{1}{10}\lambda_0^{1-\upsilon}\mathfrak{l}_{j+2}+const\tfrac{\partial\ \mathfrak{c}_{j+1}|Q|_{1,\Sigma_j}}{\partial\ t_0}+2M^{j+\frac{3}{2}}\left|Q\right|_{1,\Sigma_j}\Big]_{t=0}|\imath k_0-e(\mathbf{k})|\\
&\quad\le\tfrac{1}{4}\lambda_0^{1-\upsilon}\mathfrak{l}_j\ |\imath k_0-e(\mathbf{k})| \qquad \text{(B.8)}
\end{aligned}
$$

if α is large enough. Lemma VIII.7.i implies

$$
\begin{aligned}
|\check{v}(k;K';Q)| &= \left|\check{u}(k;K(K';Q))+\check{K}(\mathbf{k};K';Q)\nu^{(\ge j+2)}(k)\right|\\
&\le\lambda_0^{1-\upsilon}|\imath k_0-e(\mathbf{k})| \qquad \text{(B.9)}
\end{aligned}
$$

as well as

$$
\begin{aligned}
\left|\tfrac{d}{ds}\check{v}(k;K';Q+sQ')|_{s=0}\right| &= \Big|\tfrac{d}{ds}\check{u}(k;K'_{\Sigma_j}+\delta K(Q)+s\delta K(Q'))|_{s=0}\\
&\qquad+\delta\check{K}'(\mathbf{k};Q')\nu^{(\ge j+2)}(k)\Big|\\
&\le 4M^{j+\frac{3}{2}}\|\delta K(Q')\|_{1,\Sigma_j}|\imath k_0-e(\mathbf{k})|\\
&\le const\ M^{j+\frac{3}{2}}|Q'|_{1,\Sigma_j}\ |\imath k_0-e(\mathbf{k})| \qquad \text{(B.10)}
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\tfrac{d}{ds}\check{v}(k;K'+sK'';Q)|_{s=0}\right| &= \Big|\tfrac{d}{ds}\check{u}(k;K'_{\Sigma_j}+\delta K(Q)+sK''_{\Sigma_j})|_{s=0}\\
&\qquad+\check{K}''(\mathbf{k})\nu^{(\ge j+2)}(k)\Big|\\
&\le 4M^{j+\frac{3}{2}}\|K''_{\Sigma_j}\|_{1,\Sigma_j}|\imath k_0-e(\mathbf{k})|\\
&\le const\tfrac{\mathfrak{l}_j}{\mathfrak{l}_{j+1}}M^{j+\frac{3}{2}}\mathfrak{c}_{j-1}\|K''\|_{1,\Sigma_{j+1}}|\imath k_0-e(\mathbf{k})|\\
&\le const\,M^{j+\frac{3}{2}+\aleph}\mathfrak{c}_{j-1}\|K''\|_{1,\Sigma_{j+1}}|\imath k_0-e(\mathbf{k})|. \qquad \text{(B.11)}
\end{aligned}
$$

From (B.4)

$$
\begin{aligned}
&\tfrac{d}{ds}\big[\check{V}(k;K';Q+sQ')-\check{v}(k;K';Q+sQ')\big]\\
&\quad=\check{Q}'(k)\,\nu^{(\ge j+1)}(k)-\check{Q}'\big((0,\mathbf{k})\big)\nu^{(\ge j+1)}((0,\mathbf{k}))\nu^{(\ge j+2)}(k)
\end{aligned}
$$

so that

$$\begin{aligned}&\Big|\tfrac{d}{ds}\big[\check V(k;\,K';\,\mathcal{Q}+s\mathcal{Q}')-\check v(k;\,K';\,\mathcal{Q}+s\mathcal{Q}')\big]\Big|\\&\qquad\le\Big[const\,\tfrac{\partial\ \mathfrak{c}_{j+1}|\mathcal{Q}'|_{1,\Sigma_j}}{\partial\ t_0}+2M^{j+\frac32}\big|\mathcal{Q}'\big|_{1,\Sigma_j}\Big]_{t=0}|\imath k_0-e(\mathbf{k})|\end{aligned}\tag{B.12}$$

by (B.6) and (B.7). Similarly,

$$\tfrac{d}{ds}\big[\check V(k;\,K'+sK'';\,\mathcal{Q})-\check v(k;\,K'+sK'';\,\mathcal{Q})\big]=-\check K''(\mathbf{k})\nu^{(\ge j+2)}(k)$$

so that

$$\begin{aligned}&\Big|\tfrac{d}{ds}\big[\check V(k;\,K'+sK'';\,\mathcal{Q})-\check v(k;\,K'+sK'';\,\mathcal{Q})\big]\Big|\\&\quad\le 2\big|K''\big|_{1,\Sigma_{j+1}}\nu^{(j+2)}(k)\\&\quad\le 2\big|K''\big|_{1,\Sigma_{j+1}}\tfrac{|\imath k_0-e(\mathbf{k})|}{\frac{1}{\sqrt M}\frac{1}{M^{j+2}}}\\&\quad=2M^{j+\frac52}\big|K''\big|_{1,\Sigma_{j+1}}|\imath k_0-e(\mathbf{k})|\end{aligned}\tag{B.13}$$

as in (B.7).
Using (B.8) and (B.9)

$$\begin{aligned}\big|E(K';\,\mathcal{Q})\big|&=\Big|\tfrac{\nu^{(\ge j+1)}(k)}{\imath k_0-e(\mathbf{k})-\check V(k;K';\mathcal{Q})}-\tfrac{\nu^{(\ge j+1)}(k)}{\imath k_0-e(\mathbf{k})-\check v(k;K';\mathcal{Q})}\Big|\\&=\Big|\tfrac{\check V(k;K';\mathcal{Q})-\check v(k;K';\mathcal{Q})}{[\imath k_0-e(\mathbf{k})-\check V(k;K';\mathcal{Q})]\,[\imath k_0-e(\mathbf{k})-\check v(k;K';\mathcal{Q})]}\Big|\nu^{(\ge j+1)}(k)\\&\le\tfrac{\lambda_0^{1-\upsilon}\mathfrak{l}_j}{|\imath k_0-e(\mathbf{k})|}.\end{aligned}$$

The integral bound for $E_{\Sigma_j}(K';\,\mathcal{Q})$ now follows from Proposition XII.16 of [FKTo3]. Similarly,

$$\begin{aligned}&\big|\tfrac{d}{ds}E_{\Sigma_j}(K';\,\mathcal{Q}+s\mathcal{Q}')\big|_{s=0}\big|\\&\quad=\Big|\tfrac{\frac{d}{ds}\check V(k;K';\mathcal{Q}+s\mathcal{Q}')|_{s=0}}{[\imath k_0-e(\mathbf{k})-\check V(k;K';\mathcal{Q})]^2}-\tfrac{\frac{d}{ds}\check v(k;K';\mathcal{Q}+s\mathcal{Q}')|_{s=0}}{[\imath k_0-e(\mathbf{k})-\check v(k;K';\mathcal{Q})]^2}\Big|\nu^{(\ge j+1)}(k)\\&\quad\le\text{const}\,\frac{M^j|\mathcal{Q}'|_{1,\Sigma_j}+\frac{\partial\ \mathfrak{c}_j|\mathcal{Q}'|_{1,\Sigma_j}}{\partial\ t_0}\Big|_{t=0}}{|\imath k_0-e(\mathbf{k})|}\end{aligned}$$

and the first integral bound for $\frac{d}{ds}E_{\Sigma_j}(K';\,\mathcal{Q}+s\mathcal{Q}')\big|_{s=0}$ also follows from Proposition XII.16 of [FKTo3].
If if $\mathfrak{d}\in\mathfrak{N}_{d+1}$ is independent of t_0 and $|\mathcal{Q}'|_{1,\Sigma_j}\le\mathfrak{d}\mathfrak{c}_j$, then

$$\Big(M^j|\mathcal{Q}'|_{1,\Sigma_j}+\frac{\partial\ \mathfrak{c}_j|\mathcal{Q}'|_{1,\Sigma_j}}{\partial\ t_0}\Big)_{t=0}\le const\,M^j\mathfrak{d}_{\mathbf{0}}$$

and the second integral bound for $\frac{d}{ds}E_{\Sigma_j}(K';\,\mathcal{Q}+s\mathcal{Q}')\big|_{s=0}$ follows from the first.
Finally, using (B.11) and (B.13),

$$\begin{aligned}&\big|\tfrac{d}{ds}E(K'+sK'';\,\mathcal{Q})\big|_{s=0}\big|\\&\quad=\Big|\tfrac{\frac{d}{ds}\check V(k;K'+sK'';\mathcal{Q})|_{s=0}}{[\imath k_0-e(\mathbf{k})-\check V(k;K';\mathcal{Q})]^2}-\tfrac{\frac{d}{ds}\check v(k;K'+sK'';\mathcal{Q})|_{s=0}}{[\imath k_0-e(\mathbf{k})-\check v(k;K';\mathcal{Q})]^2}\Big|\nu^{(\ge j+1)}(k)\\&\quad\le\text{const}\,\frac{M^j\|K''\|_{1,\Sigma_{j+1}}}{|\imath k_0-e(\mathbf{k})|}\end{aligned}$$

and the integral bound for $\frac{d}{ds} E_{\Sigma_j}(K' + sK''; \varrho)\big|_{s=0}$ also follows from Proposition XII.16 of [FKTo3]. $\square$

Recall that $\tilde{w}_{0,2}(K'; \varrho) \in \mathcal{F}_0(2, \Sigma_j)$ was defined, following (IX.15), to be the coefficient of $\psi((\xi_1,s_1))\psi((\xi_2,s_2))$ in $:w(K(K'; \varrho)):_{-E_{\Sigma_j}(K';\varrho)}$.

Lemma B.2. *Assume that $K' \in \mathfrak{K}_{j+1}$, $\mathfrak{d} \in \mathfrak{N}_{d+1}$ is independent of t_0 and*

$$\begin{aligned} |\varrho|_{1,\Sigma_j} &\le \tfrac{\lambda_0^{1-\upsilon}}{\alpha} \tfrac{\mathfrak{l}_j}{M^j} \mathfrak{e}_j\big(\|K'\|_{1,\Sigma_{j+1}}\big), \\ |\varrho'|_{1,\Sigma_j} &\le \mathfrak{d}\mathfrak{c}_j. \end{aligned}$$

Then

$$\begin{aligned} |\tilde{w}_{0,2}(K'; \varrho)|_{1,\Sigma_j} &\le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^{6.5}} \tfrac{\mathfrak{l}_j}{M^j} \mathfrak{e}_j\big(\|K'\|_{1,\Sigma_{j+1}}\big), \\ \Big|\tfrac{d}{ds}\tilde{w}_{0,2}(K'; \varrho + s\varrho')\big|_{s=0}\Big|_{1,\Sigma_j} &\le \tfrac{\lambda_0^{1-\upsilon}}{\alpha} \mathfrak{d}\, \mathfrak{e}_j\big(\|K'\|_{1,\Sigma_{j+1}}\big), \\ \Big|\tfrac{d}{ds}\tilde{w}_{0,2}(K' + sK''; \varrho)\big|_{s=0}\Big|_{1,\Sigma_j} &\le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^{1.5}} \mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})\|K''\|_{1,\Sigma_{j+1}}. \end{aligned}$$

Proof. We use the notation of §XII of [FKTo3]. Set $\tilde{\alpha} = \frac{\alpha}{2\lambda_0^{(1-\upsilon)/2}}$, $\mathrm{b} = \sqrt{\frac{\mathrm{B}\,\mathfrak{l}_j}{M^j}}$, $X = \|K(K'; \varrho)\|_{1,\Sigma_j}$ and $\mathfrak{c} = \mathrm{const}_1\ M^j\ \mathfrak{e}_j\big(X\big)$, where const_1 is the constant of Lemma XV.5 of [FKTo3]. Let, for a sectorized Grassmann function $v = \sum_n v_n$ with $v_n \in \mathbb{C} \otimes \bigwedge^n V_\Sigma$,

$$N(v; \tilde{\alpha}) = \tfrac{1}{\mathrm{b}^2}\mathfrak{c} \sum_n \tilde{\alpha}^n\, \mathrm{b}^n\, |v_n|_{1,\Sigma}.$$

Observe that, if $V = \sum_{m,n} V_{m,n}$ with $V_{m,n} \in A_m \otimes \bigwedge^n V_\Sigma$ and $V_{0,2} = 0$, and if $v = \sum_n V_{0,n}$, then

$$N(v; 2\tilde{\alpha}) \le \tfrac{\mathrm{const}_1}{\mathrm{B}\lambda_0^{1-\upsilon}} N_j\big(V, \alpha, X\big).$$

Set, using the notation of Definition XII.6 of [FKTo3],

$$\begin{aligned} W(K'; \varrho) &= w(K(K'; \varrho))\big|_{\phi=0}, \\ W_2(K'; \varrho) &= Gr\big(w(K(K'; \varrho))_{0,2}\big), \\ W_4(K'; \varrho) &= W(K'; \varrho) - W_2(K'; \varrho). \end{aligned}$$

Then

$$\begin{aligned} \mathfrak{e}_j(X)|\tilde{w}_{0,2}(K'; \varrho)|_{1,\Sigma_j} &= \tfrac{1}{\mathrm{const}_1\,\tilde{\alpha}^2 M^j} N\big(Gr(\tilde{w}_{0,2}(K'; \varrho)); \tilde{\alpha}\big) \\ &\le \mathfrak{e}_j(X)|w(K(K'; \varrho))_{0,2}|_{1,\Sigma_j} + \tfrac{1}{\mathrm{const}_1\,\tilde{\alpha}^2 M^j} N\big(Gr(\tilde{w}_{0,2}(K'; \varrho)) - W_2(K'; \varrho); \tilde{\alpha}\big) \\ &\le \mathfrak{e}_j(X)|w(K(K'; \varrho))_{0,2}|_{1,\Sigma_j} \\ &\quad + \tfrac{1}{\mathrm{const}_1\,\tilde{\alpha}^2 M^j} N\big(:W_4(K'; \varrho):_{-E_{\Sigma_j}(K';\varrho)} - W_4(K'; \varrho); \tilde{\alpha}\big) \\ &\le \mathfrak{e}_j(X)|w(K(K'; \varrho))_{0,2}|_{1,\Sigma_j} + \mathrm{const}\,\tfrac{\lambda_0^{1-\upsilon}\mathfrak{l}_j}{\tilde{\alpha}^4 M^j} N\big(W_4(K'; \varrho); 2\tilde{\alpha}\big) \end{aligned}$$

by Corollary II.32 of [FKTr1] and Lemma B.1.iii. By the observation above

$$\begin{aligned}\tfrac{\lambda_0^{1-\upsilon}\mathfrak{l}_j}{\tilde\alpha^4 M^j} N\big(W_4(K';\varrho);2\tilde\alpha\big) &\le \text{const}\,\tfrac{\mathfrak{l}_j}{\tilde\alpha^4 M^j} N_j\big(w(K(K';\varrho))+\tfrac12\phi C^{(j)}\phi,\alpha,X\big)\\ &\le \text{const}\,\tfrac{\lambda_0^{2(1-\upsilon)}\mathfrak{l}_j}{\alpha^4 M^j} N_j\big(w(K(K';\varrho))+\tfrac12\phi C^{(j)}\phi,\alpha,X\big).\end{aligned}$$

Hence, by (O1),

$$\begin{aligned}|\tilde w_{0,2}(K';\varrho)|_{1,\Sigma_j} &\le \tfrac{\lambda_0^{1-\upsilon}\mathfrak{l}_j}{\alpha^7 M^j}\mathfrak{e}_j(X)^2+\text{const}\,\tfrac{\lambda_0^{2(1-\upsilon)}\mathfrak{l}_j}{\alpha^4 M^j}\mathfrak{e}_j(X)\\ &\le \text{const}\,\tfrac{\lambda_0^{1-\upsilon}\mathfrak{l}_j}{\alpha^7 M^j}\mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})\\ &\le \tfrac{\lambda_0^{1-\upsilon}\mathfrak{l}_j}{\alpha^{6.5} M^j}\mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})\end{aligned}$$

by Corollary A.5.ii of [FKTo1] and Lemma B.1.ii.

We now prove the bound on $\big|\tfrac{d}{ds}\tilde w_{0,2}(K';\varrho+s\varrho')\big|_{s=0}\big|_{1,\Sigma_j}$. This time we use $\alpha'=\frac{\alpha}{2}$ and, for any sectorized Grassmann function $v=\sum_n v_n$ with $v_n\in\mathbb{C}\otimes\bigwedge^n V_\Sigma$,

$$N'(v;\alpha')=\tfrac{1}{\mathrm{b}^2}\mathfrak{c}\sum_n \alpha'^n\,\mathrm{b}^n\,|v_n|_{1,\Sigma}.$$

The other notation is as in the first part of this proof. This time, if $V=\sum_{m,n}V_{m,n}$ with $V_{m,n}\in A_m\otimes\bigwedge^n V_\Sigma$ ($V_{0,2}$ need not vanish), and if $v=\sum_n V_{0,n}$, then

$$N'(v;2\alpha')\le\tfrac{\text{const}_1}{\mathrm{B}}\lambda_0^{1-\upsilon}\,N_j\big(V,\alpha,X\big).$$

Hence

$$\begin{aligned}&\mathfrak{e}_j(X)\big|\tfrac{d}{ds}\tilde w_{0,2}(K';\varrho+s\varrho')\big|_{s=0}\big|_{1,\Sigma_j}\\ &=\tfrac{1}{\text{const}_1\,\alpha'^2M^j}N'\big(\tfrac{d}{ds}Gr(\tilde w_{0,2}(K';\varrho+s\varrho'))\big|_{s=0};\alpha'\big)\\ &\le\tfrac{1}{\text{const}_1\,\alpha'^2M^j}N'\big(\tfrac{d}{ds}:W(K';\varrho+s\varrho'):_{-E_{\Sigma_j}(K';\varrho+s\varrho')}\big|_{s=0};\alpha'\big)\\ &\le\tfrac{1}{\text{const}_1\,\alpha'^2M^j}N'\big(:\tfrac{d}{ds}W(K';\varrho+s\varrho')\big|_{s=0}:_{-E_{\Sigma_j}(K';\varrho)};\alpha'\big)\\ &\quad+\tfrac{1}{\text{const}_1\,\alpha'^2M^j}N'\big(\tfrac{d}{ds}:W(K';\varrho):_{-E_{\Sigma_j}(K';\varrho+s\varrho')}\big|_{s=0};\alpha'\big)\\ &\le\tfrac{1}{\text{const}_1\,\alpha'^2M^j}N'\big(\tfrac{d}{ds}W(K';\varrho+s\varrho')\big|_{s=0};2\alpha'\big)\\ &\quad+\tfrac{1}{\text{const}_1\,\alpha'^2M^j}\tfrac{1}{(\alpha'-1)^2}\,\text{const}\,\tfrac{\mathfrak{d}_0M^j}{\mathrm{B}}N'\big(W(K';\varrho);2\alpha'\big)\\ &\le\text{const}\,\tfrac{\lambda_0^{1-\upsilon}}{\alpha^2M^j}N_j\big(\tfrac{d}{ds}W(K';\varrho+s\varrho')\big|_{s=0},\alpha,X\big)\\ &\quad+\text{const}\,\tfrac{\lambda_0^{1-\upsilon}\mathfrak{d}_0}{\alpha^4}N_j\big(W(K';\varrho),\alpha,X\big).\end{aligned}$$

In the second last inequality, we used Corollary II.32.i,iii of [FKTr1] and Lemma B.1.iii. Since

$$\begin{aligned}\tfrac{d}{ds}W(K';\varrho+s\varrho')\big|_{s=0}&=\tfrac{d}{ds}w\big(K'_{\Sigma_j}+\delta K(\varrho+s\varrho')\big)\Big|_{\substack{s=0\\ \phi=0}}\\ &=\tfrac{d}{ds}w\big(K'_{\Sigma_j}+\delta K(\varrho)+s\delta K(\varrho')\big)\Big|_{\substack{s=0\\ \phi=0}}.\end{aligned}$$

(O1) implies that

$$\begin{aligned}N_j\big(\tfrac{d}{ds}W(K';\varrho+s\varrho')\big|_{s=0},\alpha,X\big)&\le M^j\mathfrak{e}_j(X)\|\delta K(\varrho')\|_{1,\Sigma_j}\\ &\le \mathit{const}\,M^j\mathfrak{c}_{j+1}\mathfrak{e}_j(X)\|\varrho'\|_{1,\Sigma_j}.\end{aligned}$$

(O1) also implies that

$$N_j\big(W(K';\varrho),\alpha,X\big)=N_j\big(w\big(K(K';\varrho)\big)\big|_{\phi=0},\alpha,X\big)\le\mathfrak{e}_j(X).$$

Hence

$$\begin{aligned}&\mathfrak{e}_j(X)\Big|\tfrac{d}{ds}\tilde w_{0,2}(K';\varrho+s\varrho')\big|_{s=0}\Big|_{1,\Sigma_j}\\&\quad\le\text{const}\Big[\tfrac{\lambda_0^{1-\upsilon}}{\alpha^2M^j}M^j\mathfrak{c}_{j+1}\mathfrak{e}_j(X)\,\mathfrak{d}\mathfrak{c}_j+\tfrac{\lambda_0^{1-\upsilon}\mathfrak{d}_0}{\alpha^4}\mathfrak{e}_j(X)\Big]\\&\quad\le\text{const}\,\tfrac{\lambda_0^{1-\upsilon}}{\alpha^2}\mathfrak{d}\mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})\le\tfrac{\lambda_0^{1-\upsilon}}{\alpha}\mathfrak{d}\mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})\end{aligned}$$

by Lemma B.1.ii, (B.2) and Corollary A.5.ii of [FKTo1].

Finally, we prove the bound on $\Big|\frac{d}{ds}\tilde w_{0,2}(K'+sK'';\varrho)\big|_{s=0}\Big|_{1,\Sigma_j}$. We have

$$\begin{aligned}&\mathfrak{e}_j(X)\Big|\tfrac{d}{ds}\tilde w_{0,2}(K'+sK'';\varrho)\big|_{s=0}\Big|_{1,\Sigma_j}\\&=\tfrac{1}{\text{const}_1\,\alpha'^2M^j}N'\big(\tfrac{d}{ds}Gr(\tilde w_{0,2}(K'+sK'';\varrho))\big|_{s=0};\alpha'\big)\\&\le\tfrac{1}{\text{const}_1\,\alpha'^2M^j}N'\big(\tfrac{d}{ds}\,{:}W(K'+sK'';\varrho){:}_{-E_{\Sigma_j}(K'+sK'';\varrho)}\big|_{s=0};\alpha'\big)\\&\le\tfrac{1}{\text{const}_1\,\alpha'^2M^j}N'\big({:}\tfrac{d}{ds}W(K'+sK'';\varrho)\big|_{s=0}{:}_{-E_{\Sigma_j}(K';\varrho)};\alpha'\big)\\&\quad+\tfrac{1}{\text{const}_1\,\alpha'^2M^j}N'\big(\tfrac{d}{ds}\,{:}W(K';\varrho){:}_{-E_{\Sigma_j}(K'+sK'';\varrho)}\big|_{s=0};\alpha'\big)\\&\le\tfrac{1}{\text{const}_1\,\alpha'^2M^j}N'\big(\tfrac{d}{ds}W(K'+sK'';\varrho)\big|_{s=0};2\alpha'\big)\\&\quad+\tfrac{1}{\text{const}_1\,\alpha'^2M^j}\tfrac{1}{(\alpha'-1)^2}\text{ const }\tfrac{M^j\|K''\|_{1,\Sigma_{j+1}}}{\mathrm{B}}N'\big(W(K';\varrho);2\alpha'\big)\\&\le\text{const}\,\tfrac{\lambda_0^{1-\upsilon}}{\alpha^2M^j}N_j\big(\tfrac{d}{ds}W(K'+sK'';\varrho)\big|_{s=0},\alpha,X\big)\\&\quad+\text{const}\,\tfrac{\lambda_0^{1-\upsilon}}{\alpha^4}\|K''\|_{1,\Sigma_{j+1}}N_j\big(W(K';\varrho),\alpha,X\big).\end{aligned}$$

In the second last inequality, we used Corollary II.32.i,iii of [FKTr1] and Lemma B.1.iii. Since

$$\tfrac{d}{ds}W(K'+sK'';\varrho)\big|_{s=0}=\tfrac{d}{ds}w\big(K'_{\Sigma_j}+\delta K(\varrho)+sK''_{\Sigma_j}\big)\Big|_{\substack{s=0\\\phi=0}}.$$

(O1) implies that

$$\begin{aligned}N_j\big(\tfrac{d}{ds}W(K'+sK'';\varrho)\big|_{s=0},\alpha,X\big)&\le M^j\mathfrak{e}_j(X)\|K''_{\Sigma_j}\|_{1,\Sigma_j}\\&\le{\scriptstyle const}\,M^{j+\aleph}\mathfrak{c}_{j-1}\mathfrak{e}_j(X)\|K''\|_{1,\Sigma_{j+1}}\end{aligned}$$

and, as we have already observed,

$$N_j\big(w\big(K(K';\varrho)\big)\big|_{\phi=0},\alpha,X\big)\le\mathfrak{e}_j(X).$$

Hence

$$\begin{aligned}\mathfrak{e}_j(X)\Big|\tfrac{d}{ds}\tilde w_{0,2}(K'+sK'';\varrho)\big|_{s=0}\Big|_{1,\Sigma_j}&\le\text{const}\Big[\tfrac{\lambda_0^{1-\upsilon}}{\alpha^2}\mathfrak{c}_{j-1}+\tfrac{\lambda_0^{1-\upsilon}}{\alpha^4}\Big]\|K''\|_{1,\Sigma_{j+1}}\mathfrak{e}_j(X)\\&\le\text{const}\,\tfrac{\lambda_0^{1-\upsilon}}{\alpha^2}\mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})\|K''\|_{1,\Sigma_{j+1}}\\&\le\tfrac{\lambda_0^{1-\upsilon}}{\alpha^{1.5}}\mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})\|K''\|_{1,\Sigma_{j+1}}\end{aligned}$$

by Lemma B.1.ii. □

We now solve $q(K') = 2\tilde{w}_{0,2}(q(K'); K')$ by a standard contraction mapping argument. Define

$$\begin{aligned} q^{(0)} &= 0, \\ q^{(1)} &= 2\tilde{w}_{0,2}(0; K'), \\ q^{(n+1)} &= 2\tilde{w}_{0,2}(q^{(n)}; K'), \qquad\qquad n \ge 1. \end{aligned}$$

We use the shorthand notation $\mathfrak{e}_j = \mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})$.

Lemma B.3. *Let $K' \in \mathfrak{K}_{j+1}$. Then*

$$\left| q^{(n)} - q^{(n-1)} \right|_{1,\Sigma_j} \le \Big(\kappa \tfrac{\lambda_0^{1-\upsilon}}{\alpha}\Big)^{n-1} \Big(2 \tfrac{\lambda_0^{1-\upsilon}}{\alpha^{6.5}} \tfrac{\mathfrak{l}_j}{M^j}\Big) \mathfrak{e}_j$$

Proof. The proof is by induction on n. By Lemma B.2

$$\left| q^{(1)} \right|_{1,\Sigma_j} \le 2 \tfrac{\lambda_0^{1-\upsilon}}{\alpha^{6.5}} \tfrac{\mathfrak{l}_j}{M^j} \mathfrak{e}_j$$

and the conclusion of the lemma is true for $n = 1$. If the lemma is satisfied for some n, then, by Lemma B.2 with

$$\mathfrak{d} = \Big(\kappa \tfrac{\lambda_0^{1-\upsilon}}{\alpha}\Big)^{n-1} \Big(2 \tfrac{\lambda_0^{1-\upsilon}}{\alpha^{6.5}} \tfrac{\mathfrak{l}_j}{M^j}\Big) \tfrac{1}{1 - M^j \|K'\|_{1,\Sigma_{j+1}}}$$

we have

$$\begin{aligned} \left| q^{(n+1)} - q^{(n)} \right|_{1,\Sigma_j} &= 2 \left| \tilde{w}_{0,2}(q^{(n)}; K') - \tilde{w}_{0,2}(q^{(n-1)}; K') \right|_{1,\Sigma_j} \\ &\le 2 \tfrac{\lambda_0^{1-\upsilon}}{\alpha} \mathfrak{d} \mathfrak{e}_j \\ &\le 2 \tfrac{\lambda_0^{1-\upsilon}}{\alpha} \Big(\kappa \tfrac{\lambda_0^{1-\upsilon}}{\alpha}\Big)^{n-1} \Big(2 \tfrac{\lambda_0^{1-\upsilon}}{\alpha^{6.5}} \tfrac{\mathfrak{l}_j}{M^j}\Big) \mathfrak{e}_j^2 \\ &\le \Big(\kappa \tfrac{\lambda_0^{1-\upsilon}}{\alpha}\Big)^{n} \Big(2 \tfrac{\lambda_0^{1-\upsilon}}{\alpha^{6.5}} \tfrac{\mathfrak{l}_j}{M^j}\Big) \mathfrak{e}_j. \quad \square \end{aligned}$$

Proof of Lemma IX.7. Fix any $K' \in \mathfrak{K}_{j+1}$. By Corollary A.5.ii there is a constant κ such that $\mathfrak{e}_j^2 \le \frac{\kappa}{2} \mathfrak{e}_j$. If α is small enough, Lemma B.3 implies that every

$$\left| q^{(n)} \right|_{1,\Sigma_j} \le \frac{2 \frac{\lambda_0^{1-\upsilon}}{\alpha^{6.5}} \frac{\mathfrak{l}_j}{M^j}}{1 - \kappa \frac{\lambda_0^{1-\upsilon}}{\alpha}} \mathfrak{e}_j \le 4 \tfrac{\lambda_0^{1-\upsilon}}{\alpha^{6.5}} \tfrac{\mathfrak{l}_j}{M^j} \mathfrak{e}_j$$

and that the sequence $\{q^{(n)}\}_{n \ge 1}$ converges to a $q_0(K')$ also obeying

$$\left| q_0(K') \right|_{1,\Sigma_j} \le 4 \tfrac{\lambda_0^{1-\upsilon}}{\alpha^{6.5}} \tfrac{\mathfrak{l}_j}{M^j} \mathfrak{e}_j \le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^{6}} \tfrac{\mathfrak{l}_j}{M^j} \mathfrak{e}_j. \tag{B.14}$$

Fix any K'' and denote $\varrho_0 = q_0(K')$ and $\varrho' = \frac{d}{ds} q_0(K' + sK'')\big|_{s=0}$. Applying $\frac{d}{ds}\ \big|_{s=0}$ to $q_0(K' + sK'') = 2\tilde{w}_{0,2}(q_0(K' + sK''); K' + sK'')$ yields

$$\begin{aligned} \varrho' &= \tfrac{d}{ds} q_0(K' + sK'')\big|_{s=0} \\ &= 2 \tfrac{d}{ds} \tilde{w}_{0,2}(q_0(K' + sK''); K')\big|_{s=0} + 2 \tfrac{d}{ds} \tilde{w}_{0,2}(q_0(K'); K' + sK'')\big|_{s=0} \\ &= 2 \tfrac{d}{ds} \tilde{w}_{0,2}(\varrho_0 + s\varrho'; K')\big|_{s=0} + 2 \tfrac{d}{ds} \tilde{w}_{0,2}(\varrho_0; K' + sK'')\big|_{s=0}. \end{aligned}$$

As, for fixed j, $\tilde{w}_{0,2}(K';\varrho)$ is analytic in ϱ and K' and as $\mathfrak{e}_j\|K''\|_{1,\Sigma_{j+1}}$ has only finitely many finite coefficients, there is some finite β such that

$$|\varrho'|_{1,\Sigma_j} \le \beta\mathfrak{e}_j\|K''\|_{1,\Sigma_{j+1}}.$$

Choose a β that is within $\frac{\lambda_0^{1-\upsilon}}{2\alpha^{1.5}}$ of the infimum of all β's that work. By Lemma B.2, with $\mathfrak{d} = \beta\frac{\|K''\|_{1,\Sigma_{j+1}}}{1-M^j\|K'\|_{1,\Sigma_{j+1}}}$,

$$\begin{aligned} |\varrho'|_{1,\Sigma_j} &\le 2\tfrac{\lambda_0^{1-\upsilon}}{\alpha}\,\beta\tfrac{\|K''\|_{1,\Sigma_{j+1}}}{1-M^j\|K'\|_{1,\Sigma_{j+1}}}\mathfrak{e}_j + 2\tfrac{\lambda_0^{1-\upsilon}}{\alpha^{1.5}}\mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})\|K''\|_{1,\Sigma_{j+1}} \\ &\le \Big[\kappa\tfrac{\lambda_0^{1-\upsilon}}{\alpha}\beta + 2\tfrac{\lambda_0^{1-\upsilon}}{\alpha^{1.5}}\Big]\mathfrak{e}_j\|K''\|_{1,\Sigma_{j+1}} \\ &\le \Big[\tfrac{1}{4}\beta + 2\tfrac{\lambda_0^{1-\upsilon}}{\alpha^{1.5}}\Big]\mathfrak{e}_j\|K''\|_{1,\Sigma_{j+1}} \end{aligned}$$

if α is large enough. Thus $|\varrho'|_{1,\Sigma_j} \le \beta'\mathfrak{e}_j\|K''\|_{1,\Sigma_{j+1}}$ with $\beta' = \frac{1}{4}\beta + 2\frac{\lambda_0^{1-\upsilon}}{\alpha^{1.5}}$. If $\beta \ge 4\frac{\lambda_0^{1-\upsilon}}{\alpha^{1.5}}$, then

$$\beta' - \beta = 2\tfrac{\lambda_0^{1-\upsilon}}{\alpha^{1.5}} - \tfrac{3}{4}\beta \le -\tfrac{\lambda_0^{1-\upsilon}}{\alpha^{1.5}}$$

which violates the requirement that β that is within $\frac{\lambda_0^{1-\upsilon}}{2\alpha^{1.5}}$ of the infimum of all β's that work. Hence

$$|\varrho'|_{1,\Sigma_j} \le 4\tfrac{\lambda_0^{1-\upsilon}}{\alpha^{1.5}}\mathfrak{e}_j\|K''\|_{1,\Sigma_{j+1}} \le \tfrac{\lambda_0^{1-\upsilon}}{\alpha}\mathfrak{e}_j\|K''\|_{1,\Sigma_{j+1}}. \tag{B.15}$$

□

Proof of Lemma IX.8. (i) By (B.14), (B.15) and Lemma XIII.7 of [FKTo3],

$$\big\|\delta K(K')\big\|_{1,\Sigma_j} \le \mathit{const}\ \mathfrak{c}_{j+1}|q_0(K')|_{1,\Sigma_j} \le \text{const}\ \tfrac{\lambda_0^{1-\upsilon}}{\alpha^{6.5}}\tfrac{\mathfrak{l}_j}{M^j}\mathfrak{e}_j \le \tfrac{\lambda_0^{1-\upsilon}}{\alpha^{6}}\tfrac{\mathfrak{l}_j}{M^j}\mathfrak{e}_j$$

and

$$\begin{aligned} \big\|\tfrac{d}{ds}\delta K(K'+sK'')\big|_{s=0}\big\|_{1,\Sigma_j} &\le \mathit{const}\ \mathfrak{c}_{j+1}\big|\tfrac{d}{ds}q_0(K'+sK'')\big|_{s=0}\big|_{1,\Sigma_j} \\ &\le \text{const}\ \tfrac{\lambda_0^{1-\upsilon}}{\alpha^{1.5}}\tfrac{\mathfrak{l}_j}{M^j}\mathfrak{e}_j\|K''\|_{1,\Sigma_{j+1}} \\ &\le \tfrac{\lambda_0^{1-\upsilon}}{\alpha}\tfrac{\mathfrak{l}_j}{M^j}\mathfrak{e}_j\|K''\|_{1,\Sigma_{j+1}}. \end{aligned}$$

(ii) By Proposition E.10.ii of [FKTo4] and part (i),

$$\begin{aligned} \|K(K')\|_{1,\Sigma_j} &\le \mathit{const}\,\tfrac{\mathfrak{l}_j}{\mathfrak{l}_{j+1}}\mathfrak{c}_{j-1}\|K'\|_{1,\Sigma_{j+1}} + \|\delta K(K')\|_{1,\Sigma_j} \\ &\le \mathit{const}\,\tfrac{\mathfrak{l}_j}{\mathfrak{l}_{j+1}}\mathfrak{c}_{j-1}\|K'\|_{1,\Sigma_{j+1}} + \tfrac{\lambda_0^{1-\upsilon}}{\alpha^{6}}\tfrac{\mathfrak{l}_j}{M^j}\mathfrak{e}_j \end{aligned}$$

and

$$\begin{aligned} \big\|\tfrac{d}{ds}K(K'+sK'')\big|_{s=0}\big\|_{1,\Sigma_j} &= \big\|\tfrac{d}{ds}\big[K'_{\Sigma_j} + sK''_{\Sigma_j} + \delta K(K'+sK'')\big]_{s=0}\big\|_{1,\Sigma_j} \\ &\le \|K''_{\Sigma_j}\|_{1,\Sigma_j} + \big\|\tfrac{d}{ds}\delta K(K'+sK'')\big|_{s=0}\big\|_{1,\Sigma_j} \\ &\le \mathit{const}\ M^{\aleph}\mathfrak{c}_{j-1}\|K''\|_{1,\Sigma_{j+1}} \\ &\quad + \tfrac{\lambda_0^{1-\upsilon}}{\alpha}\ \mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})\|K''\|_{1,\Sigma_{j+1}} \\ &\le \mathit{const}\ M^{\aleph}\mathfrak{e}_j(\|K'\|_{1,\Sigma_{j+1}})\|K''\|_{1,\Sigma_{j+1}}. \end{aligned}$$

(iii) is contained in Lemma B.1.ii. □

Notation

Norms

Norm	Characteristics	Reference
$\vert\vert\vert \cdot \vert\vert\vert_{1,\infty}$	no derivatives, external positions, acts on functions	Definition V.3
$\Vert \cdot \Vert_{1,\infty}$	derivatives, external positions, acts on functions	Definition V.3
$\vert\vert\vert \cdot \vert\vert\vert_{\infty}$	no derivatives, external positions, acts on functions	Definition VI.7
$\vert \cdot \vert_{p,\Sigma}$	derivatives, external positions, all but p sectors summed	Definition VI.6
$\vert\vert\vert \cdot \vert\vert\vert_{1,\Sigma}$	no derivatives, all but 1 sector summed	(II.6)
$\vert\vert\vert \cdot \vert\vert\vert_{3,\Sigma}$	no derivatives, all but 3 sectors summed	(II.14)
$\Vert \cdot \Vert_{1,\Sigma}$	like $\vert \cdot \vert_{1,\Sigma}$, but for functions on $\left(\mathbb{R}^2 \times \Sigma\right)^2$	[Def'n E.3, FKTo4]
$\vert\varphi\vert_j$	$\rho_{m;n} \begin{cases} \vert\varphi\vert_{1,\Sigma_j} + \frac{1}{\mathfrak{l}_j} \vert\varphi\vert_{3,\Sigma_j} + \frac{1}{\mathfrak{l}_j^2} \vert\varphi\vert_{5,\Sigma_j} & \text{if } m = 0 \\ \frac{\mathfrak{l}_j}{M^{2j}} \vert\varphi\vert_{1,\Sigma_j} & \text{if } m \neq 0 \end{cases}$	Definition VI.6
$N_j(w, \alpha, X)$	$\frac{M^{2j}}{\mathfrak{l}_j} \mathfrak{e}_j(X) \sum_{m,n\geq 0} \alpha^n \left(\frac{\mathfrak{l}_j \mathrm{B}}{M^j}\right)^{n/2} \vert w_{m,n}\vert_j$	Definition VI.7
$N(\mathcal{G})$	$\sum_{m>0} \frac{1}{\lambda_0^{(1-\upsilon)\max\{m-2,2\}/2}} \vert\vert\vert G_m \vert\vert\vert_\infty$ Definition VI.7	

Spaces

Not'n	Description	Reference
$\mathcal{E}$	counterterm space	Definition I.1
$\mathfrak{K}_j$	space of future counterterms for scale j	Definition VI.9
$\mathcal{B}$	$\mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\} \times \{0, 1\}$ viewed as position space	before Def VII.1
$\check{\mathcal{B}}$	$\mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\} \times \{0, 1\}$ viewed as momentum space	beginning of §VI
$\mathcal{B}^{\updownarrow}$	$\mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\}$ viewed as position space	Definition VII.3
$\mathcal{F}_m(n; \Sigma)$	functions on $\mathcal{B}^m \times \left(\mathcal{B} \times \Sigma\right)^n$, internal momenta in sectors	Definition VI.3.ii
$\mathcal{D}_{\mathrm{in}}^{(j,\mathbf{form})}$	formal input data for scale j	Definition III.8
$\mathcal{D}_{\mathrm{out}}^{(j,\mathbf{form})}$	formal output data for scale j	Definition III.9
$\mathcal{D}_{\mathrm{in}}^{(j)}$	input data for scale j	Definition IX.1
$\mathcal{D}_{\mathrm{out}}^{(j)}$	output data for scale j Definition IX.2	

Other Notation

Not'n	Description	Reference
r_0	number of k_0 derivatives tracked	following (I.3)
r	number of $\mathbf{k}$ derivatives tracked	following (I.3)
M	scale parameter, $M > 1$	before Definition I.2
const	generic constant, independent of scale	
const	generic constant, independent of scale and M	
$\nu^{(j)}(k)$	j^{th} scale function	Definition I.2
$\nu^{(\geq j)}(k)$	$\sum_{i\geq j} \nu^{(j)}(k)$	Definition I.2
n_0	degree of asymmetry	Definition I.10
J	particle/hole swap operator	(III.3)
$\Omega_S(\mathcal{W})(\phi, \psi)$	$\log \frac{1}{Z} \int e^{\mathcal{W}(\phi,\psi+\zeta)}\, d\mu_S(\zeta)$	Definition III.1

Not'n	Description	Reference
$\tilde{\Omega}_C(\mathcal{W})(\phi, \psi)$	$\log \frac{1}{Z} \int e^{\phi J \zeta}\, e^{\mathcal{W}(\phi,\psi+\zeta)} d\mu_C(\zeta)$	Definition III.1
$\aleph$	$\frac{1}{2} < \aleph < \frac{2}{3}$	following Definition VI.3
λ_0	maximum allowed "coupling constant"	Theorem VIII.5
υ	$0 < \upsilon < \frac{1}{4}$, power of λ_0 eaten by bounds	Definition V.6
$\rho_{m;n}(\lambda)$	$\lambda^{-(1-\upsilon)\max\{m+n-2,2\}/2}$	Definition V.6
$\rho_{m;n}$	$\rho_{m;n}(\lambda_0)\big\{1 \text{ if } m = 0\,;\ \sqrt[4]{\mathfrak{l}_j M^j} \text{ if } m > 0$	Definition VI.6.ii
$\mathfrak{l}_j$	$= \frac{1}{M^{\aleph j}}$ = length of sectors of scale j	following Definition VI.3
Σ_j	the sectorization at scale j of length $\mathfrak{l}_j$	following Definition VI.3
B	j–independent constant	Definition VI.7
$\mathfrak{c}_j$	$= \sum_{\substack{\lvert\boldsymbol{\delta}\rvert \le r \\ \lvert\delta_0\rvert \le r_0}} M^{j\lvert\delta\rvert}\, t^\delta + \sum_{\substack{\lvert\boldsymbol{\delta}\rvert > r \\ \text{or } \lvert\delta_0\rvert > r_0}} \infty\, t^\delta \in \mathfrak{N}_{d+1}$	Definition V.2
$\mathfrak{e}_j(X)$	$= \frac{\mathfrak{c}_j}{1 - M^j X}$	Definition V.2
f_{ext}	extends $f(\mathbf{x}, \mathbf{x}')$ to $f_{\text{ext}}\big((x_0, \mathbf{x}, \sigma, a), (x_0', \mathbf{x}', \sigma', a')\big)$	[Definition E.1, FKTo4]
$*$	convolution	Definition VIII.6
$\bullet$	ladder convolution	Definition VII.2,
$\hat{\mu}$	Fourier transform	Notation V.4

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