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A Two Dimensional Fermi Liquid. Part 3: The Fermi Surface

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Abstract: We show that the particle number density derived from the thermodynamic Green's function at temperature zero constructed in the second part of this series has a jump across the Fermi curve, a basic property of a Fermi liquid. We further show that the two particle thermodynamic Green's function at temperature zero has the regularity behavior expected in a Fermi liquid.

Contents

XI. Introduction

This paper, together with [FKTf1] and [FKTf2] provides a construction of a two dimensional Fermi liquid at temperature zero. This paper contains Sects. XI through XV and

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Appendices C and D. Sections I through III and Appendix A are in [FKTf1] and Sects. IV through X and Appendix B are in [FKTf2]. Notation tables are provided at the end of the paper. The main goal of this part is the proof of the existence of a Fermi surface, stated in Theorem I.5. The proof of this theorem follows Lemma XII.4. We assume for the rest of this paper that the interaction V satisfies the reality condition (I.1) and is bar/unbar exchange invariant in the sense of (I.2). The latter is not essential[1]. It is used only for notational convenience at intermediate stages of the proof.

XII. Momentum Green's Functions

Recall that the momentum distribution function $n(\mathbf{k})$ of Theorem I.5 is expressed in terms of the Fourier transform $\check{G}_2(k)$ of the two point Green's function $G_2(x,y)$. In Theorem VIII.5, we defined a generating functional $\mathcal{G}_j^{\rm rg}(K)$. Following the statement of Theorem VIII.5, we constructed $G_2(x,y)$ as the limit of functions $2G_{j,2}^{\rm rg}((x,1),(y,0))$, where $\int d\eta_1 d\eta_2\ G_{j,2}^{\rm rg}(\eta_1,\eta_2)\ \phi(\eta_1)\phi(\eta_2) = 2\int dxdy\ G_{j,2}^{\rm rg}((x,1),(y,0))\ \phi(x,1)\phi(y,0)$ is the part of $\mathcal{G}_j^{\rm rg}(K)\big|_{K=0}$ that is homogeneous of degree two. In §XV, we shall prove the following decomposition of the Fourier transforms, $2\check{G}_{j,2}^{\rm rg}(k)$, of these functions.

Theorem XII.1. *Let $\aleph < \aleph' < \frac{2}{3}$. If the constants $\bar{\alpha},\ \bar{\lambda} > 0$ of Theorem VIII.5 are big, respectively small, enough, the data of Theorem VIII.5 may be chosen such that $2\check{G}_{j,2}^{\rm rg}(k)$ has the decomposition*

$$2\check{G}_{j,2}^{\rm rg}(k) = C_{u_j(0)}^{(\le j)}(k) + \tfrac{1}{[ik_0 - e(\mathbf{k})]^2} \sum_{i=2}^{j} \sum_{\ell=i}^{j} q^{(i,\ell)}(k).$$

Here $u_j(0)$ is the sectorized function $u_j\big((\xi_1,s_1),(\xi_2,s_2);K\big)\big|_{K=0}$ and $q^{(i,\ell)}(k)$, $\ell \ge i \ge 2$ is a family of functions with $q^{(i,\ell)}(k)$ vanishing when k is in the $(i+2)^{\rm nd}$ neighbourhood and and when $\mathbf{k}$ is not in the support of $U(\mathbf{k})$ and obeying, for each multiindex $\delta = (\delta_0,\boldsymbol{\delta})$ with $|\delta| \le 2$,

$$\sup_k \left|{\rm D}^\delta q^{(i,\ell)}(k)\right| \le 2\lambda_0^{1-2\upsilon} \tfrac{\mathfrak{l}_\ell}{M^\ell} M^{\aleph'(\ell-i)} M^{\delta_0 i} M^{|\boldsymbol{\delta}|\ell}. \tag{XII.1}$$

Furthermore $q^{(i,\ell)}(-k_0,\mathbf{k}) = \overline{q^{(i,\ell)}(k_0,\mathbf{k})}$.

For the rest of this section, we deduce consequences of Theorem XII.1. In Lemma XII.2, we describe regularity properties of $\lim\limits_{j\to\infty} \check{G}_{j,2}^{\rm rg}(k)$ and in Lemma XII.3 we show that Fourier transforms commute with the limit $j \to \infty$. From this we derive properties of the proper self–energy and use them to prove that there is a jump in the momentum distribution function $n(\mathbf{k})$, thus showing that Theorem XII.1 implies Theorem I.5.

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[1] See footnote (2) in §XV.

Lemma XII.2. *i) The sequence of functions $\check{u}_j(k;0)$ converges uniformly to a function $P(k)$ that vanishes at $k_0 = 0$ and obeys*

$$\begin{aligned}
\big|P(k)\big| &\le \lambda_0^{1-2\upsilon} \min\{\big|k_0\big|, 1\big\},\\
\big|\nabla P(k)\big| &\le \lambda_0^{1-2\upsilon},\\
\big|\nabla P(k) - \nabla P(k')\big| &\le \lambda_0^{1-2\upsilon}|k-k'|^{\frac{1}{2}},\\
\big|P(k) - \check{u}_j(k;0)\big| &\le \lambda_0^{1-2\upsilon}\mathfrak{l}_j \min\{\big|k_0\big|, 1\big\}.
\end{aligned}$$

ii) The sequence of functions $Q_j(k) = \sum_{i=2}^{j}\sum_{\ell=i}^{j} q^{(i,\ell)}(k)$ converges uniformly to a function $Q(k)$ that vanishes unless $\mathbf{k}$ is in the support of $U(\mathbf{k})$ and obeys

$$\begin{aligned}
\big|Q(k)\big| &\le \lambda_0^{1-3\upsilon} \min\{\big|ik_0 - e(\mathbf{k})\big|^{\frac{3}{2}}, 1\big\},\\
\big|\tfrac{\partial Q}{\partial k_0}(k)\big| &\le \lambda_0^{1-3\upsilon} \min\{\big|ik_0 - e(\mathbf{k})\big|^{\frac{1}{2}}, 1\big\},\\
\big|Q(k) - Q(k')\big|,\ \big|\tfrac{\partial Q}{\partial k_0}(k) - \tfrac{\partial Q}{\partial k_0}(k')\big| &\le \lambda_0^{1-3\upsilon}|k-k'|^{\frac{1}{2}},\\
\big|Q(k) - Q_j(k)\big| &\le \lambda_0^{1-3\upsilon}\mathfrak{l}_j \min\{\big|ik_0 - e(\mathbf{k})\big|, 1\big\}.
\end{aligned}$$

iii) $P(-k_0,\mathbf{k}) = \overline{P(k_0,\mathbf{k})}$ and $Q(-k_0,\mathbf{k}) = \overline{Q(k_0,\mathbf{k})}$.

Proof. i) By Lemma XII.12 of [FKTo3] and (VIII.1),

$$\sup_k \big|\mathrm{D}^\delta \check{p}^{(i)}(k)\big| \le 4\lambda_0^{1-\upsilon}\frac{\mathfrak{l}_i}{M^i} M^{i|\delta|} \tag{XII.2}$$

for all $|\delta| \le 2$. Consequently, the sequence of functions $\check{u}_j(k;0) = \sum_{i=2}^{j-1} \check{p}^{(i)}(k)$ converges uniformly to $P(k) = \sum_{i=2}^{\infty} \check{p}^{(i)}(k)$ and, by Lemma C.1, with $\alpha = \aleph$ and $\beta = 1-\aleph$,

$$\begin{aligned}
|P(k)|,\ \big|\nabla P(k)\big| &\le \mathrm{const}\,\lambda_0^{1-\upsilon} \le \lambda_0^{1-2\upsilon},\\
\big|\nabla P(k) - \nabla P(k')\big| &\le \mathrm{const}\,\lambda_0^{1-\upsilon}|k-k'|^{\aleph} \le \lambda_0^{1-2\upsilon}|k-k'|^{\frac{1}{2}},
\end{aligned}$$

if λ_0 is small enough. Since each $\check{p}^{(i)}(k)$ vanishes at $k=0$, the same is true for $P(k)$ and $|P(k)| \le \lambda_0^{1-2\upsilon}|k_0|$. Furthermore,

$$\begin{aligned}
\Big|P(k) - \sum_{i=2}^{j-1}\check{p}^{(i)}(k)\Big| &\le \sum_{i=j}^{\infty}\big|\check{p}^{(i)}(k)\big|\\
&\le \sum_{i=j}^{\infty} 4\lambda_0^{1-\upsilon}\mathfrak{l}_i \min\{\big|k_0|, 1\big\}\\
&\le \mathrm{const}\,\lambda_0^{1-\upsilon}\mathfrak{l}_j \min\{\big|k_0|, 1\big\}\\
&\le \lambda_0^{1-2\upsilon}\mathfrak{l}_j \min\{\big|k_0|, 1\big\}.
\end{aligned}$$

ii) As $\left|q^{(i,\ell)}(k)\right| \le 2\lambda_0^{1-2\upsilon}\frac{\mathfrak{l}_\ell}{M^\ell}M^{\aleph'(\ell-i)}$ and

$$\sum_{i=2}^{\infty}\sum_{\ell=i}^{\infty}\tfrac{\mathfrak{l}_\ell}{M^\ell}M^{\aleph'(\ell-i)} = \sum_{i=2}^{\infty}M^{-\aleph' i}\tfrac{1}{1-M^{-(1+\aleph-\aleph')}}M^{-(1+\aleph-\aleph')i} < \infty,$$

the sequence of functions $Q_j(k) = \sum\limits_{2\le i\le \ell\le j} q^{(i,\ell)}(k)$ converges uniformly to

$$Q(k) = \sum_{2\le i\le\ell} q^{(i,\ell)}(k).$$

By Lemma C.1, with $j = \ell - i$, $C_0 = \lambda_0^{1-2\upsilon}\frac{\mathfrak{l}_i}{M^i}$, $C_1 = \lambda_0^{1-2\upsilon}\mathfrak{l}_i$, $\alpha = 1-\aleph'+\aleph$ and $\beta = \aleph'-\aleph$,

$$\Big|\sum_{\ell=i}^{\infty}q^{(i,\ell)}(k) - \sum_{\ell=i}^{\infty}q^{(i,\ell)}(k')\Big| \le \text{const}\,\lambda_0^{1-2\upsilon}\mathfrak{l}_i|k-k'|^{1-\aleph'+\aleph},$$

and by Lemma C.1, with $j = \ell-i$, $C_0 = \lambda_0^{1-2\upsilon}\mathfrak{l}_i$, $C_1 = \lambda_0^{1-2\upsilon}\mathfrak{l}_i M^i$, $\alpha = 1-\aleph'+\aleph$ and $\beta = \aleph'-\aleph$,

$$\Big|\sum_{\ell=i}^{\infty}\tfrac{\partial q^{(i,\ell)}}{\partial k_0}(k) - \sum_{\ell=i}^{\infty}\tfrac{\partial q^{(i,\ell)}}{\partial k_0}(k')\Big| \le \text{const}\,\lambda_0^{1-2\upsilon}\mathfrak{l}_i M^{(1-\aleph'+\aleph)i}|k-k'|^{1-\aleph'+\aleph}.$$

Pick any $\frac{1}{2} < \aleph'' < \aleph$ and set $\gamma = \frac{\aleph''}{1-\aleph'+\aleph}$. Note that, since $1-\aleph' > 0$, $0 < \gamma < 1$. Taking the γ^{th} power of this bound and multiplying by the $(1-\gamma)^{\text{th}}$ power of the bound

$$\begin{aligned}\Big|\sum_{\ell=i}^{\infty}\tfrac{\partial q^{(i,\ell)}}{\partial k_0}(k) - \sum_{\ell=i}^{\infty}\tfrac{\partial q^{(i,\ell)}}{\partial k_0}(k')\Big| &\le 2\sup_k\left|\tfrac{\partial q^{(i,\ell)}}{\partial k_0}(k)\right| \le 4\sum_{\ell=i}^{\infty}\lambda_0^{1-2\upsilon}\mathfrak{l}_i\tfrac{1}{M^{(1-\aleph'+\aleph)(\ell-i)}}\\ &\le \text{const}\,\lambda_0^{1-2\upsilon}\mathfrak{l}_i\end{aligned}$$

gives

$$\Big|\sum_{\ell=i}^{\infty}\tfrac{\partial q^{(i,\ell)}}{\partial k_0}(k) - \sum_{\ell=i}^{\infty}\tfrac{\partial q^{(i,\ell)}}{\partial k_0}(k')\Big| \le \text{const}\,\lambda_0^{1-2\upsilon}\mathfrak{l}_i M^{\aleph'' i}|k-k'|^{\aleph''}.$$

Hence

$$\begin{aligned}\big|Q(k)-Q(k')\big| &\le \text{const}\,\lambda_0^{1-2\upsilon}|k-k'|^{1-\aleph'+\aleph} \le \lambda_0^{1-3\upsilon}|k-k'|^{\frac{5}{6}},\\ \Big|\tfrac{\partial Q}{\partial k_0}(k) - \tfrac{\partial Q}{\partial k_0}(k')\Big| &\le \text{const}\,\lambda_0^{1-2\upsilon}|k-k'|^{\aleph''} \le \lambda_0^{1-3\upsilon}|k-k'|^{\frac{1}{2}},\end{aligned} \tag{XII.3}$$

if λ_0 is small enough.
By hypothesis, every $q^{(i,\ell)}(k)$ vanishes on the $(i+2)^{\text{nd}}$ neighbourhood. Hence, for k in the support of $\nu^{(m)}(k)$, $q^{(i,\ell)}(k)$ vanishes when $i+2 \le m$ and

$$\begin{aligned}\big|Q(k)\big| &\le \sum_{i=m-2}^{\infty}\sum_{\ell=i}^{\infty}\big|q^{(i,\ell)}(k)\big| \le 2\lambda_0^{1-2\upsilon}\sum_{i=m-2}^{\infty}\sum_{\ell=i}^{\infty}\tfrac{\mathfrak{l}_\ell}{M^\ell}M^{\aleph'(\ell-i)}\\ &\le \text{const}\,\lambda_0^{1-2\upsilon}\tfrac{\mathfrak{l}_m}{M^m}\end{aligned}$$

$$
\begin{aligned}
&\le \text{const}\,\lambda_0^{1-2\upsilon} \min\left\{\left|ik_0 - e(\mathbf{k})\right|^{1+\aleph}, 1\right\} \\
&\le \lambda_0^{1-3\upsilon} \min\left\{\left|ik_0 - e(\mathbf{k})\right|^{\frac{3}{2}}, 1\right\}, \\
\left|\tfrac{\partial Q}{\partial k_0}(k)\right| &\le \sum_{i=m-2}^{\infty}\sum_{\ell=i}^{\infty}\left|\tfrac{\partial q^{(i,\ell)}}{\partial k_0}(k)\right| \\
&\le 2\lambda_0^{1-2\upsilon}\sum_{i=m-2}^{\infty}\sum_{\ell=i}^{\infty} \mathfrak{l}_i M^{-(1-\aleph'+\aleph)(\ell-i)} \le \text{const}\,\lambda_0^{1-2\upsilon}\mathfrak{l}_m \\
&\le \text{const}\,\lambda_0^{1-2\upsilon} \min\left\{\left|ik_0 - e(\mathbf{k})\right|^{\aleph}, 1\right\} \\
&\le \lambda_0^{1-3\upsilon} \min\left\{\left|ik_0 - e(\mathbf{k})\right|^{\frac{1}{2}}, 1\right\}. \qquad \text{(XII.4)}
\end{aligned}
$$

In general

$$
Q(k) - Q_j(k) = \sum_{i=2}^{j}\sum_{\ell=j+1}^{\infty} q^{(i,\ell)}(k) + \sum_{i=j+1}^{\infty}\sum_{\ell=i}^{\infty} q^{(i,\ell)}(k).
$$

So, for k in the support of $\nu^{(m)}(k)$,

$$
\begin{aligned}
\left|Q(k) - Q_j(k)\right| &\le \sum_{m-2\le i\le j}\ \sum_{\ell=j+1}^{\infty} 2\lambda_0^{1-2\upsilon}\tfrac{\mathfrak{l}_\ell}{M^\ell} M^{\aleph'(\ell-i)} \\
&\quad + \sum_{i=\max\{m-2,j+1\}}^{\infty}\ \sum_{\ell=i}^{\infty} 2\lambda_0^{1-2\upsilon}\tfrac{\mathfrak{l}_\ell}{M^\ell} M^{\aleph'(\ell-i)} \\
&\le \text{const}\,\lambda_0^{1-2\upsilon}\Big\{\sum_{m-2\le i\le j}\tfrac{\mathfrak{l}_j}{M^j}M^{\aleph'(j-i)} + \sum_{i=\max\{m-2,j+1\}}^{\infty}\tfrac{\mathfrak{l}_i}{M^i}\Big\} \\
&\le \text{const}\,\lambda_0^{1-2\upsilon}\Big\{\tfrac{\mathfrak{l}_{\max\{m,j\}}}{M^{\max\{m,j\}}}M^{\aleph'(\max\{m,j\}-m)} + \tfrac{\mathfrak{l}_{\max\{m,j\}}}{M^{\max\{m,j\}}}\Big\} \\
&\le \text{const}\,\lambda_0^{1-2\upsilon}\tfrac{\mathfrak{l}_j}{M^m}\Big\{M^{(\aleph'-1)(\max\{m,j\}-m)} + 1\Big\} \\
&\le \text{const}\,\lambda_0^{1-2\upsilon}\tfrac{\mathfrak{l}_j}{M^m} \\
&\le \lambda_0^{1-3\upsilon}\mathfrak{l}_j \min\left\{\left|ik_0 - e(\mathbf{k})\right|, 1\right\}.
\end{aligned}
$$

iii) That $\check{u}_j(-k_0,\mathbf{k};0) = \overline{\check{u}_j(k_0,\mathbf{k};0)}$ and $q^{(i,\ell)}(-k_0,\mathbf{k}) = \overline{q^{(i,\ell)}(k_0,\mathbf{k})}$ are consequences of Theorems VIII.5 and XII.1, respectively. □

Lemma XII.3.

$$
G_2\big((0,0,\uparrow),(x_0,\mathbf{x},\uparrow)\big) = \int \tfrac{dk_0}{2\pi}\tfrac{d^d\mathbf{k}}{(2\pi)^d}\, e^{-i(-k_0x_0+\mathbf{k}\cdot\mathbf{x})}\left\{\tfrac{U(\mathbf{k})}{ik_0-e(\mathbf{k})-P(k)} + \tfrac{Q(k)}{[ik_0-e(\mathbf{k})]^2}\right\}
$$

(with the value of $G_2\big((0,0,\uparrow),(x_0,\mathbf{x},\uparrow)\big)$ at $x_0 = 0$ defined through the limit $x_0 \to 0+$) and the Fourier transform of $G_2\big((0,0,\uparrow),(x_0,\mathbf{x},\uparrow)\big)$ is $\frac{U(\mathbf{k})}{ik_0-e(\mathbf{k})-P(k)}+\frac{Q(k)}{[ik_0-e(\mathbf{k})]^2}$, which is continuous at all points $(k_0,\mathbf{k})$ for which $ik_0 - e(\mathbf{k}) \ne 0$.

Proof. By the definitions of Q_j and $\check{u}(k)$ (Definition VI.3.iv)

$$
\begin{aligned}
2G^{\mathrm{rg}}_{j,2}\big((0,0,\uparrow,1),(x_0,\mathbf{x},\uparrow,0)\big) &= \int \tfrac{dk_0}{2\pi}\tfrac{d^d\mathbf{k}}{(2\pi)^d}\, e^{-i(-k_0x_0+\mathbf{k}\cdot\mathbf{x})} \\
&\qquad \times\left[\tfrac{\nu^{(\le j)}(k)}{ik_0-e(\mathbf{k})-\check{u}_j(k;0)} + \tfrac{Q_j(k)}{[ik_0-e(\mathbf{k})]^2}\right].
\end{aligned}
$$

Hence

$$\begin{aligned}&\int \tfrac{dk_0}{2\pi}\tfrac{d^d\mathbf{k}}{(2\pi)^d}\, e^{-i(-k_0x_0+\mathbf{k}\cdot\mathbf{x})}\Big\{\tfrac{U(\mathbf{k})}{ik_0-e(\mathbf{k})-P(k)}+\tfrac{Q(k)}{[ik_0-e(\mathbf{k})]^2}\Big\}\\ &\qquad -2G^{\rm rg}_{j,2}\big((0,0,\uparrow,1),(x_0,\mathbf{x},\uparrow,0)\big)\\ &\quad=\int \tfrac{dk_0}{2\pi}\tfrac{d^d\mathbf{k}}{(2\pi)^d}\, e^{-i(-k_0x_0+\mathbf{k}\cdot\mathbf{x})}\\ &\qquad\times\Big[\tfrac{U(\mathbf{k})-\nu^{(\le j)}(k)}{ik_0-e(\mathbf{k})-P(k)}+\tfrac{\nu^{(\le j)}(k)[P(k)-\hat u_j(k;0)]}{[ik_0-e(\mathbf{k})-\hat u_j(k;0)][ik_0-e(\mathbf{k})-P(k)]}+\tfrac{Q(k)-Q_j(k)}{[ik_0-e(\mathbf{k})]^2}\Big].\end{aligned}$$

Let $\tilde U(\mathbf{k})$ be the characteristic function of the support of $U(\mathbf{k})$. By Lemma XII.2, all three of

$$\begin{aligned}\Big|\tfrac{Q(k)-Q_j(k)}{[ik_0-e(\mathbf{k})]^2}\Big| &\le \tfrac{\mathfrak{l}_j\min\{|ik_0-e(\mathbf{k})|,1\}}{|ik_0-e(\mathbf{k})|^2}\tilde U(\mathbf{k})\\ \Big|\tfrac{U(\mathbf{k})-\nu^{(\le j)}(k)}{ik_0-e(\mathbf{k})-P(k)}\Big| &= \Big|\tfrac{\nu^{(>j)}(k)}{ik_0-e(\mathbf{k})-P(k)}\Big|\le 2\tfrac{\nu^{(>j)}(k)}{|ik_0-e(\mathbf{k})|}\\ \Big|\tfrac{\nu^{(\le j)}(k)[P(k)-\hat u_j(k;0)]}{[ik_0-e(\mathbf{k})-\hat u_j(k;0)][ik_0-e(\mathbf{k})-P(k)]}\Big| &\le 4\tfrac{\mathfrak{l}_j\min\{|k_0|,1\}}{|ik_0-e(\mathbf{k})|^2}U(\mathbf{k})\end{aligned}$$

converge to zero in $L^1(\mathbb{R}^{d+1})$ as $j\to\infty$. Consequently, $2G^{\rm rg}_{j,2}\big((0,0,\uparrow,1),(x_0,\mathbf{x},\uparrow,0)\big)$ converges uniformly to $\int \frac{dk_0}{2\pi}\frac{d^d\mathbf{k}}{(2\pi)^d}\, e^{-i(-k_0x_0+\mathbf{k}\cdot\mathbf{x})}\Big\{\frac{U(\mathbf{k})}{ik_0-e(\mathbf{k})-P(k)}+\frac{Q(k)}{[ik_0-e(\mathbf{k})]^2}\Big\}$ as j tends to infinity. In the proof of Theorem I.4, we defined $G_2((0,0,\uparrow),(x_0,\mathbf{x},\uparrow))$ as the pointwise limit of the $2G^{\rm rg}_{j,2}((0,0,\uparrow,1),(x_0,\mathbf{x},\uparrow,0))$'s. Hence

$$G_2\big((0,0,\uparrow),(x_0,\mathbf{x},\uparrow)\big)=\int \tfrac{dk_0}{2\pi}\tfrac{d^d\mathbf{k}}{(2\pi)^d}\, e^{-i(-k_0x_0+\mathbf{k}\cdot\mathbf{x})}\Big\{\tfrac{U(\mathbf{k})}{ik_0-e(\mathbf{k})-P(k)}+\tfrac{Q(k)}{[ik_0-e(\mathbf{k})]^2}\Big\}.$$

Write

$$G_2\big((0,0,\uparrow),(x_0,\mathbf{x},\uparrow)\big)=a(x_0,\mathbf{x})+b(x_0,\mathbf{x})+c(x_0,\mathbf{x})$$

with

$$\begin{aligned}a(x_0,\mathbf{x}) &= \int \tfrac{dk_0}{2\pi}\tfrac{d^d\mathbf{k}}{(2\pi)^d}\, e^{-i(-k_0x_0+\mathbf{k}\cdot\mathbf{x})}\tfrac{U(\mathbf{k})}{i(1-w(\mathbf{k}))k_0-e(\mathbf{k})},\\ b(x_0,\mathbf{x}) &= \int \tfrac{dk_0}{2\pi}\tfrac{d^d\mathbf{k}}{(2\pi)^d}\, e^{-i(-k_0x_0+\mathbf{k}\cdot\mathbf{x})}\tfrac{U(\mathbf{k})[P(k)-iw(\mathbf{k})k_0]}{[i(1-w(\mathbf{k}))k_0-e(\mathbf{k})][ik_0-e(\mathbf{k})-P(k)]},\\ c(x_0,\mathbf{x}) &= \int \tfrac{dk_0}{2\pi}\tfrac{d^d\mathbf{k}}{(2\pi)^d}\, e^{-i(-k_0x_0+\mathbf{k}\cdot\mathbf{x})}\tfrac{Q(\mathbf{k})}{[ik_0-e(\mathbf{k})]^2},\\ w(\mathbf{k}) &= \tfrac{1}{i}\tfrac{\partial P}{\partial k_0}(0,\mathbf{k}).\end{aligned}$$

By repeated use of Lemma XII.2,

$$\begin{aligned}\Big|\tfrac{Q(\mathbf{k})}{[ik_0-e(\mathbf{k})]^2}\Big| &\le \lambda_0^{1-3\upsilon}\tfrac{\min\{|ik_0-e(\mathbf{k})|^{\frac{3}{2}},1\}}{|ik_0-e(\mathbf{k})|^2}\tilde U(\mathbf{k})\in L^2(\mathbb{R}^{d+1}),\\ \Big|\tfrac{U(\mathbf{k})[P(k)-iw(\mathbf{k})k_0]}{[i(1-w(\mathbf{k}))k_0-e(\mathbf{k})][ik_0-e(\mathbf{k})-P(k)]}\Big| &\le 8\lambda_0^{1-2\upsilon}\tfrac{U(\mathbf{k})\min\{|k_0|^{3/2},|k_0|\}}{|ik_0-e(\mathbf{k})|^2}\in L^2(\mathbb{R}^{d+1}),\end{aligned}$$

so the Fourier transform of $b(x_0,\mathbf{x})+c(x_0,\mathbf{x})$ exists and equals

$$\tfrac{U(\mathbf{k})[P(k)-iw(\mathbf{k})k_0]}{[i(1-w(\mathbf{k}))k_0-e(\mathbf{k})][ik_0-e(\mathbf{k})-P(k)]}+\tfrac{Q(\mathbf{k})}{[ik_0-e(\mathbf{k})]^2}$$

Observe that

$$a(x_0,\mathbf{x}) = \tfrac{1}{1-w(\mathbf{k})}\int \tfrac{d^d\mathbf{k}}{(2\pi)^d}\; e^{-i\mathbf{k}\cdot\mathbf{x}}\, U(\mathbf{k})e^{x_0 e(\mathbf{k})/(1-w(\mathbf{k}))}\chi(\mathbf{k},x_0),$$

where

$$\chi(\mathbf{k},x_0) = \begin{cases} 1 & \text{if } e(\mathbf{k}) < 0 \text{ and } x_0 \ge 0 \\ -1 & \text{if } e(\mathbf{k}) > 0 \text{ and } x_0 < 0 \\ 0 & \text{otherwise} \end{cases}.$$

As $\tfrac{1}{1-w(\mathbf{k})}U(\mathbf{k})e^{x_0 e(\mathbf{k})/(1-w(\mathbf{k}))}$ is in $L^2(\mathbb{R}^d)$, the spatial Fourier transform $\int d^d\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}}$ $a(x_0,\mathbf{x})$ exists and equals $\tfrac{1}{1-w(\mathbf{k})}U(\mathbf{k})e^{x_0 e(\mathbf{k})/(1-w(\mathbf{k}))}\chi(\mathbf{k},x_0)$. A direct computation now shows that the temporal Fourier transform $\tfrac{1}{1-w(\mathbf{k})}\int dx_0\, e^{-ik_0x_0}\, U(\mathbf{k})e^{x_0 e(\mathbf{k})/(1-w(\mathbf{k}))}$ $\chi(\mathbf{k},x_0)$ exists and equals $\tfrac{U(\mathbf{k})}{i(1-w(\mathbf{k}))k_0-e(\mathbf{k})}$. Thus the Fourier transform of $G_2\big((0,0,\uparrow),(x_0,\mathbf{x},\uparrow)\big)$ exists and equals $\tfrac{U(\mathbf{k})}{ik_0-e(\mathbf{k})-P(k)} + \tfrac{Q(k)}{[ik_0-e(\mathbf{k})]^2}$, which, by Lemma XII.2, is continuous except when $ik_0 - e(\mathbf{k}) \neq 0$. □

Lemma XII.4. *Let $S(k_0,\mathbf{k})$ be a function that obeys*

- *$S(k_0,\mathbf{k})$ and $\tfrac{\partial S}{\partial k_0}(k_0,\mathbf{k})$ are continuous in $(k_0,\mathbf{k})$ and there are $C,\varepsilon > 0$ such that*

$$\big|\tfrac{\partial S}{\partial k_0}(k_0,\mathbf{k}) - \tfrac{\partial S}{\partial k_0}(0,\mathbf{k})\big| \le C|k_0|^{\varepsilon}.$$

- *$S(0,\mathbf{k})$ and $\tfrac{1}{i}\,\tfrac{\partial S}{\partial k_0}(0,\mathbf{k})$. are real.*
- *$\big|S(k_0,\mathbf{k})\big|$, $\Big|\tfrac{\partial S}{\partial k_0}(k_0,\mathbf{k})\Big| \le \tfrac12$ and $\big|S(0,\mathbf{k})\big| \le \tfrac12|e(\mathbf{k})|$.*

Define

$$N(\mathbf{k},\tau) = \int \tfrac{dk_0}{2\pi}\;\tfrac{e^{ik_0\tau}}{ik_0-e(\mathbf{k})-S(k_0,\mathbf{k})} \qquad N(\mathbf{k}) = \lim_{\tau\to 0+} N(\mathbf{k},\tau).$$

Then $N(\mathbf{k})$ is continuous except on the Fermi surface F. If $\bar{\mathbf{k}} \in F$, then $\lim\limits_{\substack{\mathbf{k}\to\bar{\mathbf{k}}\\ e(\mathbf{k})>0}} N(\mathbf{k})$ and $\lim\limits_{\substack{\mathbf{k}\to\bar{\mathbf{k}}\\ e(\mathbf{k})<0}} N(\mathbf{k})$ exist and obey

$$\lim_{\substack{\mathbf{k}\to\bar{\mathbf{k}}\\ e(\mathbf{k})<0}} N(\mathbf{k}) - \lim_{\substack{\mathbf{k}\to\bar{\mathbf{k}}\\ e(\mathbf{k})>0}} N(\mathbf{k}) = \big[1 - \tfrac{1}{i}\,\tfrac{\partial S}{\partial k_0}(0,\bar{\mathbf{k}})\big]^{-1}.$$

Proof. Define

$$\begin{aligned} A(\mathbf{k}) &= 1 - \tfrac{1}{i}\,\tfrac{\partial S}{\partial k_0}(0,\mathbf{k}),\\ E(\mathbf{k}) &= e(\mathbf{k}) + S(0,\mathbf{k}),\\ R(k_0,\mathbf{k}) &= S(k_0,\mathbf{k}) - S(0,\mathbf{k}) - \tfrac{\partial S}{\partial k_0}(0,\mathbf{k})\,k_0, \end{aligned}$$

and observe that

$$\begin{aligned} ik_0 - e(\mathbf{k}) - S(k_0,\mathbf{k}) &= ik_0 - \tfrac{\partial S}{\partial k_0}(0,\mathbf{k})\,k_0 - e(\mathbf{k}) - S(0,\mathbf{k}) - R(k_0,\mathbf{k})\\ &= iA(\mathbf{k})k_0 - E(\mathbf{k}) - R(k_0,\mathbf{k}). \end{aligned}$$

Hence, for any $\eta > 0$,

$$N(\mathbf{k}, \tau) = I_1(\mathbf{k}, \tau) + I_2(\mathbf{k}, \tau) + I_3(\mathbf{k}, \tau) - I_3'(\mathbf{k}, \tau) + I_4(\mathbf{k}, \tau)$$

with

$$\begin{aligned}
I_1(\mathbf{k}, \tau) &= \int_{|k_0|<\eta} \tfrac{dk_0}{2\pi}\ \tfrac{e^{ik_0\tau}}{iA(\mathbf{k})k_0 - E(\mathbf{k})},\\
I_2(\mathbf{k}, \tau) &= \int_{|k_0|<\eta} \tfrac{dk_0}{2\pi} \Big[\tfrac{e^{ik_0\tau}}{iA(\mathbf{k})k_0 - E(\mathbf{k}) - R(k_0,\mathbf{k})} - \tfrac{e^{ik_0\tau}}{iA(\mathbf{k})k_0 - E(\mathbf{k})}\Big],\\
I_3(\mathbf{k}, \tau) &= \int_{\mathbb{R}} \tfrac{dk_0}{2\pi}\ \tfrac{e^{ik_0\tau}}{ik_0 - e(\mathbf{k})},\\
I_3'(\mathbf{k}, \tau) &= \int_{|k_0|<\eta} \tfrac{dk_0}{2\pi}\ \tfrac{e^{ik_0\tau}}{ik_0 - e(\mathbf{k})},\\
I_4(\mathbf{k}, \tau) &= \int_{|k_0|\geq\eta} \tfrac{dk_0}{2\pi} \Big[\tfrac{e^{ik_0\tau}}{ik_0 - e(\mathbf{k}) - S(k_0,\mathbf{k})} - \tfrac{e^{ik_0\tau}}{ik_0 - e(\mathbf{k})}\Big].
\end{aligned}$$

We shall later fix some small η.

Control of I_1. Let $\epsilon > 0$. On the set $D_\epsilon = \big\{ (\tau, \mathbf{k}) \bigm| \tau \in \mathbb{R},\ |e(\mathbf{k})| > \epsilon \big\}$, the integrand $\frac{e^{ik_0\tau}}{iA(\mathbf{k})k_0 - E(\mathbf{k})}$ is continuous in $(\tau, \mathbf{k})$, for each fixed k_0, and is uniformly bounded by $\frac{2}{\epsilon}$. Hence, by the Lebesgue dominated convergence theorem, $I_1(\tau, \mathbf{k})$ is continuous on D_ϵ and obeys

$$\begin{aligned}
\lim_{\tau\to 0} I_1(\tau, \mathbf{k}) &= \int_{|k_0|<\eta} \tfrac{dk_0}{2\pi}\ \tfrac{1}{iA(\mathbf{k})k_0 - E(\mathbf{k})}\\
&= -\int_{|k_0|<\eta} \tfrac{dk_0}{2\pi}\ \tfrac{iA(\mathbf{k})k_0 + E(\mathbf{k})}{A(\mathbf{k})^2k_0^2 + E(\mathbf{k})^2}\\
&= -\int_{|k_0|<\eta} \tfrac{dk_0}{2\pi}\ \tfrac{E(\mathbf{k})}{A(\mathbf{k})^2k_0^2 + E(\mathbf{k})^2}\\
&= -\tfrac{\operatorname{sgn} E(\mathbf{k})}{A(\mathbf{k})} \int_{|k_0'|<\eta\left|\frac{A(\mathbf{k})}{E(\mathbf{k})}\right|} \tfrac{dk_0'}{2\pi}\ \tfrac{1}{{k_0'}^2+1}\\
&= -\tfrac{\operatorname{sgn} e(\mathbf{k})}{\pi A(\mathbf{k})} \tan^{-1}\big(\eta\big|\tfrac{A(\mathbf{k})}{E(\mathbf{k})}\big|\big).
\end{aligned}$$

Since

$$\lim_{\substack{e(\mathbf{k})\to 0\\ e(\mathbf{k})\neq 0}} \tan^{-1}\big(\eta\big|\tfrac{A(\mathbf{k})}{E(\mathbf{k})}\big|\big) = \tfrac{\pi}{2},$$

we have

$$\lim_{\substack{\mathbf{k}\to\bar{\mathbf{k}}\\ e(\mathbf{k})<0}} \lim_{\tau\to 0} I_1(\tau, \mathbf{k}) - \lim_{\substack{\mathbf{k}\to\bar{\mathbf{k}}\\ e(\mathbf{k})>0}} \lim_{\tau\to 0} I_1(\tau, \mathbf{k}) = 1/A(\bar{\mathbf{k}}). \tag{XII.5}$$

Control of I_2. Since $|iA(\mathbf{k})k_0 - E(\mathbf{k})| \geq \frac{1}{2}|k_0|$ and, for some $\tilde{k}_0$ between 0 and k_0,

$$\big|R(k_0,\mathbf{k})\big| = \big|k_0\big[\tfrac{\partial S}{\partial k_0}(\tilde{k}_0,\mathbf{k}) - \tfrac{\partial S}{\partial k_0}(0,\mathbf{k})\big]\big| \leq C|k_0|^{1+\varepsilon}$$

and

$$\begin{aligned}\big|iA(\mathbf{k})k_0 - E(\mathbf{k}) - R(k_0,\mathbf{k})\big| &\geq \big|A(\mathbf{k})k_0 - \mathrm{Im}\,R(k_0,\mathbf{k})\big| \\ &= \big|A(\mathbf{k})k_0 - \mathrm{Im}\,[R(k_0,\mathbf{k}) - R(0,\mathbf{k})]\big| \\ &= \big|k_0\big[A(\mathbf{k}) - \mathrm{Im}\,\tfrac{\partial R}{\partial k_0}(\tilde{k}_0,\mathbf{k})\big]\big| \\ &= \big|k_0\big[A(\mathbf{k}) - \mathrm{Im}\,\big\{\tfrac{\partial S}{\partial k_0}(\tilde{k}_0,\mathbf{k}) \\ &\quad -\tfrac{\partial S}{\partial k_0}(0,\mathbf{k})\big\}\big]\big| \geq \tfrac{1}{4}|k_0|.\end{aligned}$$

If η is small enough, the integrand

$$\Big[\frac{e^{ik_0\tau}}{iA(\mathbf{k})k_0 - E(\mathbf{k}) - R(k_0,\mathbf{k})} - \frac{e^{ik_0\tau}}{iA(\mathbf{k})k_0 - E(\mathbf{k})}\Big] = \frac{e^{ik_0\tau}R(k_0,\mathbf{k})}{[iA(\mathbf{k})k_0 - E(\mathbf{k})][iA(\mathbf{k})k_0 - E(\mathbf{k}) - R(k_0,\mathbf{k})]}$$

is uniformly bounded in magnitude, on the domain of integration, by the integrable function $\frac{8C}{|k_0|^{1-\varepsilon}}$. Since the integrand is, for each fixed $k_0 \neq 0$, continuous in $(\tau,\mathbf{k})$, the Lebesgue dominated convergence theorem implies that $I_2(\tau,\mathbf{k})$ is continuous $(\tau,\mathbf{k})$ and, in particular,

$$\lim_{\substack{\mathbf{k}\to\tilde{\mathbf{k}}\\ e(\mathbf{k})<0}} \lim_{\tau\to 0} I_2(\tau,\mathbf{k}) - \lim_{\substack{\mathbf{k}\to\tilde{\mathbf{k}}\\ e(\mathbf{k})>0}} \lim_{\tau\to 0} I_2(\tau,\mathbf{k}) = 0. \tag{XII.6}$$

Control of I_3 *and* I_3'. By residues, for all $\tau > 0$ and $e(\mathbf{k}) \neq 0$,

$$I_3(\tau,\mathbf{k}) = \begin{cases} 0 & \text{if } e(\mathbf{k}) > 0 \\ e^{e(\mathbf{k})\tau} & \text{if } e(\mathbf{k}) < 0 \end{cases}.$$

Hence $I_3(\tau,\mathbf{k})$ is continuous in $(\tau,\mathbf{k})$ for all $\mathbf{k}$ with $e(\mathbf{k}) \neq 0$, and obeys

$$\lim_{\substack{\mathbf{k}\to\tilde{\mathbf{k}}\\ e(\mathbf{k})<0}} \lim_{\tau\to 0+} I_3(\tau,\mathbf{k}) - \lim_{\substack{\mathbf{k}\to\tilde{\mathbf{k}}\\ e(\mathbf{k})>0}} \lim_{\tau\to 0+} I_3(\tau,\mathbf{k}) = 1. \tag{XII.7}$$

As in the "Control of I_1", but with $A(\mathbf{k}) = 1$ and $E(\mathbf{k}) = e(\mathbf{k})$, $I_3'(\tau,\mathbf{k})$ is continuous in $(\tau,\mathbf{k})$ for all $\mathbf{k}$ with $e(\mathbf{k}) \neq 0$, and obeys

$$\lim_{\substack{\mathbf{k}\to\tilde{\mathbf{k}}\\ e(\mathbf{k})<0}} \lim_{\tau\to 0} I_3'(\tau,\mathbf{k}) - \lim_{\substack{\mathbf{k}\to\tilde{\mathbf{k}}\\ e(\mathbf{k})>0}} \lim_{\tau\to 0} I_3'(\tau,\mathbf{k}) = 1. \tag{XII.8}$$

Control of I_4. Since $\left|S(k_0,\mathbf{k})\right| \le \frac{1}{2}$, $|ik_0 - e(\mathbf{k})| \ge |k_0|$ and, for some $\tilde{k}_0$ between 0 and k_0,

$$\begin{aligned}\left|ik_0 - e(\mathbf{k}) - S(k_0,\mathbf{k})\right| &\ge \left|k_0 - \mathrm{Im}\, S(k_0,\mathbf{k})\right| = \left|k_0 - \mathrm{Im}\left[S(k_0,\mathbf{k}) - S(0,\mathbf{k})\right]\right| \\ &= \left|k_0\left[1 - \mathrm{Im}\,\tfrac{\partial S}{\partial k_0}(\tilde{k}_0,\mathbf{k})\right]\right| \ge \tfrac{1}{2}|k_0|,\end{aligned}$$

the integrand

$$\left[\frac{e^{ik_0\tau}}{ik_0-e(\mathbf{k})-S(k_0,\mathbf{k})} - \frac{e^{ik_0\tau}}{ik_0-e(\mathbf{k})}\right] = \frac{e^{ik_0\tau}S(k_0,\mathbf{k})}{[ik_0-e(\mathbf{k})][ik_0-e(\mathbf{k})-S(k_0,\mathbf{k})]}$$

is uniformly bounded in magnitude, on the domain of integration, by the integrable function $\frac{1}{k_0^2}$. Since the integrand is, for each fixed k_0, continuous in $(\tau,\mathbf{k})$, the Lebesgue dominated convergence theorem implies that $I_4(\tau,\mathbf{k})$ is continuous $(\tau,\mathbf{k})$ and, in particular,

$$\lim_{\substack{\mathbf{k}\to\bar{\mathbf{k}}\\ e(\mathbf{k})<0}} \lim_{\tau\to 0} I_4(\tau,\mathbf{k}) - \lim_{\substack{\mathbf{k}\to\bar{\mathbf{k}}\\ e(\mathbf{k})>0}} \lim_{\tau\to 0} I_4(\tau,\mathbf{k}) = 0. \tag{XII.9}$$

Combining (XII.5–XII.9) gives the desired jump. □

Proof of Theorem I.5 from Theorem XII.1. Define

$$\begin{aligned} N_1(\mathbf{k},\tau) &= \int \frac{dk_0}{2\pi}\, e^{ik_0\tau} \frac{U(\mathbf{k})}{ik_0-e(\mathbf{k})-P(k)} \qquad N_1(\mathbf{k}) = \lim_{\tau\to 0+} N_1(\mathbf{k},\tau),\\ N_2(\mathbf{k},\tau) &= \int \frac{dk_0}{2\pi}\, e^{ik_0\tau} \frac{Q(k)}{[ik_0-e(\mathbf{k})]^2} \qquad N_2(\mathbf{k}) = \lim_{\tau\to 0+} N_2(\mathbf{k},\tau).\end{aligned}$$

Then, by Lemma XII.3,

$$n(\mathbf{k}) = N_1(\mathbf{k}) + N_2(\mathbf{k}).$$

By Lemma XII.2.ii, the integrand $e^{ik_0\tau}\frac{Q(k)}{[ik_0-e(\mathbf{k})]^2}$ is continuous in τ and $\mathbf{k}$, except possibly when $ik_0 - e(\mathbf{k}) = 0$, and is uniformly bounded by

$$\left|e^{ik_0\tau}\frac{Q(k)}{[ik_0-e(\mathbf{k})]^2}\right| \le \lambda_0^{1-3\upsilon}\frac{\min\{|ik_0-e(\mathbf{k})|^{\frac{3}{2}},1\}}{|ik_0-e(\mathbf{k})|^2} \le \lambda_0^{1-3\upsilon}\min\left\{\frac{1}{k_0^2},\frac{1}{\sqrt{|k_0|}}\right\} \in L^1(\mathbb{R}).$$

Hence, by the Lebesgue dominated convergence theorem, $N_2(\mathbf{k},\tau)$ is continuous. Hence, so is $N_2(\mathbf{k})$. In Lemma XII.2, parts (i) and (iii), we showed that $P(k)$ satisfies the conditions imposed on the function $S(k)$ in Lemma XII.4. So, by Lemma XII.4, with $S(k)$ replaced by $P(k)$, $N_1(\mathbf{k})$ is continuous except on F and if $\bar{\mathbf{k}} \in F$, then $\lim_{\substack{\mathbf{k}\to\bar{\mathbf{k}}\\ e(\mathbf{k})>0}} N_1(\mathbf{k})$ and $\lim_{\substack{\mathbf{k}\to\bar{\mathbf{k}}\\ e(\mathbf{k})<0}} N_1(\mathbf{k})$ exist and obey

$$\lim_{\substack{\mathbf{k}\to\bar{\mathbf{k}}\\ e(\mathbf{k})<0}} N_1(\mathbf{k}) - \lim_{\substack{\mathbf{k}\to\bar{\mathbf{k}}\\ e(\mathbf{k})>0}} N_1(\mathbf{k}) = \left[1 - \tfrac{1}{i}\,\tfrac{\partial P}{\partial k_0}(0,\bar{\mathbf{k}})\right]^{-1}.$$

Since $N_2(\mathbf{k})$ is continuous, $n(\mathbf{k})$ is continuous except on F and if $\bar{\mathbf{k}} \in F$, then $\lim_{\substack{\mathbf{k}\to\bar{\mathbf{k}} \\ e(\mathbf{k})>0}} n(\mathbf{k})$ and $\lim_{\substack{\mathbf{k}\to\bar{\mathbf{k}} \\ e(\mathbf{k})<0}} n(\mathbf{k})$ exist and obey

$$\lim_{\substack{\mathbf{k}\to\bar{\mathbf{k}} \\ e(\mathbf{k})<0}} n(\mathbf{k}) - \lim_{\substack{\mathbf{k}\to\bar{\mathbf{k}} \\ e(\mathbf{k})>0}} n(\mathbf{k}) = \left[1 - \tfrac{1}{i}\tfrac{\partial P}{\partial k_0}(0,\bar{\mathbf{k}})\right]^{-1}.$$

The remaining conclusions of Theorem I.5 were proven in Lemma XII.3. □

If $\mathbf{k}$ is such that $U(\mathbf{k}) = 1$, then

$$\tfrac{U(\mathbf{k})}{ik_0 - e(\mathbf{k}) - P(k)} + \tfrac{Q(k)}{[ik_0 - e(\mathbf{k})]^2} = \tfrac{1}{ik_0 - e(\mathbf{k}) - \Sigma(k)}$$

with

$$\Sigma(k) = \frac{P(k) + Q(k) - Q(k)\frac{P(k)}{ik_0 - e(\mathbf{k})}}{1 + \frac{Q(k)}{ik_0 - e(\mathbf{k})} - \frac{P(k)}{ik_0 - e(\mathbf{k})}\frac{Q(k)}{ik_0 - e(\mathbf{k})}}.$$

$\Sigma(k)$ is usually called the proper self–energy. For the sake of completeness, we summarize the properties of $\Sigma(k)$ proven in this paper and its relationship to the function $P(k)$ used above.

Lemma XII.5. *Let* $\frac{1}{2} < \aleph'' < \aleph < \frac{2}{3}$. *Then*

$$\begin{aligned}
\left|\Sigma(k) - P(k)\right| &\le \lambda_0^{1-4\upsilon} \min\{\left|ik_0 - e(\mathbf{k})\right|^{\frac{3}{2}}, 1\}, \\
\left|\Sigma(k)\right|, \left|\tfrac{\partial\Sigma}{\partial k_0}(k)\right| &\le \lambda_0^{1-4\upsilon}, \\
\left|\Sigma(k) - \Sigma(k')\right|, \ \left|\tfrac{\partial\Sigma}{\partial k_0}(k) - \tfrac{\partial\Sigma}{\partial k_0}(k')\right| &\le \text{const}\, \lambda_0^{1-2\upsilon}|k - k'|^{\aleph''} \le \lambda_0^{1-4\upsilon}|k - k'|^{\frac{1}{2}}.
\end{aligned}$$

Proof. Writing $E(k) = ik_0 - e(\mathbf{k})$ and differentiating

$$\Sigma(k) = \tfrac{E(k)^2 P(k) + E(k)^2 Q(k) - E(k)P(k)Q(k)}{E(k)^2 + E(k)Q(k) - P(k)Q(k)}$$

with respect to k_0 yields

$$\begin{aligned}
\tfrac{\partial\Sigma}{\partial k_0}(k) = &\frac{2iEP + E^2\frac{\partial P}{\partial k_0} + 2iEQ + E^2\frac{\partial Q}{\partial k_0} - iPQ - E\frac{\partial P}{\partial k_0}Q - EP\frac{\partial Q}{\partial k_0}}{E(k)^2 + E(k)Q(k) - P(k)Q(k)} \\
&- \frac{[E^2P + E^2Q - EPQ][2iE + iQ + E\frac{\partial Q}{\partial k_0} - \frac{\partial P}{\partial k_0}Q - P\frac{\partial Q}{\partial k_0}]}{[E(k)^2 + E(k)Q(k) - P(k)Q(k)]^2}.
\end{aligned}$$

Implementing the cancellation (of terms of the form E^nP)

$$\tfrac{2iEP}{E^2 + EQ - PQ} - \tfrac{[E^2P + E^2Q - EQP]2iE}{[E^2 + EQ - PQ]^2} = -\tfrac{2iE^3Q + 2iEP^2Q - 4iE^2PQ}{[E^2 + EQ - PQ]^2}$$

we have

$$\begin{aligned}
\tfrac{\partial\Sigma}{\partial k_0}(k) = &\frac{E^2\frac{\partial P}{\partial k_0} + 2iEQ + E^2\frac{\partial Q}{\partial k_0} - iPQ - E\frac{\partial P}{\partial k_0}Q - EP\frac{\partial Q}{\partial k_0}}{E(k)^2 + E(k)Q(k) - P(k)Q(k)} \\
&- \frac{[E^2P + E^2Q - EPQ][iQ + E\frac{\partial Q}{\partial k_0} - \frac{\partial P}{\partial k_0}Q - P\frac{\partial Q}{\partial k_0}] + 2iE^3Q + 2iEP^2Q - 4iE^2PQ}{[E(k)^2 + E(k)Q(k) - P(k)Q(k)]^2}.
\end{aligned}$$

Observe that, if we weight E, P and Q with degree one and we weight $\frac{\partial P}{\partial k_0}$ and $\frac{\partial Q}{\partial k_0}$ with degree zero, then both the numerator and denominator of the first term are homogeneous of degree two and both the numerator and denominator of the second term are homogeneous of degree four. This sort of homogeneity information can be used in place of the explicit formula for $\frac{\partial \Sigma}{\partial k_0}(k)$.

Define, for $m = 0, 1, 2$,

$$\begin{aligned}
\tilde{Q}^{(m)}(k) &= \tfrac{(ik_0)^m Q(k)}{[ik_0 - e(\mathbf{k})]^{m+1}} &&= \sum_{2\le j\le \ell} \tfrac{(ik_0)^m q^{(j,\ell)}(k)}{[ik_0 - e(\mathbf{k})]^{m+1}} &&= \sum_{2\le j\le \ell} \tilde{q}^{(j,\ell,m)}(k),\\
\tilde{Q}_0^{(m)}(k) &= \tfrac{(ik_0)^m}{[ik_0 - e(\mathbf{k})]^{m}} \tfrac{\partial Q}{\partial k_0} &&= \sum_{2\le j\le \ell} \tfrac{(ik_0)^m}{[ik_0 - e(\mathbf{k})]^{m}} \tfrac{\partial q^{(j,\ell)}}{\partial k_0} &&= \sum_{2\le j\le \ell} \tilde{q}_0^{(j,\ell,m)}(k),\\
\tilde{P}(k) &= \tfrac{P(k)}{ik_0} &&= \sum_{j=2}^{\infty} \tfrac{\check{p}^{(j)}(k)}{ik_0} &&= \sum_{j=2}^{\infty} \tilde{p}^{(j)}(k)
\end{aligned}$$

with $\tilde{q}^{(j,\ell,m)}(k) = \frac{(ik_0)^m q^{(j,\ell)}(k)}{[ik_0 - e(\mathbf{k})]^{m+1}}$ and $\tilde{q}_0^{(j,\ell,m)}(k) = \frac{(ik_0)^m}{[ik_0 - e(\mathbf{k})]^{m}} \frac{\partial q^{(j,\ell)}}{\partial k_0}$ vanishing on the $(j+2)^{\text{nd}}$ neighbourhood and obeying

$$\begin{aligned}
\sup_k \left|\mathrm{D}^\delta \tilde{q}^{(j,\ell,m)}(k)\right|, \sup_k \left|\mathrm{D}^\delta \tilde{q}_0^{(j,\ell,m)}(k)\right| &\le \text{const}\, \lambda_0^{1-2\upsilon} \tfrac{\mathfrak{l}_\ell}{M^\ell} M^{\aleph'(\ell-j)} M^j M^{j\delta_0} M^{\ell|\boldsymbol{\delta}|}\\
&\le \text{const}\, \lambda_0^{1-2\upsilon} \mathfrak{l}_j M^{-(1+\aleph-\aleph')(\ell-j)} M^{\ell|\delta|}
\end{aligned} \tag{XII.10}$$

for $|\delta| \le 1$ and $\tilde{p}^{(j)}(k) = \frac{\check{p}^{(j)}(k)}{ik_0} = -i \int_0^1 \frac{\partial \check{p}^{(j)}}{\partial k_0}(tk_0, \mathbf{k})\, dt$ obeying

$$\sup_k \left|\mathrm{D}^\delta \tilde{p}^{(j)}(k)\right| \le \text{const}\, \lambda_0^{1-\upsilon} \mathfrak{l}_j M^{j|\delta|}$$

for all $|\delta| \le 1$. The right-hand side of (XII.10) coincides with the bounds on $\frac{\partial q^{(i,\ell)}}{\partial k_0}$ that were used in the derivation of the second bounds of (XII.3) and (XII.4). Hence, for $m = 0, 1, 2$,

$$|\tilde{Q}^{(m)}(k)|, |\tilde{Q}_0^{(m)}(k)| \le \text{const}\, \lambda_0^{1-2\upsilon} \min\{|ik_0 - e(\mathbf{k})|^{\aleph}, 1\} \qquad |\tilde{P}(k)| \le \lambda_0^{1-3\upsilon} \tag{XII.11}$$

and

$$\left|\tilde{Q}^{(m)}(k) - \tilde{Q}^{(m)}(k')\right|, \left|\tilde{Q}_0^{(m)}(k) - \tilde{Q}_0^{(m)}(k')\right|, \left|\tilde{P}(k) - \tilde{P}(k')\right| \le \text{const}\, \lambda_0^{1-2\upsilon} |k - k'|^{\aleph''}. \tag{XII.12}$$

The bound on $\tilde{P}$ is a direct application of Lemma C.1 with $\alpha = \aleph$ and $\beta = 1 - \aleph$. In terms of these new functions,

$$\begin{aligned}
\Sigma(k) &= \frac{P(k) + Q(k) - P(k)\tilde{Q}^{(0)}(k)}{1 + \tilde{Q}^{(0)}(k) - \tilde{P}(k)\tilde{Q}^{(1)}(k)} \\
\Sigma(k) - P(k) &= \frac{Q(k) - 2P(k)\tilde{Q}^{(0)}(k) + P(k)\tilde{P}(k)\tilde{Q}^{(1)}(k)}{1 + \tilde{Q}^{(0)}(k) - \tilde{P}(k)\tilde{Q}^{(1)}(k)} \\
\tfrac{\partial\Sigma}{\partial k_0}(k) &= \frac{\frac{\partial P}{\partial k_0} + 2i\tilde{Q}^{(0)} + \tilde{Q}_0^{(0)} - i\tilde{P}\tilde{Q}^{(1)} - \frac{\partial P}{\partial k_0}\tilde{Q}^{(0)} - \tilde{P}\tilde{Q}_0^{(1)}}{1 + \tilde{Q}^{(0)} - \tilde{P}\tilde{Q}^{(1)}} \\
&\quad - \frac{\tilde{P}[i\tilde{Q}^{(1)} + \tilde{Q}_0^{(1)} - \frac{\partial P}{\partial k_0}\tilde{Q}^{(1)} - \tilde{P}\tilde{Q}_0^{(2)}]}{[1 + \tilde{Q}^{(0)} - \tilde{P}\tilde{Q}^{(1)}]^2} \\
&\quad - \frac{\begin{array}{c}[\tilde{Q}^{(0)} - \tilde{P}\tilde{Q}^{(1)}][i\tilde{Q}^{(0)} + \tilde{Q}_0^{(0)} - \frac{\partial P}{\partial k_0}\tilde{Q}^{(0)} - \tilde{P}\tilde{Q}_0^{(1)}] \\ +2i\tilde{Q}^{(0)} + 2i\tilde{P}^2\tilde{Q}^{(2)} - 4i\tilde{P}\tilde{Q}^{(1)}\end{array}}{[1 + \tilde{Q}^{(0)} - \tilde{P}\tilde{Q}^{(1)}]^2}.
\end{aligned} \tag{XII.13}$$

Both $\Sigma(k)$ and $\frac{\partial\Sigma}{\partial k_0}(k)$ are rational functions in the variables P, $\tilde{P}$, $\frac{\partial P}{\partial k_0}$, Q, $\tilde{Q}^{(0)}$, $\tilde{Q}^{(1)}$, $\tilde{Q}^{(2)}$, $\tilde{Q}_0^{(0)}$, $\tilde{Q}_0^{(1)}$ and $\tilde{Q}_0^{(2)}$. As all of these variables are bounded in magnitude by $\lambda_0^{1-3\upsilon}$, the numerators contain no constant terms and the denominators are bounded away from zero,

$$\big|\Sigma(k)\big|, \big|\tfrac{\partial\Sigma}{\partial k_0}(k)\big| \le \text{const}\,\lambda_0^{1-3\upsilon} \le \lambda_0^{1-4\upsilon}.$$

Each term in the numerator of $\Sigma(k) - P(k)$ contains either a factor of $Q(k)$ or a factor of $P(k)\tilde{Q}^{(m)}(k)$ so that, using the bound on $Q(k)$ in (XII.4),

$$\big|\Sigma(k) - P(k)\big| \le \text{const}\,\lambda_0^{1-2\upsilon}\min\{|ik_0 - e(\mathbf{k})|^{1+\aleph}, 1\} \le \lambda_0^{1-4\upsilon}\min\{|ik_0 - e(\mathbf{k})|^{\frac{3}{2}}, 1\}.$$

Applying a Taylor expansion with degree one remainder and with expansion point $x_0 = P(k)$, $y_0 = \tilde{P}(k), \cdots, z_0 = \tilde{Q}_0^{(2)}(k)$ and evaluation point $x = P(k')$, $y = \tilde{P}(k'), \cdots$, $z = \tilde{Q}_0^{(2)}(k')$,

$$\begin{aligned}
&\big|\Sigma(k) - \Sigma(k')\big| \\
&\quad \le \text{const}\ \max\Big\{|P(k) - P(k')|, |\tilde{P}(k) - \tilde{P}(k')|, \cdots, |\tilde{Q}_0^{(2)}(k) - \tilde{Q}_0^{(2)}(k')|\Big\} \\
&\quad \le \text{const}\,\lambda_0^{1-2\upsilon}|k - k'|^{\aleph''} \\
&\big|\tfrac{\partial\Sigma}{\partial k_0}(k) - \tfrac{\partial\Sigma}{\partial k_0}(k')\big| \\
&\quad \le \text{const}\ \max\Big\{|P(k) - P(k')|, |\tilde{P}(k) - \tilde{P}(k')|, \cdots, |\tilde{Q}_0^{(2)}(k) - \tilde{Q}_0^{(2)}(k')|\Big\} \\
&\quad \le \text{const}\,\lambda_0^{1-2\upsilon}|k - k'|^{\aleph''}. \quad \square
\end{aligned}$$

Remark XII.6. We have proven, in Lemma XII.5, only very limited regularity properties of the proper self–energy $\Sigma(k)$. It is proven in [FST3], that, to all finite orders of perturbation theory, the proper self–energy is $C^{2-\varepsilon}$ for every $\varepsilon > 0$. While we prove the convergence of perturbation theory in the course of proving Theorems I.4 and I.5, the convergence proof does not give sufficient control over derivatives to allow us to rigorously conclude that Σ is $C^{2-\varepsilon}$.

The amputation used in the input and output data of §XV multiplies all external legs by $A(k) = ik_0 - e(\mathbf{k})$. The following lemma bounds the functions used to change to the amputation in which one multiplies by $ik_0 - e(\mathbf{k}) - \Sigma(k)$, the inverse of the physical two–point function, instead.

Lemma XII.7.

i) Let $A_1(k) = \nu^{(\le i)}(k)\frac{ik_0-e(\mathbf{k})-P(k)}{ik_0-e(\mathbf{k})}$. *Then*

$$\|A_1(k)\|^{\tilde{}} \le \mathrm{const}\, \mathfrak{c}_i.$$

ii) Let $A_2(k) = \frac{ik_0-e(\mathbf{k})-\Sigma(k)}{ik_0-e(\mathbf{k})-P(k)}$. *Then*

$$\big|A_2(k)\big| \le 2 \qquad \big|A_2(k) - A_2(k')\big| \le \lambda_0^{1-3\upsilon}|k-k'|^{\frac{1}{2}}.$$

Proof. Since $\check{p}^{(j)}(k)$ is supported on the j^{th} neighbourhood,

$$\begin{aligned} A_1(k) &= \nu^{(\le i)}(k) - \nu^{(\le i)}(k)\tfrac{P(k)}{ik_0-e(\mathbf{k})} \\ &= \nu^{(\le i)}(k) - \tfrac{\nu^{(\le i)}(k)}{ik_0-e(\mathbf{k})}\sum_{j=2}^{i+1}\check{p}^{(j)}(k). \end{aligned}$$

Since $\mathfrak{c}_j \le 1 + \mathrm{const}\,\frac{M^j}{M^i}\mathfrak{c}_i$ for all $j \le i+1$,

$$\begin{aligned} \Big\|\sum_{j=2}^{i+1}\check{p}^{(j)}(k)\Big\|^{\tilde{}} &\le \sum_{j=2}^{i+1}2\lambda_0^{1-\upsilon}\tfrac{\mathfrak{l}_j}{M^j}\mathfrak{c}_j \\ &\le \mathrm{const}\,\lambda_0^{1-\upsilon}\big[1+\tfrac{1}{M^i}\mathfrak{c}_i\big]. \end{aligned}$$

Also $\big\|\nu^{(\le i)}(k)\big\|^{\tilde{}} \le \mathrm{const}\,\mathfrak{c}_i$ and $\big\|\frac{\nu^{(\le i)}(k)}{ik_0-e(\mathbf{k})}\big\|^{\tilde{}} \le \mathrm{const}\, M^i\mathfrak{c}_i$, so that

$$\sup_k\Big|\tfrac{\partial^\delta}{\partial k^\delta}\nu^{(\le i)}(k)\Big| \le \mathrm{const}\, M^{i|\delta|}$$

and

$$\begin{aligned} \sup_k\Big|\tfrac{\partial^\delta}{\partial k^\delta}\tfrac{\nu^{(\le i)}(k)}{ik_0-e(\mathbf{k})}\Big|\sup_k\Big|\tfrac{\partial^{\delta'}}{\partial k^{\delta'}}\sum_{j=2}^{i+1}\check{p}^{(j)}(k)\Big| &\le \mathrm{const}\, M^{i(|\delta|+1)}\lambda_0^{1-\upsilon}M^{i(|\delta'|-1)} \\ &\le \mathrm{const}\,\lambda_0^{1-\upsilon}M^{i|\delta+\delta'|} \end{aligned}$$

provided $\delta' \ne 0$. This handles all contributions to $\|A_1(k)\|^{\tilde{}}$ except those for which no derivatives act on $P(k)$. For those contributions, we write $P(k) = ik_0\tilde{P}(k)$ and use that $|\tilde{P}(k)| \le \lambda_0^{1-3\upsilon}$ so that

$$\sup_k\Big|P(k)\tfrac{\partial^\delta}{\partial k^\delta}\tfrac{\nu^{(\le i)}(k)}{ik_0-e(\mathbf{k})}\Big| \le \lambda_0^{1-3\upsilon}\sup_k\Big|k_0\tfrac{\partial^\delta}{\partial k^\delta}\tfrac{\nu^{(\le i)}(k)}{ik_0-e(\mathbf{k})}\Big| \le \mathrm{const}\,\lambda_0^{1-3\upsilon}M^{i|\delta|}.$$

Now we move on to

$$A_2(k) = 1 - \frac{\Sigma(k)-P(k)}{ik_0-e(\mathbf{k})-P(k)}.$$

In the notation of the proof of Lemma XII.5,

$$\begin{aligned} A_4(k) = \frac{\Sigma(k)-P(k)}{ik_0-e(\mathbf{k})-P(k)} &= \frac{Q(k)-2P(k)\tilde{Q}^{(0)}(k)+P(k)\tilde{P}(k)\tilde{Q}^{(1)}(k)}{[1+\tilde{Q}^{(0)}(k)-\tilde{P}(k)\tilde{Q}^{(1)}(k)][E(k)-P(k)]} \\ &= \frac{\tilde{Q}^{(0)}(k)-2\tilde{P}(k)\tilde{Q}^{(1)}(k)+\tilde{P}(k)^2\tilde{Q}^{(2)}(k)}{[1+\tilde{Q}^{(0)}(k)-\tilde{P}(k)\tilde{Q}^{(1)}(k)][1-\tilde{K}_0(k)\tilde{P}(k)]} \\ &= \frac{A_3(k)}{1-\tilde{K}_0(k)\tilde{P}(k)}, \end{aligned}$$

where $\tilde{K}_0(k) = \frac{ik_0}{E(k)}$ and $A_3(k) = \frac{\tilde{Q}^{(0)}(k)-2\tilde{P}(k)\tilde{Q}^{(1)}(k)+\tilde{P}(k)^2\tilde{Q}^{(2)}(k)}{1+\tilde{Q}^{(0)}(k)-\tilde{P}(k)\tilde{Q}^{(1)}(k)}$. By the bounds (XII.11) and (XII.12) on $\tilde{Q}^{(m)}$ and $\tilde{P}$

$$\begin{aligned} |A_3(k)| &\le \text{const}\,\lambda_0^{1-2\upsilon}\min\{|ik_0-e(\mathbf{k})|^{\aleph},1\} \\ \big|A_3(k)-A_3(k')\big| &\le \text{const}\,\lambda_0^{1-2\upsilon}|k-k'|^{\aleph''}. \end{aligned} \tag{XII.14}$$

To bound $\big|A_3(k)-A_3(k')\big|$, we used a Taylor expansion argument as in Lemma XII.5.

Let ℓ be such that $\frac{1}{M^\ell} \le |k-k'| \le \frac{M}{M^\ell}$. If either $|E(k)| \le \frac{1}{M^\ell}$ or $|E(k')| \le \frac{1}{M^\ell}$, then both $|E(k)|, |E(k')| \le \frac{\text{const}}{M^\ell}$ and, by (XII.11) and (XII.14),

$$|A_4(k)|, |A_4(k')| \le \text{const}\,\lambda_0^{1-2\upsilon}\frac{1}{M^{\aleph\ell}} \le \text{const}\,\lambda_0^{1-2\upsilon}|k-k'|^{\aleph}.$$

If both $|E(k)| \ge \frac{1}{M^\ell}$ and $|E(k')| \ge \frac{1}{M^\ell}$, then

$$\max\{|E(k)|,|E(k')|\} \le \text{const}\ \min\{|E(k)|,|E(k')|\}$$

and

$$\big|\tilde{K}_0(k)-\tilde{K}_0(k')\big| \le \min\left\{2, \text{const}\,\frac{|k-k'|}{\min\{|E(k)|,|E(k')|\}}\right\} \le \text{const}\left(\frac{|k-k'|}{\max\{|E(k)|,|E(k')|\}}\right)^{\aleph}.$$

Hence, by (XII.14), (XII.11) and (XII.12)

$$\begin{aligned} &\big|A_4(k)-A_4(k')\big| \\ &\quad= \left|\frac{A_3(k)-A_3(k')}{1-\tilde{K}_0(k')\tilde{P}(k')} + A_3(k)\frac{\tilde{K}_0(k)\tilde{P}(k)-\tilde{K}_0(k')\tilde{P}(k')}{[1-\tilde{K}_0(k)\tilde{P}(k)][1-\tilde{K}_0(k')\tilde{P}(k')]}\right| \\ &\quad\le \text{const}\,\lambda_0^{1-2\upsilon}|k-k'|^{\aleph''} + \text{const}\,\lambda_0^{1-2\upsilon}|E(k)|^{\aleph}\big|\tilde{K}_0(k)-\tilde{K}_0(k')\big|\big|\tilde{P}(k)\big| \\ &\qquad+\text{const}\,\lambda_0^{1-2\upsilon}\big|\tilde{K}_0(k')\big|\big|\tilde{P}(k)-\tilde{P}(k')\big| \\ &\quad\le \text{const}\,\lambda_0^{1-2\upsilon}|k-k'|^{\aleph''} + \text{const}\,\lambda_0^{1-2\upsilon}|k-k'|^{\aleph}. \end{aligned}$$

The desired bounds on A_2 follow. □

XIII. Momentum Space Norms

Theorem XII.1 is concerned with the Fourier transforms of the two point functions $G^{\rm rg}_{j,2}(\eta_1,\eta_2)$. In the proof of Theorem VIII.5, $G^{\rm rg}_{j,2}$ is built up recursively from contributions to effective interactions with two, one or no external fields. To control $\check G^{\rm rg}_{j,2}$, we control the partial Fourier transforms, with respect to the external variables, of these contributions. In this chapter, we provide the notation to do this. In particular, we specify norms for functions that depend on "external" momentum and "internal" sectorized position variables. These norms coincide with those of §VI when there are no external legs. When there are external legs, the new norms sup over the momenta of external as well as internal arguments and also allow momentum space derivatives to act on external arguments. We do this because, in this part, we are interested in the momentum space regularity properties of the two and four point functions. This contrasts with the norms of §VI, which were used to prove the existence of all n–point functions but which provided very limited momentum space regularity.

Definition XIII.1 (Partial Fourier transforms). *Let $f(\eta_1,\cdots,\eta_m;\xi_1,\cdots,\xi_n)$ be a translation invariant function on $\mathcal{B}^m\times\mathcal{B}^n$. If $n\ge 1$, the partial Fourier transform $f^\sim$ is defined, using the notation of Definition VI.1, by*

$$f^\sim(\check\eta_1,\cdots,\check\eta_m;\xi_1,\cdots,\xi_n)=\int\Big(\prod_{i=1}^m E_+(\check\eta_i,\eta_i)\,d\eta_i\Big)\,f(\eta_1,\cdots,\eta_m;\xi_1,\cdots,\xi_n)$$

or, equivalently, by

$$f(\eta_1,\cdots,\eta_m;\xi_1,\cdots,\xi_n)=\int\Big(\prod_{i=1}^m E_-(\check\eta_i,\eta_i)\,\tfrac{d\check\eta_i}{(2\pi)^{d+1}}\Big)\,f^\sim(\check\eta_1,\cdots,\check\eta_m;\xi_1,\cdots,\xi_n).$$

If $n=0$, we set $f^\sim=\check f$.

Definition XIII.2. *If $\phi(\eta)$ is a Grassmann field, we set for, $\check\eta=(k,\sigma,a)\in\check{\mathcal{B}}$,*

$$\check\phi(\check\eta)=\int d\eta\;E_-(\check\eta,\eta)\phi(\eta)=\int dx_0d^d\mathbf{x}\;e^{-(-1)^a\imath<k,x>_-}\phi(x_0,\mathbf{x},\sigma,a).$$

Remark XIII.3. A translation invariant sectorized Grassmann function $\mathcal{W}$ can be uniquely written in the form

$$\mathcal{W}(\phi,\psi)=\sum_{m,n}\int d\eta_1\cdots d\eta_m\,d\xi_1\cdots d\xi_n\,W_{m,n}(\eta_1,\cdots,\eta_m\,;\xi_1,\cdots,\xi_n)\;\phi(\eta_1)\cdots\phi(\eta_m)\\ \times\psi(\xi_1)\cdots\psi(\xi_n)$$

with $W_{m,n}$ antisymmetric separately in the η and in the ξ variables. Then, under the Fourier transform Definitions XIII.1 and XIII.2,

$$\begin{aligned}\mathcal{W}(\phi,\psi)\\ &=\sum_m\int\prod_{i=1}^m\tfrac{d\check\eta_i}{(2\pi)^{d+1}}\;W^\sim_{m,0}(\check\eta_1,\cdots,\check\eta_m)\;(2\pi)^{d+1}\delta(\check\eta_1+\cdots+\check\eta_m)\;\check\phi(\check\eta_1)\cdots\check\phi(\check\eta_m)\\ &\quad+\sum_{\substack{m,n\\ n\ge1}}\int\prod_{i=1}^m\tfrac{d\check\eta_i}{(2\pi)^{d+1}}\prod_{i=1}^n d\xi_i\;W^\sim_{m,n}(\check\eta_1,\cdots,\check\eta_m\,;\xi_1,\cdots,\xi_n)\;\check\phi(\check\eta_1)\cdots\check\phi(\check\eta_m)\psi(\xi_1)\cdots\psi(\xi_n).\end{aligned}$$

Definition XIII.4. *A function* $f(\check\eta_1,\cdots,\check\eta_m;\xi_1,\cdots,\xi_n)$ *on* $\check{\mathcal{B}}^m \times \mathcal{B}^n$*, with* $n \ge 1$*, is said to be translation invariant if*

$$f(\check\eta_1,\cdots,\check\eta_m;\xi_1+t,\cdots,\xi_n+t) = e^{\imath\langle \check\eta_1+\cdots+\check\eta_m,t\rangle_-}\, f(\check\eta_1,\cdots,\check\eta_m;\xi_1,\cdots,\xi_n)$$

for all $t \in \mathbb{R}\times\mathbb{R}^d$*. A distribution* $f(\check\eta_1,\cdots,\check\eta_m)$ *on* $\check{\mathcal{B}}^m$ *is said to be translation invariant if it is supported on* $\check\eta_1+\cdots+\check\eta_m = 0$*. Recall, from just before Definition VI.1, that if* $\check\eta = (k,\sigma,a)$, $\check\eta' = (k',\sigma',a') \in \check{\mathcal{B}}$*, then* $\check\eta + \check\eta' = (-1)^a k + (-1)^{a'} k' \in \mathbb{R}\times\mathbb{R}^d$.

The partial Fourier transform of a translation invariant function on $\mathcal{B}^m \times \mathcal{B}^n$ is a translation invariant function on $\check{\mathcal{B}}^m \times \mathcal{B}^n$.

Definition XIII.5 (Differential–decay operators). *Let* $m, n \ge 0$*. If* $n \ge 1$*, let* f *be a function on* $\check{\mathcal{B}}^m \times \mathcal{B}^n$*. If* $n = 0$*, let* f *be a function on*

$$\check{\mathcal{B}}_m = \left\{ (\check\eta_1,\cdots,\check\eta_m) \in \check{\mathcal{B}}^m \;\middle|\; \check\eta_1+\cdots+\check\eta_m = 0 \right\}.$$

i) For $1 \le j \le m$ *and a multiindex* δ *set*

$$\begin{aligned} \mathrm{D}_j^\delta f\,&((p_1,\tau_1,b_1),\cdots,(p_m,\tau_m,b_m);\xi_1,\cdots,\xi_n) \\ &= \left[\imath(-1)^{b_j}\right]^{\delta_0} \prod_{\ell=1}^{d}\left[-\imath(-1)^{b_j}\right]^{\delta_\ell} \frac{\partial^{\delta_0}}{\partial p_{j,0}^{\delta_0}} \frac{\partial^{\delta_1}}{\partial \mathbf{p}_{j,1}^{\delta_1}} \cdots \frac{\partial^{\delta_d}}{\partial \mathbf{p}_{j,d}^{\delta_d}} \\ &\quad \times f((p_1,\tau_1,b_1),\cdots,(p_m,\tau_m,b_m);\xi_1,\cdots,\xi_n). \end{aligned}$$

ii) Let $1 \le i \ne j \le m+n$ *and* δ *a multiindex. Set*

$$\begin{aligned} &\mathrm{D}_{i;j}^\delta f = (\mathrm{D}_i - \mathrm{D}_j)^\delta f && \text{if } 1 \le i < j \le m,\\ &\mathrm{D}_{i;j}^\delta f = (\mathrm{D}_i - \xi_{j-m})^\delta f && \text{if } 1 \le i \le m,\ m+1 \le j \le m+n,\\ &\mathrm{D}_{i;j}^\delta f = (\xi_i - \mathrm{D}_{j-m})^\delta f && \text{if } m+1 \le i \le m+n,\ 1 \le j \le m,\\ &\mathrm{D}_{i;j}^\delta f = (\xi_{i-m} - \xi_{j-m})^\delta f = \mathcal{D}_{i-m,j-m}^\delta f && \text{if } m+1 \le i < j \le m+n. \end{aligned}$$

The decay operator $\mathcal{D}_{i,j}$ *was defined in Definition V.1.*

iii) A differential–decay operator (dd–operator) of type (m,n)*, with* $m+n \ge 2$*, is an operator* D *of the form*

$$\mathrm{D} = \mathrm{D}_{i_1;j_1}^{\delta^{(1)}} \cdots \mathrm{D}_{i_r;j_r}^{\delta^{(r)}}$$

with $1 \le i_\ell \ne j_\ell \le m+n$ *for all* $1 \le \ell \le r$*. A dd–operator of type* $(1,0)$ *is an operator of the form* $\mathrm{D} = \mathrm{D}_1^{\delta^{(1)}} \cdots \mathrm{D}_1^{\delta^{(r)}}$*. The total order of* D *is* $\delta(\mathrm{D}) = \delta^{(1)}+\cdots+\delta^{(r)}$.

Remark XIII.6.

i) For a translation invariant function φ on $\mathcal{B}^m \times \mathcal{B}^n$,

$$\mathrm{D}_{i;j}(\varphi^\sim) = (\mathcal{D}_{i,j}\varphi)^\sim.$$

In particular, Leibniz's rule also applies for differential–decay operators.

ii) Let f be a translation invariant function on $\check{\mathcal{B}}^m \times \mathcal{B}$. Then, for $\xi = (x_0, \mathbf{x}, \sigma, a) \in \mathcal{B}$,

$$f(\check{\eta}_1, \cdots, \check{\eta}_m; \xi) = e^{\imath \langle \check{\eta}_1 + \cdots + \check{\eta}_m, (x_0, \mathbf{x}) \rangle_-} f(\check{\eta}_1, \cdots, \check{\eta}_m; (0, \sigma, a)).$$

Consequently, for $1 \le i \le m$ and a multiindex δ

$$\mathrm{D}^\delta_{i;m+1} f(\check{\eta}_1, \cdots, \check{\eta}_m; \xi) = e^{\imath \langle \check{\eta}_1 + \cdots + \check{\eta}_m, (x_0, \mathbf{x}) \rangle_-} \mathrm{D}^\delta_i f(\check{\eta}_1, \cdots, \check{\eta}_m; (0, \sigma, a)).$$

Definition XIII.7. *For a function f on $\check{\mathcal{B}}_m$, set*

$$\| f \|^{\tilde{}} = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^2} \frac{1}{\delta!} \max_{\substack{\mathrm{D}\ \text{dd}-\text{operator} \\ \text{with } \delta(\mathrm{D}) = \delta}} \sup_{\check{\eta}_1, \cdots, \check{\eta}_m \in \check{\mathcal{B}}} \left| \mathrm{D} f\, (\check{\eta}_1, \cdots, \check{\eta}_m) \right| t^\delta.$$

Let f be a function on $\check{\mathcal{B}}^m \times \mathcal{B}^n$ with $n \ge 1$. Set

$$\| f \|^{\tilde{}} = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^2} \frac{1}{\delta!} \max_{\substack{\mathrm{D}\ \text{dd}-\text{operator} \\ \text{with } \delta(\mathrm{D}) = \delta}} \sup_{\check{\eta}_1, \cdots, \check{\eta}_m \in \check{\mathcal{B}}} \left|\!\left|\!\left| \mathrm{D} f\, (\check{\eta}_1, \cdots, \check{\eta}_m; \xi_1, \cdots, \xi_n) \right|\!\right|\!\right|_{1,\infty} t^\delta.$$

The norm $|||\cdot|||_{1,\infty}$ of Definition V.3 refers to the variables $\xi_1, \cdots, \xi_n$. That is,

$$\left|\!\left|\!\left| \mathrm{D} f\, (\check{\eta}_1, \cdots, \check{\eta}_m; \xi_1, \cdots, \xi_n) \right|\!\right|\!\right|_{1,\infty} = \max_{1 \le j_0 \le n} \sup_{\xi_{j_0} \in \mathcal{B}} \int \prod_{\substack{j=1, \cdots, n \\ j \ne j_0}} d\xi_j \, |\mathrm{D} f\, (\check{\eta}_1, \cdots, \check{\eta}_m; \xi_1, \cdots, \xi_n)|.$$

Remark XIII.8. In the case $m = 0$ the norm $\|\cdot\|_{1,\infty}$ of Definition V.3 and the norm $\|\cdot\|^{\tilde{}}$ of Definition XIII.7 agree.

Definition XIII.9. *We amputate a Grassmann function by applying the Fourier transform $\hat{A}$, in the sense of Notation V.4, of $A(k) = ik_0 - e(\mathbf{k})$ to its external arguments. Precisely, if $\mathcal{W}(\phi, \psi)$ is a Grassmann function, then*

$$\mathcal{W}^a(\phi, \psi) = \mathcal{W}(\hat{A}\phi, \psi),$$

where

$$(\hat{A}\phi)(\xi) = \int d\xi' \, \hat{A}(\xi, \xi')\phi(\xi').$$

Remark XIII.10. If $C(\xi, \xi')$ is the covariance associated to $C(k)$ in the sense of (III.1) and (III.2) and J is the particle hole swap operator of (III.3), then, by parts (i) and (ii) of Lemma IX.5 of [FKTo2],

$$\int d\eta d\eta' \, C(\xi, \eta) J(\eta, \eta') \hat{A}(\eta', \xi) = \hat{E}(\xi, \xi'),$$

where $\hat{E}$ is the Fourier transform of $\big(ik_0 - e(\mathbf{k})\big)C(k)$ in the sense of Notation V.4.

Integrating out the first few scales is controlled much as in Theorem V.8.

Theorem XIII.11. *There are (M and j_0–dependent) constants $\bar\lambda$, μ and β_0 such that, for all $\lambda_0 < \bar\lambda$ and $\beta_0 \le \beta \le \frac{1}{\lambda_0^{\upsilon/5}}$, the following holds: Let $X \in \mathfrak{N}_{d+1}$ with $X_{\mathbf{0}} < \mu$, $\delta e \in \mathcal{E}$ with $\|\delta\hat e\|_{1,\infty} \le X$ and*

$$\mathcal{V}(\psi) = \int_{\mathcal{B}^4} d\xi_1\cdots d\xi_4\; V(\xi_1,\cdots,\xi_4)\,\psi(\xi_1)\cdots\psi(\xi_1)$$

with an antisymmetric function V fulfilling

$$\|V\|_{1,\infty} \le \lambda_0\,\mathfrak{e}_0(X).$$

Write

$$\tilde\Omega_{C_{-\delta e}^{(\le j_0)}}\big(\mathcal{V}(\psi)\big)(\phi,\psi) = \mathcal{V}(\psi) + \tfrac12 \phi J C_{-\delta e}^{(\le j_0)} J\phi + \sum_{\substack{m,n\ge 0\\ m+n \text{ even}}} \int_{\mathcal{B}^{m+n}} d\eta_1\cdots d\eta_m\; d\xi_1\cdots d\xi_n$$
$$\times\, W_{m,n}(\eta_1\cdots\eta_m,\xi_1,\cdots,\xi_n;\,\delta e)\,\phi(\eta_1)\cdots\phi(\eta_m)\,\psi(\xi_1)\cdots\psi(\xi_n)$$

with kernels $W_{m,n}$ that are separately antisymmetric under permutations of their η and ξ arguments. Then

$$\sum_{\substack{m+n\ge 2\\ m+n \text{ even}}} \beta^{m+n}\,\tilde\rho_{m;n}\,\|W_{m,n}^{a\sim}(\delta e)\|^{\tilde{}} \le \beta^4\lambda_0^{\upsilon}\,\mathfrak{e}_0(X)$$

and

$$\sum_{\substack{m+n\ge 2\\ m+n \text{ even}}} \beta^{m+n}\,\tilde\rho_{m;n}\,\Big\|\tfrac{d}{ds} W_{m,n}^{a\sim}(\delta e + s\delta e')\big|_{s=0}\Big\|^{\tilde{}} \le \beta^4\lambda_0^{\upsilon}\,\mathfrak{e}_0(X)\,\|\delta\hat e'\|_{1,\infty},$$

where

$$\tilde\rho_{m;n} = \frac{\lambda_0^{m\upsilon/7}}{\lambda_0^{(1-\upsilon)\max\{m+n-2,2\}/2}}$$

and υ was fixed in Definition V.6.

Proof. Apply Theorem X.12 of [FKTo2] with $\rho_{m,n} = \tilde\rho_{m;n}$, $\varepsilon = {\scriptstyle const}\,\beta^4\lambda_0^{\upsilon} \le {\scriptstyle const}\,\lambda_0^{\upsilon/5}$ and $\varepsilon' = \lambda_0^{\upsilon/7}$. Observe that, $\tilde\rho_{m;n}$ is $\lambda_0^{m\upsilon/7}$ times the $\rho_{m;n}$ of Remark VIII.7.iii of [FKTo2]. So the hypotheses of Theorem X.12 concerning $\rho_{m;n}$ are fulfilled. Choosing $\bar\lambda$ small enough ensures that the hypotheses $\varepsilon, \varepsilon' \le \varepsilon_0$ of Theorem X.12 are satisfied. □

In the course of exhibiting control over amputated Green's functions in momentum space, we must deal with kernels having external arguments in momentum space and internal arguments sectorized and in position space.

Definition XIII.12 (Momentum Space Norms). *Let p be a natural number and Σ a sectorization.*

(i) For a function f on $\check{\mathcal{B}}_m$ we define

$$|f|_{p,\Sigma}^{\tilde{}} = \begin{cases} \|f\|^{\tilde{}} & \text{if } p = m-1,\, m = 2,4. \\ 0 & \text{otherwise} \end{cases}.$$

(ii) For a translation invariant function f on $\check{\mathcal{B}}^m \times (\mathcal{B} \times \Sigma)^n$ with $n \geq 1$, we set $|f|^{\sim}_{p,\Sigma} = 0$ when $p > m+n$ or $p < m$, and

$$|f|^{\sim}_{p,\Sigma} = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^2} \sup_{\substack{1 \le i_1 < \cdots < i_{p-m} \le n \\ s_{i_1}, \cdots, s_{i_{p-m}} \in \Sigma \\ \check\eta_1, \cdots, \check\eta_m \in \check{\mathcal{B}}}} \sum_{\substack{s_i \in \Sigma \text{ for} \\ i \ne i_1, \cdots i_{p-m}}} \frac{1}{\delta!} \times \max_{\substack{\mathrm{D}\ \mathrm{dd-operator} \\ \text{with } \delta(\mathrm{D}) = \delta}} \left|\!\left|\!\left| \mathrm{D} f\,(\check\eta_1, \cdots, \check\eta_m; (\xi_1, s_1), \cdots, (\xi_n, s_n)) \right|\!\right|\!\right|_{1,\infty} \mathfrak{t}^{\delta}$$

when $m \leq p \leq m+n$. The norm $|||\cdot|||_{1,\infty}$ of Definition V.3 refers to the variables $\xi_1, \cdots, \xi_n$.

Remark XIII.13. In the case $m = 0$ the norm $|\cdot|_{p,\Sigma}$ of Definition VI.6 and the norm $|\cdot|^{\sim}_{p,\Sigma}$ of Definition XIII.12 agree.

Definition XIII.14. *Let $m, n \geq 0$ and Σ a sectorization. For $n \geq 1$, denote by $\check{\mathcal{F}}_m(n; \Sigma)$ the space of all translation invariant, complex valued functions $f(\check\eta_1, \cdots, \check\eta_m; (\xi_1, s_1), \cdots, (\xi_n, s_n))$ on $\check{\mathcal{B}}^m \times \left(\mathcal{B} \times \Sigma\right)^n$ whose Fourier transform $\check f(\check\eta_1, \cdots, \check\eta_m; (\check\xi_1, s_1), \cdots, (\check\xi_n, s_n))$ vanishes unless $k_i \in \tilde s_i$ for all $1 \leq j \leq n$. Here, $\check\xi_i = (k_i, \sigma_i, a_i)$. Also, let $\check{\mathcal{F}}_m(0; \Sigma)$ be the space of all momentum conserving, complex valued functions $f(\check\eta_1, \cdots, \check\eta_m)$ on $\check{\mathcal{B}}^m$.*

We now provide the analogue of Definition VI.6.ii for the $|\ |^{\sim}$-norms. Let $j \geq 2$ and let Σ_j be the sectorization of scale j and length $\mathfrak{l}_j = \frac{1}{M^{\aleph j}}$ fixed just before Definition VI.4.

Definition XIII.15.

i) For $f \in \check{\mathcal{F}}_m(n; \Sigma_j)$ set

$$|f|^{\sim}_j = \tilde\rho_{m;n} \begin{cases} |f|^{\sim}_{1,\Sigma_j} + |f|^{\sim}_{2,\Sigma_j} + \frac{1}{\mathfrak{l}_j} |f|^{\sim}_{3,\Sigma_j} \\ \quad + \frac{1}{\mathfrak{l}_j} |f|^{\sim}_{4,\Sigma_j} + \frac{1}{\mathfrak{l}_j^2} |f|^{\sim}_{5,\Sigma_j} + \frac{1}{\mathfrak{l}_j^2} |f|^{\sim}_{6,\Sigma_j} & \text{if } m \ne 0 \\ |f|^{\sim}_{1,\Sigma_j} + \frac{1}{\mathfrak{l}_j} |f|^{\sim}_{3,\Sigma_j} + \frac{1}{\mathfrak{l}_j^2} |f|^{\sim}_{5,\Sigma_j} & \text{if } m = 0 \end{cases},$$

$$|f|^{\sim}_{p,\Sigma_j,\tilde\rho} = \tilde\rho_{m;n} |f|^{\sim}_{p,\Sigma_j}.$$

ii) An even sectorized Grassmann function w can be uniquely written in the form

$$w(\phi, \psi) = \sum_{m,n} \int d\eta_1 \cdots d\eta_m\, d\xi_1 \cdots d\xi_n\; w_{m,n}(\eta_1, \cdots, \eta_m\, (\xi_1, s_1), \cdots, (\xi_n, s_n)) \times \phi(\eta_1) \cdots \phi(\eta_m)\; \psi((\xi_1, s_1)) \cdots \psi((\xi_n, s_n))$$

with $w_{m,n}$ antisymmetric separately in the η and in the ξ variables. Set, for $\alpha > 0$ and $X \in \mathfrak{N}_{d+1}$,

$$N^{\sim}_j(w, \alpha,\ X) = \frac{M^{2j}}{\mathfrak{l}_j}\, \mathfrak{e}_j(X) \sum_{m,n \geq 0} \alpha^{m+n} \left(\frac{\mathfrak{l}_j\, \mathrm{B}}{M^j}\right)^{(m+n)/2} |w^{\sim}_{m,n}|^{\sim}_j,$$

where $w^{\sim}_{m,n}$ is the partial Fourier transform of $w_{m,n}$ of Definition XIII.1 and $\mathrm{B} = 4\max\{4\mathrm{B}_3, \mathrm{B}_4\}$ with B_3, B_4 being the constants of Proposition XVI.8 of [FKTo3].

Remark XIII.16. In particular, for the "pure ϕ" part of w,

$$N_j^{\sim}(w(\phi,0),\alpha,X) = \mathfrak{e}_j(X)\Big[\tfrac{\alpha^2 \mathrm{B}}{\lambda_0^{1-9\upsilon/7}} M^j |w_{2,0}^{\sim}|_{1,\Sigma_j}^{\sim} + \tfrac{\alpha^4 \mathrm{B}^2}{\lambda_0^{1-11\upsilon/7}} |w_{4,0}^{\sim}|_{3,\Sigma_j}^{\sim}\Big].$$

The sectorized version of Theorem XIII.11 is

Theorem XIII.17. *For $K \in \mathfrak{K}_{j_0}$, set*

$$u(K) = -\big[K_{\rm ext}\big]_{\Sigma_{j_0}} \in \mathcal{F}_0(2,\Sigma_{j_0}),$$

where $K_{\rm ext}$ was defined in Definition E.3 of [FKTo4]. Then there exist constants $\bar\alpha, \bar\lambda > 0$ such that for all $0 \le \lambda_0 < \bar\lambda$, $\bar\alpha < \alpha < \frac{1}{\lambda_0^{\upsilon/10}}$ and all

$$K \in \mathfrak{K}_{j_0} \qquad \|V\|_{1,\infty} \le \lambda_0 \mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big)$$

the Σ_{j_0}–sectorized representative $w(\phi,\psi;K)$ of $\tilde\Omega_{C_{u(K)}^{(\le j_0)}}\big(\mathcal{V}(\psi)\big)(\phi,\psi) - \frac12 \phi J C_{u(K)}^{(\le j_0)} J\phi$ constructed in Theorem VI.12 may be chosen to obey

$$N_{j_0}^{\sim}\big(w^a(K),\alpha,\|K\|_{1,\Sigma_{j_0}}\big) \le \text{const } \alpha^4 \lambda_0^{\upsilon}\, \mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big),$$
$$N_{j_0}^{\sim}\big(\tfrac{d}{ds} w^a(K+sK')\big|_{s=0},\alpha,\ \|K\|_{1,\Sigma_{j_0}}\big) \le \text{const } \alpha^4 \lambda_0^{\upsilon}\, \mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big)\, \|K'\|_{1,\Sigma_{j_0}}$$

for all K'. Furthermore $w^a(\phi,\psi,K) - w^a(0,\psi,K)$ vanishes unless $\hat\nu^{(\le j_0)}\phi$ is nonzero.

Proof. Write

$$\tilde\Omega_{C_{u(K)}^{(\le j_0)}}\big(\mathcal{V}(\psi)\big)(\phi,\psi) = \mathcal{V}(\psi) + \tfrac12 \phi J C_{u(K)}^{(\le j_0)} J\phi + \sum_{\substack{m,n\ge 0\\ m+n \text{ even}}} \int_{\mathcal{B}^{m+n}} d\eta_1\cdots d\eta_m\ d\xi_1\cdots d\xi_n$$
$$\times W_{m,n}(\eta_1,\cdots,\eta_m,\xi_1,\cdots,\xi_n)\phi(\eta_1)\cdots\phi(\eta_m)\ \psi(\xi_1)\cdots\psi(\xi_n)$$

and set, as in Theorem VI.12,

$$w_{m,n} = \begin{cases} \big(W_{m,n}\big)_{\Sigma_{j_0}} & \text{if}(m,n) \ne (0,4) \\ \big(W_{0,4}+V\big)_{\Sigma_{j_0}} & \text{if}(m,n) = (0,4) \end{cases}$$

using the sectorization f_Σ of Definition XIX.14 of [FKTo4], By Proposition XIX.15 of [FKTo4],

$$N_{j_0}^{\sim}\big(w^a(K),\alpha,\|K\|_{1,\Sigma_{j_0}}\big) = \tfrac{M^{2j_0}}{\mathfrak{l}_{j_0}}\, \mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big) \sum_{m,n\ge 0} \alpha^{m+n} \big(\tfrac{\mathfrak{l}_{j_0}\,\mathrm{B}}{M^{j_0}}\big)^{(m+n)/2} |w_{m,n}^{a\sim}|_{j_0}^{\sim}$$
$$\le \text{const } \mathfrak{c}_{j_0}\, \mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big)\Big[\tfrac{\alpha^4}{\lambda_0^{1-\upsilon}}\ \|V\|_{1,\infty}$$
$$+ \sum_{m,n\ge 0} (\text{const}\,\alpha)^{m+n}\ \tilde\rho_{m;n}\ \|W_{m,n}^{a\sim}\|^{\sim}\Big].$$

By hypothesis

$$\tfrac{\alpha^4}{\lambda_0^{1-\upsilon}}\ \|V\|_{1,\infty} \le \alpha^4 \lambda_0^{\upsilon} \mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big),$$

and by Theorem XIII.11, with $\delta e = -\check{u}$, $X = \mathrm{const}\, \|K\|_{1,\Sigma_{j_0}}$ and $\beta = \mathrm{const}\,\alpha$,

$$\sum_{m,n\geq 0} (\mathrm{const}\,\alpha)^{m+n}\, \tilde{\rho}_{m;n}\, \|W^{a\sim}_{m,n}\|^{\tilde{}} \leq \beta^4 \lambda_0^{\upsilon} \mathfrak{e}_0(X) \leq \mathrm{const}\,\alpha^4 \lambda_0^{\upsilon}\, \mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big).$$

Therefore, by Corollary A.5.ii of [FKTo1],

$$\begin{aligned} N_{j_0}^{\sim}\big(w^a(K), \alpha, \|K\|_{1,\Sigma_{j_0}}\big) &\leq \mathrm{const}\ \alpha^4 \lambda_0^{\upsilon}\ \mathfrak{c}_{j_0}\, \mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big)^2 \\ &\leq \mathrm{const}\ \alpha^4 \lambda_0^{\upsilon}\ \mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big). \end{aligned}$$

The proof of the bound on $N_{j_0}^{\sim}\big(\frac{d}{ds} w^a(K+sK')\big|_{s=0}, \alpha,\ \|K\|_{1,\Sigma_{j_0}}\big)$ is similar.

By Lemma VII.3 of [FKTo2],

$$\tilde{\Omega}_{C_{u(K)}^{(\leq j_0)}}\big(\mathcal{V}(\psi)\big)(\phi,\psi) - \tfrac{1}{2}\phi J C_{u(K)}^{(\leq j_0)} J\phi = \Omega_{C_{u(K)}^{(\leq j_0)}}\big(\mathcal{V}(\psi)\big)(\phi, \psi + C_{u(K)}^{(\leq j_0)} J\phi).$$

Since $\Omega_{C_{u(K)}^{(\leq j_0)}}\big(\mathcal{V}(\psi)\big)(\phi,\psi)$ is independent of ϕ and since $C_{u(K)}^{(\leq j_0)} J\phi$ vanishes unless $\hat{\nu}^{(\leq j_0)}\phi$ is nonzero, $w^a(\phi,\psi,K) - w^a(0,\psi,K)$ vanishes unless $\hat{\nu}^{(\leq j_0)}\phi$ is nonzero. □

XIV. Ladders with External Momenta

In the proof of Theorem VIII.5, ladders and iterated particle hole ladders needed special attention. Due to the "external improvement" of Lemma XII.19 of [FKTo3], we needed to consider only ladders all of whose "ends" correspond to ψ fields and are integrated out at a later scale. This is not the case here. We consider ladders some of whose "ends" correspond to ψ fields and have sectorized position space arguments $(\xi, s) \in \mathcal{B} \times \Sigma_j$, and some of whose ends correspond to ϕ fields and have momentum space arguments $\check{\eta} \in \check{\mathcal{B}}$. To do this, we extend the definitions and estimates of ladders and iterated particle hole ladders from §VII.

The following definition extends Definition VII.3.

Definition XIV.1. *Let Σ be a sectorization.*

i) Define

$$\check{\mathcal{B}}^{\updownarrow} = \big\{\ (k_0, \mathbf{k}, \sigma)\ \big|\ k_0 \in \mathbb{R},\ \mathbf{k} \in \mathbb{R}^2,\ \sigma \in \{\uparrow, \downarrow\}\ \big\}$$

and the disjoint unions

$$\begin{aligned} \mathfrak{Y}_{\Sigma}^{\updownarrow} &= \check{\mathcal{B}}^{\updownarrow} \cup (\mathcal{B}^{\updownarrow} \times \Sigma), \\ \mathfrak{X}_{\Sigma} &= \check{\mathcal{B}} \cup (\mathcal{B} \times \Sigma). \end{aligned}$$

ii) Let $z \in \mathfrak{X}_{\Sigma}$. Then we define its undirected part $u(z) \in \mathfrak{Y}_{\Sigma}^{\updownarrow}$ and its creation/annihilation index $b(z) \in \{0,1\}$ by

$$u(z) = \begin{cases} (k,\sigma) & \text{if } z = (k,\sigma,b) \in \check{\mathcal{B}} \\ (x,\sigma,s) & \text{if } z = (x,\sigma,b,s) \in \mathcal{B} \times \Sigma, \end{cases}$$

$$b(z) = \begin{cases} b & \text{if } z = (k,\sigma,b) \in \check{\mathcal{B}} \\ b & \text{if } z = (x,\sigma,b,s) \in \mathcal{B} \times \Sigma. \end{cases}$$

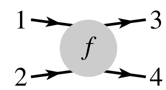
iii) Let $z' \in \mathfrak{Y}_\Sigma^\updownarrow$ and $b \in \{0, 1\}$. Then we define $\iota_b(z')$ as the unique point $z \in \mathfrak{X}_\Sigma$ with $u(z) = z'$ and $b(z) = b$.

With this notation, Definition VII.4 and Lemma VII.5.ii about particle–particle and particle–hole reductions and values carry over almost verbatim.

Definition XIV.2.

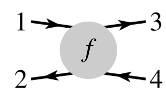
i) Let f be a four legged kernel over $\mathfrak{X}_\Sigma$. When f is a rung, its particle–particle reduction is the four legged kernel over $\mathfrak{Y}_\Sigma^\updownarrow$ given by

$$f^{\rm pp}(z_1',z_2',z_3',z_4') = f\big(\iota_0(z_1'),\iota_0(z_2'),\iota_1(z_3'),\iota_1(z_4')\big) =$$



and its particle–hole reduction is

$$f^{\rm ph}(z_1',z_2',z_3',z_4') = f\big(\iota_0(z_1'),\iota_1(z_2'),\iota_1(z_3'),\iota_0(z_4')\big) =$$



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When f is a bubble propagator, the corresponding reductions are

$$\begin{aligned} {}^{\rm pp}f(z_1',z_2',z_3',z_4') &= f\big(\iota_1(z_1'),\iota_1(z_2'),\iota_0(z_3'),\iota_0(z_4')\big),\\ {}^{\rm ph}f(z_1',z_2',z_3',z_4') &= f\big(\iota_1(z_1'),\iota_0(z_2'),\iota_0(z_3'),\iota_1(z_4')\big). \end{aligned}$$

ii) Let f' be a four legged kernel over $\mathfrak{Y}_\Sigma^\updownarrow$. Its particle–particle value is the four legged kernel over $\mathfrak{X}_\Sigma$ given by

$$\begin{aligned} V_{\rm pp}(f')(z_1,z_2,z_3,z_4) = {}& \delta_{b(z_1),0}\delta_{b(z_2),0}\delta_{b(z_3),1}\delta_{b(z_4),1}\, f'\big(u(z_1),u(z_2),u(z_3),u(z_4)\big)\\ &+\delta_{b(z_1),1}\delta_{b(z_2),1}\delta_{b(z_3),0}\delta_{b(z_4),0}\, f'\big(u(z_3),u(z_4),u(z_1),u(z_2)\big), \end{aligned}$$

and its particle–hole value is

$$\begin{aligned} V_{\rm ph}(f')(z_1,z_2,z_3,z_4) = {}& \delta_{b(z_1),0}\delta_{b(z_2),1}\delta_{b(z_3),1}\delta_{b(z_4),0}\, f'\big(u(z_1),u(z_2),u(z_3),u(z_4)\big)\\ &+\delta_{b(z_1),1}\delta_{b(z_2),0}\delta_{b(z_3),0}\delta_{b(z_4),1}\, f'\big(u(z_2),u(z_1),u(z_4),u(z_3)\big)\\ &-\delta_{b(z_1),1}\delta_{b(z_2),0}\delta_{b(z_3),1}\delta_{b(z_4),0}\, f'\big(u(z_2),u(z_1),u(z_3),u(z_4)\big)\\ &-\delta_{b(z_1),0}\delta_{b(z_2),1}\delta_{b(z_3),0}\delta_{b(z_4),1}\, f'\big(u(z_1),u(z_2),u(z_4),u(z_3)\big). \end{aligned}$$

Lemma XIV.3. *Let f be an antisymmetric, particle number preserving, four legged kernel over $\mathfrak{X}_\Sigma$. Then*

$$f = V_{\rm pp}(f^{\rm pp}) + V_{\rm ph}(f^{\rm ph}).$$

In §XIII we developed norms for functions on $\check{\mathcal{B}}^m \times (\mathcal{B}\times\Sigma)^n$, since we usually write Grassmann monomials in a way such that all ϕ fields stand before all ψ fields. However the "ends" of ladders have a natural ordering, and ϕ and ψ fields may be arbitrarily distributed among them. Therefore we extend some of the notation of §XIII to this situation and repeat the detailed Definition of §XVI of [FKTo3].

Definition XIV.4. *Set $\mathfrak{X}_0 = \check{\mathcal{B}}$ and $\mathfrak{X}_1 = \mathcal{B}\times\Sigma$. Let $\vec{i} = (i_1,\cdots,i_n) \in \{0,1\}^n$.*

i) The inclusions of $\mathfrak{X}_{i_j}$, $j = 1,\cdots,n$, in $\mathfrak{X}_\Sigma$ induce an inclusion of $\mathfrak{X}_{i_1}\times\cdots\times\mathfrak{X}_{i_n}$ in $\mathfrak{X}_\Sigma^n$. We identify $\mathfrak{X}_{i_1}\times\cdots\times\mathfrak{X}_{i_n}$ with its image in $\mathfrak{X}_\Sigma^n$.

ii) Set $m(\vec{i}) = n - (i_1 + \cdots + i_n)$*. Clearly,* $m(\vec{i})$ *is the number of copies of* $\check{\mathcal{B}}$ *in* $\mathfrak{X}_{i_1} \times \cdots \times \mathfrak{X}_{i_n}$*.*

iii) If f *is a function on* $\mathfrak{X}_{i_1} \times \cdots \times \mathfrak{X}_{i_n}$*, then* $\mathrm{Ord}\, f$ *is the function on* $\check{\mathcal{B}}^{m(\vec{i})} \times (\mathcal{B} \times \Sigma)^{n-m(\vec{i})}$ *obtained from* f *by shifting all of the* $\check{\mathcal{B}}$ *arguments before all of the* $\mathcal{B} \times \Sigma$ *arguments, while preserving the relative order of the* $\check{\mathcal{B}}$ *arguments and the relative order of the* $\mathcal{B} \times \Sigma$ *arguments and multiplying by the sign of the permutation that implements the reordering of the arguments. That is,* $\mathrm{Ord}\, f(x_1, \cdots, x_n) = \mathrm{sgn}\pi\, f(x_{\pi(1)}, \cdots, x_{\pi(n)})$*, where the permutation* $\pi \in S_n$ *is determined by* $\pi(j) < \pi(j')$ *if* $i_j < i_{j'}$ *or* $i_j = i_{j'}$ $j < j'$*.*

Remark XIV.5. Using the identification of Definition XIV.4.i,

$$\mathfrak{X}_\Sigma^n = \dot{\bigcup_{i_1,\cdots,i_n \in \{0,1\}}} \mathfrak{X}_{i_1} \times \cdots \times \mathfrak{X}_{i_n},$$

where, on the right-hand side we have a disjoint union. If f is a function on $\mathfrak{X}_\Sigma^n$ and $\vec{i} = (i_1, \cdots, i_n) \in \{0,1\}^n$, we denote by $f\big|_{\vec{i}}$ the restriction of f to $\mathfrak{X}_{i_1} \times \cdots \times \mathfrak{X}_{i_n}$.

Definition XIV.6. *i) We denote by* $\check{\mathcal{F}}_{n;\Sigma}$ *the set of functions on* $\mathfrak{X}_\Sigma^n$ *with the property that for each* $\vec{i} = (i_1, \cdots, i_n) \in \{0,1\}^n$ *with* $m(\vec{i}) < n$*,*

$$\mathrm{Ord}\left(f\big|_{\vec{i}}\right) \in \check{\mathcal{F}}_{m(\vec{i})}(n - m(\vec{i}); \Sigma),$$

and as such there is a function g *on* $\check{\mathcal{B}}_n$ *such that*

$$f\big|_{(0,\cdots,0)}(\check{\eta}_1, \cdots, \check{\eta}_n) = (2\pi)^3 \delta(\check{\eta}_1 + \cdots + \check{\eta}_n)\, g(\check{\eta}_1, \cdots, \check{\eta}_n).$$

ii) For a function $f \in \check{\mathcal{F}}_{n;\Sigma}$ *and a natural number* p *we set*

$$|f|^{\sim}_{p,\Sigma} = |g|^{\sim}_{p,\Sigma} + \sum_{\substack{\vec{i} \in \{0,1\}^n \\ m(\vec{i}) \neq 0}} \left|\mathrm{Ord}\left(f\big|_{\vec{i}}\right)\right|^{\sim}_{p,\Sigma},$$

$$|f|^{\sim}_{p,\Sigma,\vec{\rho}} = |g|^{\sim}_{p,\Sigma,\vec{\rho}} + \sum_{\substack{\vec{i} \in \{0,1\}^n \\ m(\vec{i}) \neq 0}} \left|\mathrm{Ord}\left(f\big|_{\vec{i}}\right)\right|^{\sim}_{p,\Sigma,\vec{\rho}},$$

where g *is the function on* $\check{\mathcal{B}}_n$ *such that*

$$f\big|_{(0,\cdots,0)}(\check{\eta}_1, \cdots, \check{\eta}_n) = (2\pi)^3 \delta(\check{\eta}_1 + \cdots + \check{\eta}_n)\, g(\check{\eta}_1, \cdots, \check{\eta}_n).$$

We extend Definition VII.2 of ladders to the case that the rungs are all defined over $\mathfrak{X}_\Sigma = \check{\mathcal{B}} \cup (\mathcal{B} \times \Sigma)$ and the bubble propagators are defined over $\mathcal{B}$.

Definition XIV.7. *Let* Σ *be a sectorization.*

i) Let P *be a bubble propagator over* $\mathcal{B}$ *and* r *a rung over* $\mathfrak{X}_\Sigma$*. We set*

$$(r \bullet P)(y_1,y_2;x_3,x_4) = \sum_{s_1',s_2' \in \Sigma} \int_{\mathcal{B}\times\mathcal{B}} dx_1' dx_2'\; r\big(y_1,y_2,(x_1',s_1'),(x_2',s_2')\big)\; P(x_1',x_2';x_3,x_4).$$

$(r \bullet P)$ is a function on $\mathfrak{X}_\Sigma^2 \times \mathcal{B}^2$. For a general function F on $\mathfrak{X}_\Sigma^2 \times \mathcal{B}^2$, define the rung $(F \bullet r)$ over $\mathfrak{X}_\Sigma$ by

$$(F \bullet r)(y_1, y_2, y_3, y_4) = \sum_{s'_1, s'_2 \in \Sigma} \int_{\mathcal{B}\times\mathcal{B}} dx'_1 dx'_2 \; F(y_1, y_2; x'_1, x'_2) \; r((x'_1, s'_1), (x'_2, s'_2), y_3, y_4)$$

if at least one of the arguments $y_1, \cdots, y_4$ lies in $\mathcal{B}\times\Sigma \subset \mathfrak{X}_\Sigma$, and for $\check\eta_1, \check\eta_2, \check\eta_3, \check\eta_4 \in \check{\mathcal{B}} \subset \mathfrak{X}_\Sigma$,

$$\begin{aligned}&(F \bullet r)(\check\eta_1, \check\eta_2, \check\eta_3, \check\eta_4)\, (2\pi)^3 \delta(\check\eta_1 + \check\eta_2 + \check\eta_3 + \check\eta_4)\\ &\quad = \sum_{s'_1, s'_2 \in \Sigma} \int_{\mathcal{B}\times\mathcal{B}} dx'_1 dx'_2 \; F(\check\eta_1, \check\eta_2; x'_1, x'_2) \; r((x'_1, s'_1), (x'_2, s'_2), \check\eta_3, \check\eta_4).\end{aligned}$$

ii) Let $\ell \ge 1$, $r_1, \cdots, r_{\ell+1}$ rungs over $\mathfrak{X}_\Sigma$ and $P_1, \cdots, P_\ell$ bubble propagators over $\mathcal{B}$. The ladder with rungs $r_1, \cdots, r_{\ell+1}$ and bubble propagators $P_1, \cdots, P_\ell$ is defined to be

$$r_1 \bullet P_1 \bullet r_2 \bullet P_2 \bullet \cdots \bullet r_\ell \bullet P_\ell \bullet r_{\ell+1}.$$

If r is a rung over $\mathfrak{X}_\Sigma$ and A, B are propagators over $\mathcal{B}$, we define $L_\ell(r; A, B)$ as the ladder with $\ell + 1$ rungs r and ℓ bubble propagators $\mathcal{C}(A, B)$.

iii) Definitions (i) and (ii) apply verbatim to the situation when creation/annihilation indices are ignored, that is, when $\mathcal{B}$ and $\mathfrak{X}_\Sigma$ are replaced by $\mathcal{B}^\updownarrow$ and $\mathfrak{Y}_\Sigma$, respectively.

Remark XIV.8. The ladder $r_1 \bullet P_1 \bullet r_2 \bullet P_2 \bullet \cdots \bullet r_\ell \bullet P_\ell \bullet r_{\ell+1}$ depends only on $r_1\big|_{\mathfrak{X}_\Sigma^2 \times (\mathcal{B}\times\Sigma)^2}$, $r_2\big|_{(\mathcal{B}\times\Sigma)^4}, \cdots, r_\ell\big|_{(\mathcal{B}\times\Sigma)^4}$ and $r_{\ell+1}\big|_{(\mathcal{B}\times\Sigma)^2\times\mathfrak{X}_\Sigma^2}$. If $r_1, \cdots, r_{\ell+1}$ are supported on $(\mathcal{B}\times\Sigma)^4 \subset \mathfrak{X}_\Sigma^4$ then Definitions VII.2.ii and XIV.7.ii agree. Also Lemma VII.5.i carries over verbatim.

The analog of Proposition VII.6 is

Proposition XIV.9. *Let $0 < \Lambda < \frac{\tau_2}{2M^j}$, where τ_2 is the constant of Lemma XIII.6 of [FKTo3]. Let $u((\xi,s), (\xi',s')) \in \mathcal{F}_0(2, \Sigma_j)$ be an antisymmetric, spin independent, particle number conserving function whose Fourier transforms obeys $|\check u(k)| \le \frac{1}{2}|\imath k_0 - e(k)|$ and such that $|u|_{1,\Sigma_j} \le \Lambda \mathfrak{c}_j$. Furthermore let $f \in \check{\mathcal{F}}_{4,\Sigma_j}$. Then for all $\ell \ge 1$,*

$$\begin{aligned}\big|L_\ell\big(f; C_u^{(j)}, C_u^{(\ge j+1)}\big)\big|_{3,\Sigma_j}^{\sim} &\le \big(\text{const } \mathfrak{c}_j\big)^\ell \; |f|_{3,\Sigma_j}^{\sim\,\ell+1},\\ \Big|V_{\rm pp}\Big(L_\ell\big(f; C_u^{(j)}, C_u^{(\ge j+1)}\big)^{\rm pp}\Big)\Big|_{3,\Sigma_j}^{\sim} &\le \big(\text{const } \mathfrak{l}_j^{1/n_0} \mathfrak{c}_j\big)^\ell \; |f|_{3,\Sigma_j}^{\sim\,\ell+1},\end{aligned}$$

if the Fermi curve F is strongly asymmetric in the sense of Definition I.10. Here, n_0 is the constant of Definition I.10.

Proof. The first inequality is a direct consequence of Remark D.8 of [FKTo3] with $X = 2\Lambda M^j \mathfrak{c}_j$ and $v' = u' = v = u$ followed by Corollary A.5.i of [FKTo1] with $X = \tau_2$, $\mu = 1$. If we set $\mathcal{C} = \mathcal{C}\big(C_u^{(j)}, C_u^{(\ge j+1)}\big)$, then, by Lemma VII.5.i and Remark XIV.8,

$$L_\ell\big(f; C_u^{(j)}, C_u^{(\ge j+1)}\big)^{\rm pp} = f^{\rm pp} \bullet {}^{\rm pp}\mathcal{C} \bullet \cdots \bullet {}^{\rm pp}\mathcal{C} \bullet f^{\rm pp}.$$

Thus the second inequality follows from Theorem XXII.7 of [FKTo4]. □

Remark XIV.10. By Remarks XIII.13 and XIV.8, Proposition VII.6 is a special case of Proposition XIV.9.

Definition VII.7 and Theorem VII.8 carry over almost verbatim to the present situation.

Definition XIV.11. *Let* $\vec{F} = \big\{ F^{(i)} \bigm| i = 2, 3, \cdots \big\}$ *be a family of antisymmetric functions in* $\check{\mathcal{F}}_{4,\Sigma_i}$. *Let* $\vec{p} = \big(p^{(2)}, p^{(3)}, \cdots\big)$ *be a sequence of antisymmetric, spin independent, particle number conserving functions* $p^{(i)}((\xi,s),(\xi',s')) \in \mathcal{F}_0(2, \Sigma_i)$. *We define, recursively on* $0 \le j < \infty$, *the iterated particle hole (or wrong way) ladders up to scale* j, *denoted by* $\mathcal{L}^{(j)}(\vec{p}, \vec{F})$, *as*

$$\mathcal{L}^{(0)}(\vec{p}, \vec{F}) = 0$$

$$\mathcal{L}^{(j+1)}(\vec{p}, \vec{F}) = \mathcal{L}^{(j)}(\vec{p}, \vec{F})_{\Sigma_j} + 2 \sum_{\ell=1}^{\infty} (-1)^{\ell}(12)^{\ell+1} L_\ell\big(w_j; C_{u_j}^{(j)}, C_{u_j}^{(\ge j+1)}\big)^{\mathrm{ph}}$$

where $u_j = \sum_{i=2}^{j-1} p_{\Sigma_j}^{(i)}$ *and* $w_j = \sum_{i=2}^{j} F_{\Sigma_j}^{(i)} + \frac{1}{8}\mathrm{Ant}\Big(V_{\mathrm{ph}}\big(\mathcal{L}^{(j)}(\vec{p}, \vec{F})\big)\Big)_{\Sigma_j}$.

Theorem XIV.12. *For every* $\varepsilon > 0$ *there are constants* ρ_0, const *such that the following holds. Let* $\vec{F} = \big(F^{(2)}, F^{(3)}, \cdots\big)$ *be a sequence of antisymmetric, spin independent, particle number conserving functions* $F^{(i)} \in \check{\mathcal{F}}_{4,\Sigma_i}$ *and* $\vec{p} = \big(p^{(2)}, p^{(3)}, \cdots\big)$ *be a sequence of antisymmetric, spin independent, particle number conserving functions* $p^{(i)} \in \mathcal{F}_0(2, \Sigma_i)$. *Assume that there is* $\rho \le \rho_0$ *such that for* $i \ge 2$,

$$|F^{(i)}|_{3,\Sigma_i}^{\sim} \le \tfrac{\rho}{M^{\varepsilon i}}\, \mathfrak{c}_i \qquad |p^{(i)}|_{1,\Sigma_i} \le \tfrac{\rho\, \mathfrak{l}_i}{M^i}\, \mathfrak{c}_i \qquad \check{p}^{(i)}(0, \mathbf{k}) = 0.$$

Then for all $j \ge 2$,

$$\big|V_{\mathrm{ph}}\big(\mathcal{L}^{(j)}(\vec{p}, \vec{F})_{\Sigma_j}\big)\big|_{3,\Sigma_j}^{\sim} \le \mathrm{const}\, \rho^2\, \mathfrak{c}_j.$$

Theorem XIV.12 follows from Theorem D.2 below. Theorem D.2, in turn, is proven in [FKTl].

Remark XIV.13. With the same argument as in Remark XIV.10, Theorem VII.8 is a special case of Theorem XIV.12 .

Theorem XIV.12 is not general enough for controlling the effect of a renormalization group step on the $|\cdot|_{3,\Sigma_j}^{\sim}$ norms. Consider an iterated particle–hole ladder $\mathcal{L}^{(j)}(\vec{p}, \vec{f})$. Integrating out subsequent scales can result in a propagator and a ϕ field being hooked to the ψ legs of the ladder. See the $\psi + C_{u,\Sigma_j}^{(j)} J\phi$ in Remark IX.6.iii. The resulting object is no longer an iterated particle–hole ladder. In §IX, this was harmless because "external improvement" with respect to the $|\cdot|_{3,\Sigma_j}$ norm (see Lemma XII.19 of [FKTo3]) led to bounds that were summable over scales. With the more sensitive norm $|\cdot|_{3,\Sigma_j}^{\sim}$, we have to control the "shear" from ψ to ϕ fields in particle–hole ladders. Under a shear transformation, a Grassmann function $\mathcal{W}(\phi, \psi)$ is mapped to $\mathcal{W}(\phi, \psi + \hat{B}\phi)$.

Definition XIV.14. *Let* Σ *be a sectorization,* $B(k)$ *a function on* $\mathbb{R} \times \mathbb{R}^2$ *and* $f \in \check{\mathcal{F}}_{n;\Sigma}$.

i) The **shear** *of f with respect to B is the element* $\mathrm{shear}(f,B)\in\check{\mathcal{F}}_{n;\Sigma}$ *defined by*

$$\mathrm{shear}(f,B)\big|_{\vec{i}}(y_1,\cdots,y_n)=\sum_{\substack{\vec{j}\in\{0,1\}^n\\ j_p\ge i_p,\ 1\le p\le n}}\ \prod_{\substack{1\le\nu\le n\\ i_\nu=0,\ j_\nu=1}}\Big\{\sum_{s_\nu\in\Sigma}\int_{\mathcal{B}}d\xi_\nu\ E_+(\check{\eta}_\nu,\xi_\nu)B(k_\nu)\Big\}$$
$$\times f\big|_{\vec{j}}(z_1,\cdots,z_n)\Big|_{\substack{z_\nu=(\xi_\nu,s_\nu)\ \text{if}\ i_\nu=0,\ j_\nu=1\\ z_\nu=y_\nu\quad\ \text{if}\ i_\nu=j_\nu,}},$$

where, for $i_\nu=0$, $y_\nu=\check{\eta}_\nu=(k_\nu,\sigma_\nu,a_\nu)\in\check{\mathcal{B}}$ and E_+ was defined before Definition VI.1.

ii) We use $Gr(\phi,\psi;f)$ to denote the Grassmann function with kernel f. That is,

$$Gr(\phi,\psi;f)=\sum_{\vec{i}\in\{0,1\}^n}\int dy_1\cdots dy_n\ \hat{f}\big|_{\vec{i}}(y_1,\cdots,y_n)\prod_{p=1}^{n}\begin{cases}\phi(y_p) & \text{if}\,i_p=0\\ \psi(y_p) & \text{if}\,i_p=1\end{cases},$$

where factors in the product are in the order specified by the index p, $\hat{f}$ is the Fourier transform of f with respect to its ϕ arguments and y_ν runs over $\mathcal{B}$ when $\nu=0$ and over $\mathcal{B}\times\Sigma$ when $\nu=1$.

The definition of shear has been chosen so that

$$Gr(\phi,\psi;\mathrm{shear}(f,B))=Gr(\phi,\psi+\hat{B}\phi;f)\tag{XIV.1}$$

where, with some abuse of notation, we set $(\hat{B}\phi)(\xi,s)=\int_{\mathcal{B}}d\xi'\ \hat{B}(\xi,\xi')\phi(\xi')$, for all $s\in\Sigma$, retaining the $\hat{B}$ defined in Notation V.4.

Corollary XIV.15 (to Theorem XIV.12). *For every $\varepsilon>0$ and $c_B>0$ there are constants ρ_0,* const *such that the following holds. Let $\vec{v}=\big(v^{(2)},v^{(3)},\cdots\big)$ be a sequence of antisymmetric, spin independent, particle number conserving functions $v^{(i)}\in\check{\mathcal{F}}_{4,\Sigma_i}$ and $\vec{p}=\big(p^{(2)},p^{(3)},\cdots\big)$ be a sequence of antisymmetric, spin independent, particle number conserving functions $p^{(i)}\in\mathcal{F}_0(2,\Sigma_i)$. Assume that there is $\rho\le\rho_0$ such that for $i\ge 2$,*

$$|v^{(i)}|^{\sim}_{3,\Sigma_i}\le\tfrac{\rho}{M^{\varepsilon i}}\mathfrak{c}_i,\qquad |p^{(i)}|_{1,\Sigma_i}\le\tfrac{\rho\,\mathfrak{l}_i}{M^i}\mathfrak{c}_i,\qquad \check{p}^{(i)}(0,\mathbf{k})=0.$$

Let $B(k)$ be a function obeying $\|B(k)\|^{\sim}\le c_B\mathfrak{c}_j$, set $f^{(i)}=\mathrm{shear}\big(v^{(i)},\nu^{(\ge i)}B\big)\in\check{\mathcal{F}}_{4,\Sigma_i}$ and let $\vec{f}=\big(f^{(2)},f^{(3)},\cdots\big)$.

i) For all $j\ge 2$,

$$\big|V_{\mathrm{ph}}\big(\mathcal{L}^{(j)}(\vec{p},\vec{f})_{\Sigma_j}\big)\big|^{\sim}_{3,\Sigma_j}\le\mathrm{const}\,\rho^2\,\mathfrak{c}_j.$$

ii) Let $B'(k)$ obey $\|B'(k)\|^{\sim}\le c'c_B\mathfrak{c}_j$, set $f_s^{(i)}=\mathrm{shear}\big(v^{(i)},\nu^{(\ge i)}(B+sB')\big)\in\check{\mathcal{F}}_{4,\Sigma_i}$ and let $\vec{f}_s=\big(f_s^{(2)},f_s^{(3)},\cdots\big)$. For all $j\ge 2$ and all $c'>0$,

$$\big|\tfrac{d}{ds}V_{\mathrm{ph}}\big(\mathcal{L}^{(j)}(\vec{p},\vec{f}_s)_{\Sigma_j}\big)\big|_{s=0}\big|^{\sim}_{3,\Sigma_j}\le\mathrm{const}\,c'\rho^2\,\mathfrak{c}_j.$$

In the proof, which follows Remark XIV.20, we will use auxiliary external fields, named ϕ'. We now extend the notation of Definitions XIV.4 and XIV.6 to include them.

Definition XIV.16. *Let Σ be a sectorization. Set $\mathfrak{X}_{-1} = \mathfrak{X}_0 = \check{\mathcal{B}}$, $\mathfrak{X}_1 = \mathcal{B} \times \Sigma$ and $\mathfrak{X}'_\Sigma = \mathfrak{X}_{-1} \cup \mathfrak{X}_0 \cup \mathfrak{X}_1$. Let $\vec{i} = (i_1, \cdots, i_n) \in \{-1, 0, 1\}^n$.*

i) The inclusions of $\mathfrak{X}_{i_j}$, $j = 1, \cdots, n$, in $\mathfrak{X}'_\Sigma$ induce an inclusion of $\mathfrak{X}_{i_1} \times \cdots \times \mathfrak{X}_{i_n}$ in $\mathfrak{X}'^n_\Sigma$. We identify $\mathfrak{X}_{i_1} \times \cdots \times \mathfrak{X}_{i_n}$ with its image in $\mathfrak{X}'^n_\Sigma$.

ii) Set $m'(\vec{i}) = \#\big\{ 1 \le j \le n \bigm| i_j = -1 \big\}$ and $m(\vec{i}) = \#\big\{ 1 \le j \le n \bigm| i_j = 0 \big\}$.

iii) If f is a function on $\mathfrak{X}_{i_1} \times \cdots \times \mathfrak{X}_{i_n}$, then $\mathrm{Ord}\, f$ is the function on $\check{\mathcal{B}}^{m'(\vec{i})} \times \check{\mathcal{B}}^{m(\vec{i})} \times (\mathcal{B} \times \Sigma)^{n-m'(\vec{i})-m(\vec{i})}$ obtained from f by permuting the arguments so that all $\mathfrak{X}_{-1}$ arguments appear before all $\mathfrak{X}_0$ arguments and all $\mathfrak{X}_0$ arguments appear before all $\mathfrak{X}_1$ arguments, while preserving the relative order of the $\mathfrak{X}_j$ arguments, $j = -1, 0, 1$, and multiplying by the sign of the permutation that implements the reordering of the arguments. That is, $\mathrm{Ord}\, f(x_1, \cdots, x_n) = \mathrm{sgn}\,\sigma\ f(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$, where the permutation $\sigma \in S_n$ is determined by $\sigma(j) < \sigma(j')$ if $i_j < i_{j'}$ or $i_j = i_{j'}$, $j < j'$.

iv) Using the identification of part (i),

$$\mathfrak{X}'^n_\Sigma = \dot{\bigcup_{i_1, \cdots, i_n \in \{-1,0,1\}}} \mathfrak{X}_{i_1} \times \cdots \times \mathfrak{X}_{i_n},$$

where, on the right-hand side we have a disjoint union. If f is a function on $\mathfrak{X}'^n_\Sigma$ and $\vec{i} = (i_1, \cdots, i_n) \in \{-1, 0, 1\}^n$, we denote by $f\big|_{\vec{i}}$ the restriction of f to $\mathfrak{X}_{i_1} \times \cdots \times \mathfrak{X}_{i_n}$.

Definition XIV.17. *Let Σ be a sectorization and $m', m, n \ge 0$.*

i) For $n \ge 1$, denote by $\check{\mathcal{F}}_{m',m}(n; \Sigma)$ the space of all translation invariant, complex valued functions $f(\check{\eta}_1, \cdots, \check{\eta}_{m'+m}; (\xi_1, s_1), \cdots, (\xi_n, s_n))$ on $\mathfrak{X}_{-1}^{m'} \times \mathfrak{X}_0^m \times \big(\mathcal{B} \times \Sigma\big)^n$ whose Fourier transform $\check{f}(\check{\eta}_1, \cdots, \check{\eta}_m; (\check{\xi}_1, s_1), \cdots, (\check{\xi}_n, s_n))$ vanishes unless $k_i \in \tilde{s}_i$ for all $1 \le j \le n$. Here, $\check{\xi}_i = (k_i, \sigma_i, a_i)$. Also, let $\check{\mathcal{F}}_{m',m}(0; \Sigma)$ be the space of all momentum conserving, complex valued functions $f(\check{\eta}_1, \cdots, \check{\eta}_{m'+m})$ on $\mathfrak{X}_{-1}^{m'} \times \mathfrak{X}_0^m$.

ii) We denote by $\check{\mathcal{F}}'_{n;\Sigma}$ the set of functions on $\mathfrak{X}'^n_\Sigma$ with the property that for each $\vec{i} = (i_1, \cdots, i_n) \in \{-1, 0, 1\}^n$ with $m'(\vec{i}) + m(\vec{i}) < n$,

$$\mathrm{Ord}\,\big(f\big|_{\vec{i}}\big) \in \check{\mathcal{F}}_{m'(\vec{i}),m(\vec{i})}(n - m'(\vec{i}) - m(\vec{i}); \Sigma),$$

and such that for each $\vec{i} = (i_1, \cdots, i_n) \in \{-1, 0, 1\}^n$ with $m'(\vec{i}) + m(\vec{i}) = n$ there is a function $g_{m',m} \in \check{\mathcal{F}}_{m',m}(0; \Sigma)$ such that

$$\mathrm{Ord}\,\big(f\big|_{\vec{i}}\big)(\check{\eta}_1, \cdots, \check{\eta}_n) = (2\pi)^3 \delta(\check{\eta}_1 + \cdots + \check{\eta}_n)\, g_{m',m}(\check{\eta}_1, \cdots, \check{\eta}_n).$$

iii) There is a natural identification $\Pi : \check{\mathcal{F}}_{m',m}(n; \Sigma) \to \check{\mathcal{F}}_{m'+m}(n; \Sigma)$ obtained by identifying $\mathfrak{X}_{-1}^{m'} \times \mathfrak{X}_0^m = \check{\mathcal{B}}^{m'} \times \check{\mathcal{B}}^m$ with $\mathfrak{X}_0^{m'+m} = \check{\mathcal{B}}^{m'+m}$. Similarly, if $\vec{i} = (i_1, \cdots, i_n) \in \{-1, 0, 1\}^n$ and f is a function on $\mathfrak{X}_{i_1} \times \cdots \times \mathfrak{X}_{i_n}$ then the function $\Pi(f)$ on $\mathfrak{X}_{\pi(i_1)} \times \cdots \times \mathfrak{X}_{\pi(i_n)}$, where $\pi(-1) = \pi(0) = 0$ and $\pi(1) = 1$, is obtained by identifying $\mathfrak{X}_{-1}$ with $\mathfrak{X}_0$. We extend the map to $\Pi : \check{\mathcal{F}}'_{n;\Sigma} \to \check{\mathcal{F}}_{n;\Sigma}$ by

$$\Pi(f)\big|_{\vec{i}} = \sum_{\substack{\vec{j} \in \{-1,0,1\}^n \\ \pi(\vec{j}) = \vec{i}}} \Pi\big(f\big|_{\vec{j}}\big) \qquad \text{for all } \vec{i} \in \{0, 1\}.$$

iv) For a function $f \in \check{\mathcal{F}}_{m',m}(n;\Sigma)$ and a natural number p we set

$$|f|^{\tilde{}}_{p,\Sigma} = |\Pi(f)|^{\tilde{}}_{p,\Sigma}.$$

For a function $f \in \check{\mathcal{F}}'_{n;\Sigma}$ and a natural number p we set

$$|f|^{\tilde{}}_{p,\Sigma} = \sum_{\vec{i}\in\{-1,0,1\}^n} \big|\Pi\big(f\big|_{\vec{i}}\big)\big|^{\tilde{}}_{p,\Sigma}.$$

Lemma XIV.18. *For $\vec{\kappa} \in \mathbb{C}^n$, define $S_{\vec{\kappa}} : \check{\mathcal{F}}'_{n,\Sigma} \to \check{\mathcal{F}}'_{n,\Sigma}$ by*

$$\big(S_{\vec{\kappa}} f\big)\big|_{\vec{i}} = \Big(\prod_{j=1}^n \kappa_j^{1-i_j}\Big) f\big|_{\vec{i}} \qquad \text{for all } \vec{i} \in \{-1,0,1\}^n.$$

Then, for all $f \in \check{\mathcal{F}}'_{n,\Sigma}$,

$$|\Pi(f)|^{\tilde{}}_{p,\Sigma} \le |f|^{\tilde{}}_{p,\Sigma} \le 3^n \sup_{\substack{\vec{\kappa}\in\mathbb{C}^n \\ |\kappa_j|\le 1,\ 1\le j\le n}} |\Pi(S_{\vec{\kappa}} f)|^{\tilde{}}_{p,\Sigma}.$$

Proof. The first bound is just the triangle inequality,

$$|\Pi(f)|^{\tilde{}}_{p,\Sigma} = \sum_{\vec{i}\in\{0,1\}^n} \big|\Pi(f)\big|_{\vec{i}}\big|^{\tilde{}}_{p,\Sigma}. \le \sum_{\vec{i}\in\{0,1\}^n} \sum_{\substack{\vec{j}\in\{-1,0,1\}^n \\ \pi(\vec{j})=\vec{i}}} \big|\Pi\big(f\big|_{\vec{j}}\big)\big|^{\tilde{}}_{p,\Sigma} = |f|^{\tilde{}}_{p,\Sigma}.$$

For the second bound, we just use the Cauchy integral formula

$$f\big|_{\vec{i}} = \Big(\prod_{j=1}^n \tfrac{1}{(1-i_j)!} \tfrac{d^{1-i_j}}{d\kappa_j^{1-i_j}}\Big) S_{\vec{\kappa}} f\Big|_{\vec{\kappa}=\vec{0}} = \Big(\prod_{j=1}^n \oint_{|\kappa_j|=1} \tfrac{d\kappa_j}{2\pi\imath} \tfrac{1}{\kappa_j^{2-i_j}}\Big) S_{\vec{\kappa}} f$$

to prove that, for each $\vec{i} \in \{-1,0,1\}^n$,

$$\big|f\big|_{\vec{i}}\big|^{\tilde{}}_{p,\Sigma} = \big|\Pi\big(f\big|_{\vec{i}}\big)\big|^{\tilde{}}_{p,\Sigma} \le \sup_{\substack{\vec{\kappa}\in\mathbb{C}^n \\ |\kappa_j|\le 1,\ 1\le j\le n}} |\Pi(S_{\vec{\kappa}} f)|^{\tilde{}}_{p,\Sigma}$$

and then sum over $\vec{i} \in \{-1,0,1\}^n$. □

Let $B(k)$ be a kernel which is the product of two other kernels $B_1(k)$ and $B_2(k)$. Then, for any Grassmann function $\mathcal{W}(\phi,\psi)$,

$$\mathcal{W}(\phi,\psi+\hat{B}\phi) = \mathcal{W}(\phi,\psi+\hat{B}_1\phi')\big|_{\phi'=\hat{B}_2\phi}.$$

Thus the shear transformation with respect to B may be written as the composition of another shear–like transformation with respect to B_1 and a “scaling transformation” with respect to B_2. To make this precise, we have

Definition XIV.19. *Let Σ be a sectorization and $B(k)$ a function on $\mathbb{R}\times\mathbb{R}^2$.*

i) If $f \in \check{\mathcal{F}}_{n;\Sigma}$*, then the element* $\mathrm{shear}'(f,B) \in \check{\mathcal{F}}'_{n;\Sigma}$ *is defined by*

$$\mathrm{shear}'(f,B)\Big|_{\vec{i}}(y_1,\cdots,y_n) = \prod_{\substack{1\le\nu\le n\\ i_\nu=-1}} \Big\{ \sum_{s_\nu\in\Sigma} \int_{\mathcal{B}} d\xi_\nu\ E_+(\check{\eta}'_\nu,\xi_\nu) B(k'_\nu) \Big\} \times f\Big|_{|\vec{i}|}(z_1,\cdots,z_n)\Big|_{\substack{z_\nu=(\xi_\nu,s_\nu) \text{ if } i_\nu=-1\\ z_\nu=y_\nu \text{ if } i_\nu\ne-1}},$$

where $|\vec{i}| = (|i_1|,\cdots,|i_n|)$ *and* $y_\nu = \check{\eta}'_\nu = (k'_\nu,\sigma_\nu,a_\nu) \in \check{\mathcal{B}}$ *when* $i_\nu = -1$.

ii) If $f \in \check{\mathcal{F}}'_{n;\Sigma}$*, then the element* $\mathrm{sct}'(f,B) \in \check{\mathcal{F}}'_{n;\Sigma}$ *is defined by*

$$\mathrm{sct}'(f,B)\Big|_{\vec{i}}(y_1,\cdots,y_n) = \Big\{ \prod_{\substack{1\le\nu\le n\\ i_\nu=-1}} B(k'_\nu) \Big\} f\Big|_{\vec{i}}(y_1,\cdots,y_n),$$

where $y_\nu = (k'_\nu,\sigma_\nu,a_\nu) \in \check{\mathcal{B}}$ *if* $i_\nu = -1$.

iii) If $f \in \check{\mathcal{F}}_{n;\Sigma}$*, then the element* $\mathrm{sct}(f,B) \in \check{\mathcal{F}}_{n;\Sigma}$ *is defined by*

$$\mathrm{sct}(f,B)\Big|_{\vec{i}}(y_1,\cdots,y_n) = \Big\{ \prod_{\substack{1\le\nu\le n\\ i_\nu=0}} B(k_\nu) \Big\} f\Big|_{\vec{i}}(y_1,\cdots,y_n),$$

where $y_\nu = (k_\nu,\sigma_\nu,a_\nu) \in \check{\mathcal{B}}$ *if* $i_\nu = 0$.

Remark XIV.20. i) If $B_1(k)$ and $B_2(k)$ are functions on $\mathbb{R}\times\mathbb{R}^2$ and $f \in \check{\mathcal{F}}_{n;\Sigma}$, then

$$\mathrm{shear}(f,B_1B_2) = \Pi\Big(\mathrm{sct}'\big(\mathrm{shear}'(f,B_1),B_2\big)\Big).$$

ii) Let $f \in \check{\mathcal{F}}_{n;\Sigma_i}$, $f' \in \check{\mathcal{F}}'_{n;\Sigma_i}$ and $B(k)$ be a function obeying $\|B(k)\|^{\tilde{}} \le c_B\mathfrak{c}_i$. Then, by repeated application of Eq. (XVII.3) of [FKTo3], with j replaced by i, $X=0$, $X_B=1$, there is a constant const, depending on n, such that

$$\begin{aligned} |\mathrm{shear}'(f,B)|^{\tilde{}}_{p,\Sigma_i} &\le \mathrm{const}\ \max\{1,c_B\}^n\mathfrak{c}_i\ |f|^{\tilde{}}_{p,\Sigma_i},\\ |\mathrm{sct}(f,B)|^{\tilde{}}_{p,\Sigma_i} &\le \mathrm{const}\ \max\{1,c_B\}^n\mathfrak{c}_i\ |f|^{\tilde{}}_{p,\Sigma_i},\\ |\mathrm{sct}'(f',B)|^{\tilde{}}_{p,\Sigma_i} &\le \mathrm{const}\ \max\{1,c_B\}^n\mathfrak{c}_i\ |f'|^{\tilde{}}_{p,\Sigma_i}. \end{aligned}$$

Proof of Corollary XIV.15. i) Let $\tilde{v}^{(i)} = \mathrm{shear}'(v^{(i)}, \nu^{(\ge i)})$. Then, by Remark XIV.20.i,

$$f^{(i)} = \Pi\big(\mathrm{sct}'(\tilde{v}^{(i)},B)\big)$$

and, extending the definition of $\mathcal{L}^{(j)}(\vec{p},\vec{F})$ to $F^{(i)} \in \check{\mathcal{F}}'_{n;\Sigma}$ in the obvious way (just replace $\check{\mathcal{B}}$ by $\check{\mathcal{B}} \cup \check{\mathcal{B}}$ in Definitions XIV.7 and XIV.11),

$$\begin{aligned} V_{\mathrm{ph}}\big(\mathcal{L}^{(j)}(\vec{p},\vec{f}\,)_{\Sigma_j}\big) &= V_{\mathrm{ph}}\Big(\mathcal{L}^{(j)}\big(\vec{p},\Pi(\mathrm{sct}'(\vec{\tilde{v}},B))\big)_{\Sigma_j}\Big)\\ &= \Pi\Big(\mathrm{sct}'\Big(V_{\mathrm{ph}}\big(\mathcal{L}^{(j)}\big(\vec{p},\vec{\tilde{v}}\,)\big)_{\Sigma_j},B\Big)\Big). \end{aligned}$$

Hence, by Lemma XIV.18 and Remark XIV.20.ii,

$$\begin{aligned}\big|V_{\rm ph}\big(\mathcal{L}^{(j)}(\vec p,\vec f\,)_{\Sigma_j}\big)\big|^{\sim}_{3,\Sigma_j} &\le \Big|\mathrm{sct}'\Big(V_{\rm ph}\big(\mathcal{L}^{(j)}\big(\vec p,\vec{\tilde v}\,\big)\big)_{\Sigma_j},B\Big)\Big|^{\sim}_{3,\Sigma_j}\\ &\le const\ \mathfrak{c}_j\ \big|V_{\rm ph}\big(\mathcal{L}^{(j)}\big(\vec p,\vec{\tilde v}\,\big)\big)_{\Sigma_j}\big|^{\sim}_{3,\Sigma_j}\\ &\le const\ \mathfrak{c}_j \sup_{\substack{\vec\kappa\in\mathbb{C}^4\\ |\kappa_\nu|\le1,\ 1\le\nu\le4}} \Big|\Pi\Big(S_{\vec\kappa}V_{\rm ph}\big(\mathcal{L}^{(j)}\big(\vec p,\vec{\tilde v}\,\big)\big)_{\Sigma_j}\Big)\Big|^{\sim}_{3,\Sigma_j}.\end{aligned}$$

Observe that

$$\Pi\Big(S_{\vec\kappa}\ V_{\rm ph}\big(\mathcal{L}^{(j)}\big(\vec p,\vec{\tilde v}\,\big)\big)_{\Sigma_j}\Big) = V_{\rm ph}\Big(\mathcal{L}^{(j)}\big(\vec p,\Pi(S_{\vec\kappa}\vec{\tilde v}\,)\big)\Big)_{\Sigma_j}.$$

By Definition I.2.ii, $\|v^{(\ge i)}(k)\|^{\sim}\le const\ \mathfrak{c}_i$. Hence, by Lemma XIV.18 and Remark XIV.20.ii,

$$\begin{aligned}\sup_{\substack{\vec\kappa\in\mathbb{C}^4\\ |\kappa_\nu|\le1,\ 1\le\nu\le4}}\big|\Pi\big(S_{\vec\kappa}\tilde v^{(i)}\big)\big|^{\sim}_{3,\Sigma_i} &\le \sup_{\substack{\vec\kappa\in\mathbb{C}^4\\ |\kappa_\nu|\le1,\ 1\le\nu\le4}}\big|S_{\vec\kappa}\tilde v^{(i)}\big|^{\sim}_{3,\Sigma_i}\\ &\le \big|\tilde v^{(i)}\big|^{\sim}_{3,\Sigma_i}\le const\ \mathfrak{c}_i\big|v^{(i)}\big|^{\sim}_{3,\Sigma_i}\le const\ \tfrac{\rho}{M^{\varepsilon i}}\mathfrak{c}_i^2\\ &\le const\ \tfrac{\rho}{M^{\varepsilon i}}\mathfrak{c}_i.\end{aligned}$$

So Theorem XIV.12 gives

$$\begin{aligned}\big|V_{\rm ph}\big(\mathcal{L}^{(j)}(\vec p,\vec f\,)_{\Sigma_j}\big)\big|^{\sim}_{3,\Sigma_j} &\le const\ \mathfrak{c}_j \sup_{\substack{\vec\kappa\in\mathbb{C}^4\\ |\kappa_\nu|\le1,\ 1\le\nu\le4}}\Big|V_{\rm ph}\Big(\mathcal{L}^{(j)}\big(\vec p,\Pi(S_{\vec\kappa}\vec{\tilde v})\big)\Big)_{\Sigma_j}\Big|^{\sim}_{3,\Sigma_j}\\ &\le \mathrm{const}\ \rho^2\ \mathfrak{c}_j.\end{aligned}$$

ii) Part (i), with c_B replaced by $2c_B$ and B replaced by $B+sB'$, implies that, for all $s\in\mathbb{C}$ obeying $|s|\le\frac{1}{c'}$,

$$\big|V_{\rm ph}\big(\mathcal{L}^{(j)}(\vec p,\vec f_s\,)_{\Sigma_j}\big)\big|^{\sim}_{3,\Sigma_j}\le \mathrm{const}\ \rho^2\ \mathfrak{c}_j.$$

Since $V_{\rm ph}\big(\mathcal{L}^{(j)}(\vec p,\vec f_s\,)_{\Sigma_j}$ is a polynomial in s, the desired bound now follows from the Cauchy integral formula

$$\tfrac{d}{ds}V_{\rm ph}\big(\mathcal{L}^{(j)}(\vec p,\vec f_s\,)_{\Sigma_j}\big)\big|_{s=0}=\tfrac{1}{2\pi i}\oint_{|s|=\frac{1}{c'}}ds\ \tfrac{1}{s^2}V_{\rm ph}\big(\mathcal{L}^{(j)}(\vec p,\vec f_s\,)_{\Sigma_j}.$$

XV. Recursion Step for Momentum Green's Functions

This section provides the analog of §IX for the $|\ \ |^{\sim}$–norms. Recall, from §XI, that we are assuming that the interaction V satisfies the reality condition (I.1) and is bar/unbar exchange invariant in the sense of (I.2).

1. More Input and Output Data. We now supplement the conditions on the input and output data of Definitions IX.1 and IX.2 in order get more detailed information on the behaviour of the two– and four–point functions. Recall from Theorem XII.1 that $\frac{1}{2} < \aleph < \aleph' < \frac{2}{3}$. We generalize the notation $\mathfrak{c}_j$ and $\mathfrak{e}_j(X)$ of Definition V.2 to

$$\begin{aligned}\mathfrak{c}_{i,j} &= \sum_{\substack{|\boldsymbol{\delta}|\le r\\ |\delta_0|\le r_0}} M^{i\delta_0} M^{j|\boldsymbol{\delta}|}\, t^{\delta} + \sum_{\substack{|\boldsymbol{\delta}|> r\\ \text{or }|\delta_0|> r_0}} \infty\, t^{\delta} \in \mathfrak{N}_{d+1},\\ \mathfrak{e}_{i,j}(X) &= \tfrac{\mathfrak{c}_{i,j}}{1-M^j X},\end{aligned} \tag{XV.1}$$

so that we can track different degrees of smoothness in temporal and spatial directions.

Definition XV.1 (More Input Data). *Let $\tilde{\mathcal{D}}_{\rm in}^{(j)}$ be the set of interaction quadruples, $(\mathcal{W}, \mathcal{G}, u, \vec{p})$, that fulfill Definitions VIII.1, IX.1 and the following. Let $w(\phi,\psi;K)$ be the Σ_j–sectorized representative of $\mathcal{W}(K)$ specified in (I1) and $w^a(\phi,\psi;K)$ its amputation in the sense of Definition XIII.9.*

(I$^\sim$1) The Grassmann function $w^a(\phi,\psi;K) - w^a(0,\psi;K)$ vanishes unless $\hat{\nu}^{(<j)}\phi$ is nonzero and

$$N_j^\sim\big(w^a(K), 64\alpha, \|K\|_{1,\Sigma_j}\big) \le \mathfrak{e}_j\big(\|K\|_{1,\Sigma_j}\big),$$
$$N_j^\sim\big(\tfrac{d}{ds} w^a(K+sK')\big|_{s=0}, 64\alpha,\ \|K\|_{1,\Sigma_j}\big) \le M^j\ \mathfrak{e}_j\big(\|K\|_{1,\Sigma_j}\big)\, \|K'\|_{1,\Sigma_j}$$

for all $K \in \mathfrak{K}_j$ and all K'.

(I$^\sim$2) There is a family $\vec{v}$ of antisymmetric kernels

$$v^{(i)} \in \check{\mathcal{F}}_{4,\Sigma_i}, \qquad 2 \le i \le j-1$$

(independent of K) and an antisymmetric kernel $\delta f^{(j)}(K) \in \check{\mathcal{F}}_{4,\Sigma_j}$ obeying

$$\begin{aligned}\big|v^{(i)}\big|^\sim_{3,\Sigma_i,\tilde\rho} &\le \tfrac{\mathfrak{l}_i^{1/n_0}}{\alpha^7}\,\mathfrak{c}_i \qquad \text{for all } 2 \le i \le j-1,\\ \big|\delta f^{(j)}(0)\big|^\sim_{3,\Sigma_j,\tilde\rho} &\le \tfrac{\mathfrak{l}_j^{1/n_0}}{\alpha^7}\,\mathfrak{c}_j,\\ \big|\tfrac{d}{ds}\delta f^{(j)}(K+sK')\big|_{s=0}\big|^\sim_{3,\Sigma_j,\tilde\rho} &\le \tfrac{1}{64\alpha^4 {\rm B}^2}\,\mathfrak{e}_j(\|K\|_{1,\Sigma_j}) M^j \|K'\|_{1,\Sigma_j}\end{aligned}$$

for all $K \in \mathfrak{K}_j$, from which the quartic parts of w^a and $\mathcal{G}^a$ are built as follows. Let $f^{(i)} = \operatorname{shear}\big(v^{(i)}, C_{u(0)}^{[i,j)}(k)A(k)\big) \in \check{\mathcal{F}}_{4,\Sigma_i}$, $2 \le i \le j-1$, where $\operatorname{shear}(f,B)$ was defined in Definition XIV.14. Then the kernel of the quartic part of $w^a + \mathcal{G}^a$ is

$$\delta f^{(j)}(K) + \sum_{i=2}^{j-1} f^{(i)}_{\Sigma_j} + \tfrac{1}{8}\operatorname{Ant}\Big(V_{\rm ph}\big(\mathcal{L}^{(j)}(\vec{p},\vec{f})\big)\Big)_{\Sigma_j},$$

where the particle–hole value $V_{\rm ph}$ was defined in Definition XIV.2. The kernel $v^{(i)}$ vanishes unless all of its ϕ momenta are in the support of $\nu^{(<i)}$.

($I^{\sim}3$) Let $\int d\eta_1 d\eta_2\, G_2(\eta_1,\eta_2;K)\,\phi(\eta_1)\phi(\eta_2)$ be the part of $\mathcal{G}(K)$ that is homogeneous of degree two. Then $2\check{G}_2(k;K)$ has the decomposition

$$2\check{G}_2(k;K) = C^{(<j)}_{u(0)}(k) + \tfrac{1}{(ik_0-e(\mathbf{k}))^2}\sum_{i=2}^{j-1}\Big\{\delta q^{(i)}(k;K) + \sum_{\ell=i}^{j} q^{(i,\ell)}(k)\Big\}$$

with $q^{(i,\ell)}(k)$, $i\le\ell\le j$ and $\delta q^{(i)}(k;K)$ vanishing in the $(i+2)^{\rm nd}$ neighbourhood and when $\mathbf{k}$ is not in the support of $U(\mathbf{k})$, $\delta q^{(i)}(k;0)=0$ for $2\le i\le j-1$ and

$$\big\|q^{(i,\ell)}(k)\big\|^{\tilde{}} \le \lambda_0^{1-2\upsilon}\,\tfrac{\mathfrak{l}_\ell}{M^\ell}\,M^{\aleph'(\ell-i)}\mathfrak{c}_{i,\ell},$$
$$\big\|\tfrac{d}{ds}\delta q^{(i)}(K+sK')\big|_{s=0}\big\|^{\tilde{}} \le M^{\aleph'(j-i)}\mathfrak{e}_{i+\frac12,j+\frac12}\big(\|K\|_{1,\Sigma_j}\big)\,\|K'\|_{1,\Sigma_j}.$$

Let $w^a_{1,1}(\eta,(\xi,s);K)$ be the kernel of the part of $w^a(K)$ that is of degree 1 in ϕ and degree 1 in ψ. Then

$$|w^{a\sim}_{1,1}(K)|^{\tilde{}}_{1,\Sigma_j,\tilde\rho} \le \tfrac{1}{\alpha^6}\tfrac{\mathfrak{l}_j}{M^j}\mathfrak{e}_j(\|K\|_{1,\Sigma_j}).$$

($I^{\sim}4$) $\mathcal{W}$, $\mathcal{G}$, the Σ_j–sectorized representative $w(\phi,\psi;K)$ of $\mathcal{W}(K)$ and all of the $F^{(i)}$'s and $v^{(i)}$'s are bar/unbar invariant in the sense of Definition B.1.B of [FKTo2]. If K is real, then $\mathcal{W}$, $\mathcal{G}$, the Σ_j–sectorized representative $w(\phi,\psi;K)$ of $\mathcal{W}(K)$ and all of the $q^{(i,\ell)}$'s are k_0–reversal real in the sense of Definition B.1.R of [FKTo2].

Remark XV.2. By Remark B.3.ii of [FKTo2], for any interaction quadruple $(\mathcal{W},\mathcal{G},u,\vec p)$, u and every $p^{(i)}$ is bar/unbar invariant.

Definition XV.3 (More Output Data). *Let $\tilde{\mathcal{D}}^{(j)}_{\rm out}$ be the set of interaction quadruples $(\mathcal{W},\mathcal{G},u,\vec p)$ that fulfill Definitions VIII.1, IX.2 and the following. Let $w(\phi,\psi;K)$ be the Σ_j–sectorized representative of $\mathcal{W}(K)$ specified in (O1) and $w^a(\phi,\psi;K)$ its amputation in the sense of Definition XIII.9.*

($O^{\sim}1$) The Grassmann function $w^a(\phi,\psi;K)-w^a(0,\psi;K)$ vanishes unless $\hat\nu^{(\le j)}\phi$ is nonzero and

$$N^{\sim}_j\big(w^a(K),\alpha,\|K\|_{1,\Sigma_j}\big) \le \mathfrak{e}_j\big(\|K\|_{1,\Sigma_j}\big),$$
$$N^{\sim}_j\big(\tfrac{d}{ds}w^a(K+sK')\big|_{s=0},\alpha,\ \|K\|_{1,\Sigma_j}\big) \le M^j\,\mathfrak{e}_j\big(\|K\|_{1,\Sigma_j}\big)\,\|K'\|_{1,\Sigma_j}$$

for all $K\in\mathfrak{K}_j$ and all K'.

($O^{\sim}2$) There is a family $\vec v$ of antisymmetric kernels

$$v^{(i)}\in\check{\mathcal{F}}_{4,\Sigma_i},\qquad 2\le i\le j$$

(independent of K) and an antisymmetric kernel $\delta f^{(j+1)}(K)\in\check{\mathcal{F}}_{4,\Sigma_j}$ obeying

$$\big|v^{(i)}\big|^{\tilde{}}_{3,\Sigma_i,\tilde\rho} \le \tfrac{\mathfrak{l}_i^{1/n_0}}{\alpha^7}\mathfrak{c}_i \qquad \text{for all } 2\le i\le j,$$
$$\big|\delta f^{(j+1)}(0)\big|^{\tilde{}}_{3,\Sigma_j,\tilde\rho} \le \tfrac{\mathfrak{l}_{j+1}^{1/n_0}}{\alpha^8}\mathfrak{c}_j,$$
$$\big|\tfrac{d}{ds}\delta f^{(j+1)}(K+sK')\big|_{s=0}\big|^{\tilde{}}_{3,\Sigma_j,\tilde\rho} \le \tfrac{1}{\alpha^4\mathrm{B}^2}\mathfrak{e}_j(\|K\|_{1,\Sigma_j})M^j\|K'\|_{1,\Sigma_j}$$

for all $K \in \mathfrak{K}_j$, from which the quartic parts of w^a and $\mathcal{G}^a$ are built as follows. Let $f^{(i)} = \mathrm{shear}\big(v^{(i)}, C^{[i,j]}_{u(0)}(k)A(k)\big) \in \check{\mathcal{F}}_{4,\Sigma_i},\ 2 \le i \le j$. Then the kernel of the quartic part of $w^a + \mathcal{G}^a$ is

$$\delta f^{(j+1)}(K) + \sum_{i=2}^{j} f^{(i)}_{\Sigma_j} + \tfrac{1}{8}\mathrm{Ant}\Big(V_{\mathrm{ph}}\big(\mathcal{L}^{(j+1)}(\vec{p}, \vec{f})\big)\Big).$$

The kernel $v^{(i)}$ vanishes unless all of its ϕ momenta are in the support of $\nu^{(<i)}$.

($O^\sim 3$) Let $\int d\eta_1 d\eta_2\ G_2(\eta_1, \eta_2; K)\ \phi(\eta_1)\phi(\eta_2)$ be the part of $\mathcal{G}(K)$ that is homogeneous of degree two. Then $2\check{G}_2(k; K)$ has the decomposition

$$2\check{G}_2(k; K) = C^{(\le j)}_{u(0)}(k) + \tfrac{1}{(ik_0 - e(\mathbf{k}))^2} \sum_{i=2}^{j} \Big\{ \delta q^{(i)}(k; K) + \sum_{\ell=i}^{j} q^{(i,\ell)}(k) \Big\}$$

with $q^{(i,\ell)}(k)$, $i \le \ell \le j$ and $\delta q^{(i)}(k; K)$ vanishing in the $(i+2)^{\mathrm{nd}}$ neighbourhood and when $\mathbf{k}$ is not in the support of $U(\mathbf{k})$, $\delta q^{(i)}(k; 0) = 0$ for $2 \le i \le j$ and

$$\begin{aligned}
\big\| q^{(i,\ell)}(k) \big\|^{\tilde{}} &\le \lambda_0^{1-2\upsilon} \tfrac{\mathfrak{l}_\ell}{M^\ell} M^{\aleph'(\ell-i)} \mathfrak{c}_{i,\ell},\\
\big\| \tfrac{d}{ds}\delta q^{(i)}(K + sK')\big|_{s=0} \big\|^{\tilde{}} &\le M^{\aleph'(j-i)} \mathfrak{e}_{i+\frac12, j+\frac12}\big(\|K\|_{1,\Sigma_j}\big)\, \|K'\|_{1,\Sigma_j} \ \text{ for } i < j,\\
\big\| \tfrac{d}{ds}\delta q^{(j)}(K + sK')\big|_{s=0} \big\|^{\tilde{}} &\le \sqrt{M^{\aleph'-\aleph}}\, \mathfrak{e}_{j+\frac12}\big(\|K\|_{1,\Sigma_j}\big)\, \|K'\|_{1,\Sigma_j}.
\end{aligned}$$

Let $w^a_{1,1}(\eta, (\xi, s); K)$ be the kernel of the part of $w^a(K)$ that is of degree 1 in ϕ and degree 1 in ψ. Then

$$\begin{aligned}
\big| w^{a\sim}_{1,1}(K)_{\Sigma_{j+1}} \big|^{\tilde{}}_{1,\Sigma_{j+1},\tilde{\rho}} &\le \tfrac{1}{\alpha^7} \tfrac{\mathfrak{l}_j}{M^j} \mathfrak{e}_j(\|K\|_{1,\Sigma_j}),\\
\big| \tfrac{d}{ds} w^{a\sim}_{1,1}(K + sK')_{\Sigma_{j+1}} \big|_{s=0} \big|^{\tilde{}}_{1,\Sigma_{j+1},\tilde{\rho}} &\le \big(\tfrac{1}{\alpha} + \lambda_0^{\upsilon/8}\big) \tfrac{1}{\alpha^2 \mathrm{B}}\, \mathfrak{e}_j(\|K\|_{1,\Sigma_j})\, \|K'\|_{1,\Sigma_j}.
\end{aligned}$$

($O^\sim 4$) $\mathcal{W}$, $\mathcal{G}$, the Σ_j–sectorized representative $w(\phi, \psi; K)$ of $\mathcal{W}(K)$ and all of the $F^{(i)}$'s and $v^{(i)}$'s are bar/unbar invariant. If K is real, then $\mathcal{W}$, $\mathcal{G}$, the Σ_j–sectorized representative $w(\phi, \psi; K)$ of $\mathcal{W}(K)$ and all of the $q^{(i,\ell)}$'s are k_0–reversal real in the sense of Definition B.1.R of [FKTo2].

2. Integrating Out a Scale.

Theorem XV.4. *If $(\mathcal{W}, \mathcal{G}, u, \vec{p}) \in \tilde{\mathcal{D}}^{(j)}_{\mathrm{in}}$, then $\Omega_j(\mathcal{W}, \mathcal{G}, u, \vec{p}) \in \tilde{\mathcal{D}}^{(j)}_{\mathrm{out}}$.*

The rest of this subsection is devoted to the proof of this theorem. Let

$$w = \sum_{m,n} \omega_{m,n}(K)$$

with

$$\begin{aligned}
\omega_{m,n}(\phi, \psi; K) = \sum_{s_1,\cdots,s_n \in \Sigma_j} \int & d\eta_1 \cdots d\eta_m\, d\xi_1 \cdots d\xi_n\ w_{m,n}(\eta_1, \cdots, \eta_m\, (\xi_1, s_1), \cdots, (\xi_n, s_n); K)\\
&\times \phi(\eta_1) \cdots \phi(\eta_m)\ \psi((\xi_1, s_1)) \cdots \psi((\xi_n, s_n))
\end{aligned}$$

be the Σ_j–sectorized representative of $\mathcal{W}$ specified in (I1). Let $(\mathcal{W}', \mathcal{G}', u, \vec{p}) = \Omega_j(\mathcal{W}, \mathcal{G}, u, \vec{p})$ and choose the sectorized representative w' of $\mathcal{W}'$ as in Remark IX.6.iii. Define z by

$$\begin{aligned} {:}z(\phi, \psi; K){:}_{\psi, D_j(u;K)_{\Sigma_j}} &= \Omega_{C^{(j)}_{u,\Sigma_j}}\big({:}w^a - \omega^a_{1,1}{:}_{\psi, C_j(u;K)_{\Sigma_j}}\big) \\ &\quad \times (\phi, \psi + \hat{B}\phi + \widehat{\delta B}\phi), \end{aligned} \tag{XV.2}$$

where

$$\begin{aligned} B(k) &= \big(ik_0 - e(\mathbf{k})\big)C^{(j)}_{u(0)}(k), \\ \delta B(k; K) &= \big(ik_0 - e(\mathbf{k})\big)\big\{C^{(j)}_{u(K)}(k)\big(1 + \check{w}_{1,1}(k; K)\big) - C^{(j)}_{u(0)}(k)\big\}, \end{aligned}$$

and $\check{w}^a_{1,1}(k; K)$ is the Fourier transform, as in the last part of Definition VI.1, of the function $(\eta, \xi) \mapsto \sum_{s\in\Sigma_j} w^a_{1,1}(\eta, (\xi, s); K)$. As in Proposition XII.8 of [FKTo3], with some abuse of notation,

$$(\hat{B}\phi)(\xi, s) = \int d\xi'\ \hat{B}(\xi, \xi')\phi(\xi')$$

for all $s \in \Sigma_j$. Recall that if, for each $s, s' \in \Sigma_j$, $c\big((\,\cdot\,, s), (\,\cdot\,, s')\big)$ is the Fourier transform, as in (III.1) and (III.2), of $\chi_s(k)C(k)\chi_{s'}(k)$, then $C_{\Sigma_j}\big((\xi, s), (\xi', s')\big) = \sum_{\substack{t,t'\in\Sigma_j \\ s\cap t\neq\emptyset \\ s'\cap t'\neq\emptyset}} c\big((\xi, t), (\xi', t')\big)$. The integration and initial/final Wick ordering covariances $C^{(j)}_{u,\Sigma_j}$, $C_j(u; K)_{\Sigma_j}$, $D_j(u; K)_{\Sigma_j}$ are constructed in this way from the $C^{(j)}_u(k)$, $C_j(u; K)(k)$, $D_j(u; K)(k)$ specified in Definition VI.11.

Lemma XV.5.

$$\begin{aligned} w'^a(\phi, \psi; K) &= z(\phi, \psi; K) + \omega^a_{1,1}(\phi, \psi; K) - z(\phi, 0; K), \\ \mathcal{G}'^a(\phi) &= \mathcal{G}^a(\phi) + z(\phi, 0; K), -\tfrac{1}{2}(J(1+\check{w}_{1,1})\hat{\ }\hat{A}\phi)C^{(j)}_{u(K)}J(1+\check{w}_{1,1})\hat{\ }\hat{A}\phi. \end{aligned}$$

Proof. In this proof, set $C = C^{(j)}_{u,\Sigma_j}$ and $C_j = C_j(u; K)_{\Sigma_j}$. Define, for each $s \in \Sigma_j$, $B_s(k)$ to be the Fourier transform, as in the last part of Definition VI.1, of the function $(\eta, \xi) \mapsto w_{1,1}(\eta, (\xi, s); K)$. Observe that

$$\sum_{s\in\Sigma_j} B_s(k) = \check{w}_{1,1}(k; K).$$

By Lemma B.6[2] of [FKTo2],

$$\omega_{1,1}(\phi, \psi) = \sum_{s\in\Sigma_j} \int d\eta d\xi\ \psi(\xi, s)(J\hat{B}_s)(\xi, \eta)\phi(\eta). \tag{XV.3}$$

[2] This is the only step in the proof that substantially uses bar/unbar invariance. Without it, the argument $\hat{B}\phi + \widehat{\delta B}\phi$ of (XV.2) would have a more general, but still controllable form.

Consequently, by Lemma C.1 of [FKTo2],

$$\begin{aligned}
\Omega_C\big(:w:_{\psi,C_j}\big)(\phi,\psi) &= \log \tfrac{1}{Z}\int e^{:w(\phi,\psi+\zeta;K):_{\psi,C_j}}\, d\mu_C(\zeta)\\
&= \log \tfrac{1}{Z}\int e^{\omega_{1,1}(\phi,\psi)+\omega_{1,1}(\phi,\zeta)}\, e^{:(w-\omega_{1,1})(\phi,\psi+\zeta;K):_{\psi,C_j}}\, d\mu_C(\zeta)\\
&= \omega_{1,1}(\phi,\psi) + \log \tfrac{1}{Z}\int e^{\Sigma_s \zeta(s) J\hat{B}_s\phi}\\
&\quad \times e^{:(w-\omega_{1,1})(\phi,\psi+\zeta;K):_{\psi,C_j}}\, d\mu_C(\zeta)\\
&= \omega_{1,1}(\phi,\psi) - \tfrac{1}{2}\sum_{s,s'\in\Sigma_j}(J\hat{B}_s\phi)C(s,s')(J\hat{B}_{s'}\phi)\\
&\quad + \log\tfrac{1}{Z}\int e^{:(w-\omega_{1,1})(\phi,\psi+\Sigma_s C(\,\cdot\,,s)J\hat{B}_s\phi+\zeta;K):_{\psi,C_j}}\, d\mu_C(\zeta)\\
&= \omega_{1,1}(\phi,\psi) - \tfrac{1}{2}\sum_{s,s'\in\Sigma_j}(J\hat{B}_s\phi)C(s,s')(J\hat{B}_{s'}\phi)\\
&\quad + \Omega_C\big(:w-\omega_{1,1}:_{\psi,C_j}\big)(\phi,\psi+\Sigma_s C(\,\cdot\,,s)J\hat{B}_s\phi)\\
&= \omega_{1,1}(\phi,\psi) - \tfrac{1}{2}(J\widehat{\check{w}_{1,1}}\phi)C_u^{(j)}(J\widehat{\check{w}_{1,1}}\phi)\\
&\quad + \Omega_C\big(:w-\omega_{1,1}:_{\psi,C_j}\big)(\phi,\psi+C_u^{(j)}J\widehat{\check{w}_{1,1}}\phi).
\end{aligned}$$

Therefore the Grassmann function w'' of Remark IX.6.iii is determined by

$$\begin{aligned}
:w''(\phi,\psi;K):_{\psi,D_j(u;K)_{\Sigma_j}} &= \tfrac{1}{2}\phi J C_u^{(j)} J\phi + \Omega_C(:w:_{\psi,C_j})(\phi,\psi+C_u^{(j)}J\phi)\\
&= -\tfrac{1}{2}(J\phi)C_u^{(j)}J\phi + \omega_{1,1}(\phi, C_u^{(j)}J\phi)\\
&\quad -\tfrac{1}{2}(J\widehat{\check{w}_{1,1}}\phi)C_u^{(j)}(J\widehat{\check{w}_{1,1}}\phi) + \omega_{1,1}(\phi,\psi)\\
&\quad + \Omega_C\big(:w-\omega_{1,1}:_{\psi,C_j}\big)(\phi,\psi+C_u^{(j)}J(1+\check{w}_{1,1})\hat{}\phi).
\end{aligned}$$

By (XV.3), the antisymmetry of $C_u^{(j)}$ and repeated use of Lemma IX.5 of [FKTo2],

$$\begin{aligned}
\omega_{1,1}(\phi, C_u^{(j)}J\phi) &= \sum_s (C_u^{(j)}J\phi)J\hat{B}_s\phi = (C_u^{(j)}J\phi)J\widehat{\check{w}_{1,1}}\phi\\
&= -(J\phi)C_u^{(j)}(J\widehat{\check{w}_{1,1}}\phi) = -(J\widehat{\check{w}_{1,1}}\phi)C_u^{(j)}(J\phi),
\end{aligned}$$

so that

$$\begin{aligned}
:w''(\phi,\psi;K):_{\psi,D_j(u;K)_{\Sigma_j}} &= -\tfrac{1}{2}(J(1+\check{w}_{1,1})\hat{}\phi)C_u^{(j)}J(1+\check{w}_{1,1})\hat{}\phi + \omega_{1,1}(\phi,\psi)\\
&\quad + \Omega_C\big(:w-\omega_{1,1}:_{\psi,C_j}\big)(\phi,\psi+[C_u^{(j)}(k)(1+\check{w}_{1,1})]\hat{}\phi).
\end{aligned}$$

Amputating and applying parts (i) and (ii) of Lemma IX.5 of [FKTo2],

$$w''^{a}(\phi,\psi;K) = -\tfrac{1}{2}(J(1+\check{w}_{1,1})\hat{}\hat{A}\phi)C_u^{(j)}J(1+\check{w}_{1,1})\hat{}\hat{A}\phi + \omega^a_{1,1}(\phi,\psi) + z(\phi,\psi;K).$$

The lemma now follows by Remark IX.6.iii. □

Lemma XV.6.

$$(J(1+\check{w}_{1,1})\hat{}\hat{A}\phi)C^{(j)}_{u(K)}J(1+\check{w}_{1,1})\hat{}\hat{A}\phi = (J\hat{A}\phi)C^{(j)}_{u(0)}(J\hat{A}\phi) + \phi J\hat{E}^{(j)}\phi$$

with

$$E^{(j)}(k;K) = -A(k)^2\big\{C^{(j)}_{u(K)}(k)\check{w}_{1,1}(k;K)\big(2+\check{w}_{1,1}(k;K)\big) + C^{(j)}_{u(K)}(k) - C^{(j)}_{u(0)}(k)\big\}.$$

Proof. Set

$$B'(k;K) = 1 + \check{w}_{1,1}(k;K).$$

By repeated use of Lemma IX.5 of [FKTo2] and the facts that $J^2 = -\mathbb{1}$ and $J^t = -J$,

$$\begin{aligned}(J\hat{B}'\hat{A}\phi)C^{(j)}_{u(K)}(J\hat{B}'\hat{A}\phi) &= -(J\widehat{B'A}J^2\phi)C^{(j)}_{u(K)}(J\widehat{B'A}\phi) = (\widehat{B'A}^t J\phi)C^{(j)}_{u(K)}(J\widehat{B'A}\phi)\\ &= (J\phi)\widehat{B'A}C^{(j)}_{u(K)}J\widehat{B'A}\phi \qquad = -\phi J\widehat{B'A}C^{(j)}_{u(K)}J\widehat{B'A}\phi.\end{aligned}$$

Similarly,

$$(J\hat{A}\phi)C^{(j)}_{u(0)}(J\hat{A}\phi) = -\phi J\hat{A}C^{(j)}_{u(0)}J\hat{A}\phi.$$

Subtracting and observing that

$$\begin{aligned}-A(k)^2\big\{C^{(j)}_{u(K)}(k)B'(k)^2 - C^{(j)}_{u(0)}(k)\big\} &= -A(k)^2\big\{C^{(j)}_{u(K)}(k)\big(B'(k)^2-1\big)\\ &\qquad + C^{(j)}_{u(K)}(k) - C^{(j)}_{u(0)}(k)\big\}\\ &= E^{(j)}(k;K)\end{aligned}$$

yields

$$(J\hat{B}'\hat{A}\phi)C^{(j)}_{u(K)}(J\hat{B}'\hat{A}\phi) - (J\hat{A}\phi)C^{(j)}_{u(0)}(J\hat{A}\phi) = \phi J\hat{E}^{(j)}\phi. \quad \square$$

Lemma XV.7.

i)

$$\begin{aligned}\big\|C^{(j)}_{u(K)}(k)\big\|^{\tilde{}} &\le \text{const}\, M^j\, \mathfrak{e}_j\big(\|K\|_{1,\Sigma_j}\big),\\ \big\|\big(ik_0-e(\mathbf{k})\big)C^{(j)}_{u(K)}(k)\big\|^{\tilde{}} &\le \text{const}\, \mathfrak{e}_j\big(\|K\|_{1,\Sigma_j}\big),\\ \big\|\tfrac{d}{ds}C^{(j)}_{u(K+sK')}(k)\big|_{s=0}\big\|^{\tilde{}} &\le \text{const}\, M^{2j}\, \mathfrak{e}_j\big(\|K\|_{1,\Sigma_j}\big)\,\|K'\|_{1,\Sigma_j},\\ \big\|\tfrac{d}{ds}(ik_0-e(\mathbf{k}))C^{(j)}_{u(K+sK')}(k)\big|_{s=0}\big\|^{\tilde{}} &\le \text{const}\, M^j\, \mathfrak{e}_j\big(\|K\|_{1,\Sigma_j}\big)\,\|K'\|_{1,\Sigma_j},\\ \big\|\tfrac{d}{ds}(ik_0-e(\mathbf{k}))^2C^{(j)}_{u(K+sK')}(k)\big|_{s=0}\big\|^{\tilde{}} &\le \mathit{const}\, \mathfrak{e}_{j+\frac{1}{2}}\big(\|K\|_{1,\Sigma_j}\big)\,\|K'\|_{1,\Sigma_j}.\end{aligned}$$

ii)

$$\begin{aligned}\|B\|^{\tilde{}} &\le \text{const}\, \mathfrak{c}_j,\\ \|\delta B(K)\|^{\tilde{}} &\le \text{const}\, \mathfrak{e}_j\big(\|K\|_{1,\Sigma_j}\big)\Big[M^j\|K\|_{1,\Sigma_j} + \tfrac{\lambda_0^{1-2\upsilon}\mathfrak{l}_j}{\alpha^6}\Big],\\ \big\|\tfrac{d}{ds}\delta B(K+sK')\big|_{s=0}\big\|^{\tilde{}} &\le \text{const}\, M^j\, \mathfrak{e}_j\big(\|K\|_{1,\Sigma_j}\big)\|K'\|_{1,\Sigma_j}.\end{aligned}$$

iii)

$$\|E^{(j)}(k;0)\|^{\tilde{}} \le \text{const}\, \tfrac{\lambda_0^{1-8\upsilon/7}}{\alpha^6}\, \tfrac{\mathfrak{l}_j}{M^j}\, \mathfrak{c}_j\,,$$
$$\big\| \tfrac{d}{ds} E^{(j)}(K+sK')\big|_{s=0}\big\|^{\tilde{}} \le const\ \mathfrak{e}_{j+\frac12}\big(\|K\|_{1,\Sigma_j}\big)\,\|K'\|_{1,\Sigma_j}\,.$$

Proof. i) Observe that

$$(ik_0 - e(\mathbf{k}))C^{(j)}_{u(K)}(k) = (ik_0 - e(\mathbf{k}))\tfrac{\nu^{(j)}(k)}{ik_0 - e(\mathbf{k}) - \check u(k;K)} = \frac{\nu^{(j)}(k)}{1 - \frac{\check u(k;K)}{ik_0 - e(\mathbf{k})}}$$

and

$$\begin{aligned}
&\tfrac{d}{ds}(ik_0 - e(\mathbf{k}))^2 C^{(j)}_{u(K+sK')}(k)\\
&\quad = (ik_0 - e(\mathbf{k}))^2 \tfrac{d}{ds}\tfrac{\nu^{(j)}(k)}{ik_0 - e(\mathbf{k}) - \check u(k;K+sK')}\\
&\quad = (ik_0 - e(\mathbf{k}))^2 \tfrac{\nu^{(j)}(k)}{[ik_0 - e(\mathbf{k}) - \check u(k;K+sK')]^2}\tfrac{d}{ds}\check u(k;K+sK')\\
&\quad = \frac{\nu^{(j)}(k)\tfrac{d}{ds}\check u(k;K+sK')}{\big[1 - \frac{\check u(k;K+sK')}{ik_0 - e(\mathbf{k})}\big]^2}.
\end{aligned}$$

Next use the fact that, on the support of $\nu^{(j)}$, $\frac{1}{\sqrt{M}\,M^j} \le |ik_0 - e(\mathbf{k})| \le \frac{\sqrt{2M}}{M^j}$ to show that, if $f(k)$ vanishes except on the support of $\nu^{(\le j)}(k)$,

$$\begin{aligned}
\big\|\nu^{(j)}(k)\big\|^{\tilde{}} &\le const\ \mathfrak{c}_{j+\frac12}\,,\\
\big\|\tfrac{f(k)}{ik_0 - e(\mathbf{k})}\big\|^{\tilde{}} &\le const\ M^{j+\frac12}\,\mathfrak{c}_{j+\frac12}\,\|f(k)\|^{\tilde{}}.
\end{aligned}\tag{XV.4}$$

Also use Lemma VIII.7 and Lemma XII.12 of [FKTo3] to show that

$$\begin{aligned}
\big\|\check u(K)\big\|^{\tilde{}} &\le 2|u(K)|_{1,\Sigma_j} \le const\Big[\tfrac{\lambda_0^{1-\upsilon}}{M^{j-1}} + \|K\|_{1,\Sigma_j}\Big]\mathfrak{e}_j(\|K\|_{1,\Sigma_j})\\
&\le const\ \tfrac{1}{M^j}\,\mathfrak{e}_j(\|K\|_{1,\Sigma_j}),\\
\big\|\tfrac{d}{ds}\check u(K+sK')\big|_{s=0}\big\|^{\tilde{}} &\le 2\big|\tfrac{d}{ds}u(K+sK')\big|_{s=0}\big|_{1,\Sigma_j}\\
&\le const\ \mathfrak{e}_j(\|K\|_{1,\Sigma_j})\,\|K'\|_{1,\Sigma_j}.
\end{aligned}\tag{XV.5}$$

Since $\Big|\frac{\check u(k;K)}{ik_0 - e(\mathbf{k})}\Big| \le \frac12$ on the support of $\nu^{(j)}$, we have

$$\begin{aligned}
\big\|(ik_0 - e(\mathbf{k}))C^{(j)}_{u(K)}(k)\big\|^{\tilde{}} &\le const\ \mathfrak{e}_{j+\frac12}\big(\|K\|_{1,\Sigma_j}\big),\\
\big\|\tfrac{d}{ds}(ik_0 - e(\mathbf{k}))^2 C^{(j)}_{u(K+sK')}(k)\big|_{s=0}\big\|^{\tilde{}} &\le const\ \mathfrak{e}_{j+\frac12}\big(\|K\|_{1,\Sigma_j}\big)\,\|K'\|_{1,\Sigma_j}.
\end{aligned}$$

The remaining bounds follow from these, the second bound of (XV.4), Corollary A.5 of [FKTo1] and the observation that $\mathfrak{e}_{j+\frac12}(\|K\|_{1,\Sigma_j}) \le \text{const}\ \mathfrak{e}_j(\|K\|_{1,\Sigma_j})$ for some M, r_0 and r dependent constant.

ii) By $(I^{\sim}3)$, Remark XIII.6 and $(I^{\sim}1)$,

$$\begin{aligned}\big\|\big(ik_0 - e(\mathbf{k})\big)\check{w}_{1,1}(k;\,K)\big\|^{\tilde{}} &\le 2|w^{a\sim}_{1,1}(K)|^{\tilde{}}_{1,\Sigma_j}\\ &= 2\lambda_0^{1-8\upsilon/7}|w^{a\sim}_{1,1}(K)|^{\tilde{}}_{1,\Sigma_j,\tilde\rho}\\ &\le 2\tfrac{\lambda_0^{1-8\upsilon/7}}{\alpha^6}\tfrac{\mathfrak{l}_j}{M^j}\mathfrak{e}_j(\|K\|_{1,\Sigma_j}),\\ \big\|\big(ik_0 - e(\mathbf{k})\big)\tfrac{d}{ds}\check{w}_{1,1}(k;\,K+sK')\big|_{s=0}\big\|^{\tilde{}} &\le \tfrac{\lambda_0^{1-8\upsilon/7}}{\mathrm{B}\alpha^2}\mathfrak{e}_j(\|K\|_{1,\Sigma_j})\,\|K'\|_{1,\Sigma_j}.\end{aligned} \tag{XV.6}$$

All three bounds follow by using Leibniz and Corollary A.5.ii of [FKTo1] to combine (XV.6) with the bounds of part (i).

iii) For the first bound, just apply $\big\|A(k)C^{(j)}_{u(0)}(k)\big\|^{\tilde{}} \le \text{const}\,\mathfrak{c}_j$ from part (i), the first bound of (XV.6) and

$$\big\|\check{w}_{1,1}(k;\,K)\big\|^{\tilde{}} \le \text{const}\,\frac{\lambda_0^{1-8\upsilon/7}}{\alpha^6}\mathfrak{l}_j\mathfrak{e}_j(\|K\|_{1,\Sigma_j}), \tag{XV.7}$$

which follows from (XV.6), (XV.4) and the fact that $\check{w}_{1,1}(k;\,K)$ vanishes except on the support of $\nu^{(>j)}$.

Now for the second bound. By part (i),

$$\big\|\tfrac{d}{ds}A(k)^2C^{(j)}_{u(K+sK')}(k)\big|_{s=0}\big\|^{\tilde{}} \le const\;\mathfrak{e}_{j+\frac12}\big(\|K\|_{1,\Sigma_j}\big)\,\|K'\|_{1,\Sigma_j}.$$

By part (i), (XV.6) and (XV.7),

$$\begin{aligned}&\big\|A(k)\check{w}_{1,1}(k;K)\big(2+\check{w}_{1,1}(k;K)\big)\tfrac{d}{ds}A(k)C^{(j)}_{u(K+sK')}(k)\big|_{s=0}\big\|^{\tilde{}}\\ &\le \text{const}\frac{\lambda_0^{1-\frac87\upsilon}}{\alpha^6}\mathfrak{l}_j\mathfrak{e}_j(\|K\|_{1,\Sigma_j})\|K'\|_{1,\Sigma_j},\\ &\le \frac{\lambda_0^{1-2\upsilon}}{\alpha^6}\mathfrak{l}_j\,\mathfrak{e}_j(\|K\|_{1,\Sigma_j})\,\|K'\|_{1,\Sigma_j}.\\ &\quad\big\|A(k)C^{(j)}_{u(K)}(k)\big(1+\check{w}_{1,1}(k;K)\big)\tfrac{d}{ds}A(k)\check{w}_{1,1}(k;K+sK')\big|_{s=0}\big\|^{\tilde{}}\\ &\le \text{const}\,\frac{\lambda_0^{1-\frac87\upsilon}}{\alpha^2}\,\mathfrak{e}_j(\|K\|_{1,\Sigma_j})\,\|K'\|_{1,\Sigma_j}\\ &\le \frac{\lambda_0^{1-2\upsilon}}{\alpha^2}\,\mathfrak{e}_j(\|K\|_{1,\Sigma_j})\,\|K'\|_{1,\Sigma_j}. \quad\square\end{aligned}$$

Proof of Theorem XV.4. We have already proven in Theorem IX.5 that $(\mathcal{W}', \mathcal{G}', u, \vec{p}) \in \mathcal{D}^{(j)}_{\rm out}$. We shall use $\omega_{m,n},\ \mathfrak{z}_{m,n},\ \cdots$ to denote the part of $w,\ z,\ \cdots$ that is of degree m in ϕ and degree n in ψ and $w_{m,n},\ z_{m,n},\ \cdots$ to denote the corresponding kernels. We shall also use $\omega_n,\ \mathfrak{z}_n,\ \cdots$ to denote the part of $w,\ z,\ \cdots$ that is of degree n in ϕ and ψ combined and $w_n,\ z_n,\ \cdots$ to denote the corresponding kernels. Recall that

$$:z(\phi,\psi;K):_{\psi,D_j(u;K)_{\Sigma_j}} = \Omega_{C^{(j)}_{u,\Sigma_j}}\big(:w^a-\omega^a_{1,1}:_{\psi,C_j(u;K)_{\Sigma_j}}\big)(\phi,\psi+\hat{B}\phi+\hat{\delta}B\phi),$$

and define z'' by

$$:z''(\phi,\psi;K):_{\psi,D_j(u;K)_{\Sigma_j}} = \Omega_{C^{(j)}_{u,\Sigma_j}}\big(:w^a-\omega^a_{1,1}:_{\psi,C_j(u;K)_{\Sigma_j}}\big)(\phi,\psi)$$

so that

$$z(\phi,\psi;K) = z''(\phi,\psi + \hat{B}\phi + \widehat{\delta B}(K)\phi; K). \tag{XV.8}$$

Preparation for the verification of ($O^\sim 1$), ($O^\sim 2$) *and* ($O^\sim 3$). We now apply Theorems XVII.3 and XVII.6 of [FKTo3] to bound z. By ($I^\sim 1$), parts (i) and (ii) of Lemma VIII.7 and Lemma XV.7.ii, the hypotheses of Theorem XVII.3 of [FKTo3], with $w = w^a - \omega^a_{1,1}$, B replaced by $B + \delta B$, $w' = z$, $w'' = z''$, $\mu = \mathit{const}$, $\Lambda = \frac{\lambda_0^{1-\upsilon}}{M^{j-1}}$, $\gamma = \lambda_0^{\upsilon/7}$ and $X = \|K\|_{1,\Sigma_j}$, are fulfilled. Therefore,

$$N_j^\sim(z - w^a + \omega^a_{1,1}, \alpha, \|K\|_{1,\Sigma_j}) \le \text{const}\left(\tfrac{1}{\alpha} + \lambda_0^{\upsilon/7}\right) \frac{N_j^\sim(w^a,64\alpha,\|K\|_{1,\Sigma_j})}{1-\frac{\text{const}}{\alpha}N_j^\sim(w^a,64\alpha,\|K\|_{1,\Sigma_j})}$$

$$|z_{2,0}|^\sim_{1,\Sigma_j,\tilde\rho},\ |z_{1,1}|^\sim_{1,\Sigma_j,\tilde\rho} \le \frac{\text{const}}{\alpha^8}\,\frac{\mathfrak{l}_j}{M^j}\,\frac{N_j^\sim(w^a,64\alpha,\|K\|_{1,\Sigma_j})^2}{1-\frac{\text{const}}{\alpha}N_j^\sim(w^a,64\alpha,\|K\|_{1,\Sigma_j})} \tag{XV.9}$$

and

$$\Big|z''_4 - w^a_4 - \tfrac{1}{4}\sum_{\ell=1}^{\infty}(-1)^\ell(12)^{\ell+1}\,\mathrm{Ant}\,L_\ell(w^a_4;\ C_u^{(j)},D_j)\Big|^\sim_{3,\Sigma_j,\tilde\rho}$$
$$\le \frac{\text{const}}{\alpha^{10}}\,\mathfrak{l}_j\,\frac{N_j^\sim(w^a,64\alpha,\|K\|_{1,\Sigma_j})^2}{1-\frac{\text{const}}{\alpha}N_j^\sim(w^a,64\alpha,\|K\|_{1,\Sigma_j})}. \tag{XV.10}$$

The hypotheses of Theorem XVII.6 of [FKTo3], with, in addition, $Y = \mathit{const}\,\|K'\|_{1,\Sigma_j}$, $Z = \text{const}\,M^j\,\|K'\|_{1,\Sigma_j}$ and $\varepsilon = \text{const}\,M^j\,\|K'\|_{1,\Sigma_j}\big|_{t=0}$, are fulfilled. Hence

$$\begin{aligned}
&N_j^\sim\Big(\tfrac{d}{ds}\big[z(K+sK') + \omega^a_{1,1}(K+sK') - w^a(K+sK')\big]_{s=0}, \alpha,\ \|K\|_{1,\Sigma_j}\Big)\\
&\quad\le \text{const}\left\{\lambda_0^{\upsilon/7} + \frac{1}{\alpha^2}\,\frac{N_j^\sim(w^a,64\alpha,\ \|K\|_{1,\Sigma_j})}{1-\frac{\text{const}}{\alpha^2}N_j^\sim(w^a,64\alpha,\ \|K\|_{1,\Sigma_j})}\right\}\\
&\qquad\times N_j^\sim\Big(\tfrac{d}{ds}w^a(K+sK')\big|_{s=0}, 16\alpha,\ \|K\|_{1,\Sigma_j}\Big)\\
&\qquad+ \text{const}\,\frac{N_j^\sim(w^a,64\alpha,\ \|K\|_{1,\Sigma_j})}{1-\frac{\text{const}}{\alpha^2}N_j^\sim(w^a,64\alpha,\ \|K\|_{1,\Sigma_j})}\\
&\qquad\times\left\{\frac{1}{\alpha^2}N_j^\sim(w^a,64\alpha,\ \|K\|_{1,\Sigma_j}) + \lambda_0^{\upsilon/7}\right\}M^j\|K'\|_{1,\Sigma_j}.
\end{aligned} \tag{XV.11}$$

By ($I^\sim 1$) and Corollary A.5.ii of [FKTo1],

$$\frac{N_j^\sim(w^a,64\alpha,\ \|K\|_{1,\Sigma_j})}{1-\frac{\text{const}}{\alpha}N_j^\sim(w^a,64\alpha,\ \|K\|_{1,\Sigma_j})} \le \frac{\mathfrak{e}_j(\|K\|_{1,\Sigma_j})}{1-\frac{\text{const}}{\alpha}\mathfrak{e}_j(\|K\|_{1,\Sigma_j})} \le \text{const}\,\mathfrak{e}_j(\|K\|_{1,\Sigma_j})$$

and

$$\frac{N_j^\sim(w^a,64\alpha,\ \|K\|_{1,\Sigma_j})^2}{1-\frac{\text{const}}{\alpha}N_j^\sim(w^a,64\alpha,\ \|K\|_{1,\Sigma_j})} \le \text{const}\,\mathfrak{e}_j(\|K\|_{1,\Sigma_j})^2 \le \text{const}\,\mathfrak{e}_j(\|K\|_{1,\Sigma_j}).$$

Therefore, by (XV.9) and ($I^\sim 1$), recalling that w^a vanishes when $\psi = 0$,

$$\begin{aligned}
N_j^\sim(z + \omega^a_{1,1}, \alpha,\ \|K\|_{1,\Sigma_j}) &\le N_j^\sim(w^a, \alpha,\ \|K\|_{1,\Sigma_j}) + \text{const}\left(\tfrac{1}{\alpha} + \lambda_0^{\upsilon/7}\right)\mathfrak{e}_j(\|K\|_{1,\Sigma_j})\\
&\le \tfrac{1}{64}N_j^\sim(w^a, 64\alpha,\ \|K\|_{1,\Sigma_j}) + \tfrac{1}{2}\,\mathfrak{e}_j(\|K\|_{1,\Sigma_j})\\
&\le \mathfrak{e}_j(\|K\|_{1,\Sigma_j})
\end{aligned} \tag{XV.12}$$

and

$$|z_{2,0}|^{\sim}_{1,\Sigma_j,\tilde\rho},\ |z_{1,1}|^{\sim}_{1,\Sigma_j,\tilde\rho} \le \tfrac{\rm const}{\alpha^8}\,\tfrac{\mathfrak{l}_j}{M^j}\,\mathfrak{e}_j(\|K\|_{1,\Sigma_j}) \le \tfrac{1}{\alpha^7}\,\tfrac{\mathfrak{l}_j}{M^j}\,\mathfrak{e}_j(\|K\|_{1,\Sigma_j}) \quad \text{(XV.13)}$$

and

$$\Big|z_4'' - w_4^a - \tfrac14 \sum_{\ell=1}^{\infty} (-1)^\ell (12)^{\ell+1} \mathrm{Ant}\, L_\ell(w_4^a;\ C_u^{(j)}, D_j)\Big|^{\sim}_{3,\Sigma_j,\tilde\rho} \le \tfrac{\rm const}{\alpha^{10}}\,\mathfrak{l}_j\,\mathfrak{e}_j(\|K\|_{1,\Sigma_j}). \quad \text{(XV.14)}$$

Furthermore, by (XV.11), $(I^\sim 1)$ and Corollary A.5.ii of [FKTo1],

$$\begin{aligned} &N_j^\sim\Big(\tfrac{d}{ds}\big[z(K+sK') + \omega_{1,1}^a(K+sK') - w^a(K+sK')\big]_{s=0}, \alpha,\ \|K\|_{1,\Sigma_j}\Big)\\ &\quad \le \text{const}\,\big(\tfrac{1}{\alpha^2} + \lambda_0^{\upsilon/7}\big)\,\mathfrak{e}_j(\|K\|_{1,\Sigma_j})\,M^j\,\|K'\|_{1,\Sigma_j}. \end{aligned} \quad \text{(XV.15)}$$

and

$$\begin{aligned} &N_j^\sim\Big(\tfrac{d}{ds}\big[z(K+sK') + \omega_{1,1}^a(K+sK')\big]_{s=0}, \alpha,\ \|K\|_{1,\Sigma_j}\Big)\\ &\le N_j^\sim\Big(\tfrac{d}{ds} w^a(K+sK')\big|_{s=0}, \alpha,\ \|K\|_{1,\Sigma_j}\Big)\\ &\quad + \text{const}\,\big(\tfrac{1}{\alpha^2} + \lambda_0^{\upsilon/7}\big)\,\mathfrak{e}_j(\|K\|_{1,\Sigma_j})\,M^j\,\|K'\|_{1,\Sigma_j}\\ &\le \tfrac{1}{64} N_j^\sim\Big(\tfrac{d}{ds} w^a(K+sK')\big|_{s=0}, 64\alpha,\ \|K\|_{1,\Sigma_j}\Big) + \tfrac12\,\mathfrak{e}_j(\|K\|_{1,\Sigma_j})\,M^j\,\|K'\|_{1,\Sigma_j}\\ &\le M^j\,\mathfrak{e}_j(\|K\|_{1,\Sigma_j})\,\|K'\|_{1,\Sigma_j}. \end{aligned} \quad \text{(XV.16)}$$

Verification of ($O^\sim 1$). Observe that $B(k)$ and $\delta B(k;\ K)$ vanish unless k is in the support of $\nu^{(j)}$. Also, by $(I^\sim 1)$, $w^a(\phi,\psi;\ K) - w^a(0,\psi;\ K)$ vanishes unless $\hat\nu^{(<j)}\phi$ is nonzero. Consequently, $z(\phi,\psi;\ K) - z(0,\psi;\ K)$, and hence $w'^a(\phi,\psi;\ K) - w'^a(0,\psi;\ K)$, vanishes unless $\hat\nu^{(\le j)}\phi$ is nonzero. The bounds of $(O^\sim 1)$ follow from Lemma XV.5, (XV.12) and (XV.16).

Verification of ($O^\sim 2$). By Lemma XV.5,

$$\begin{aligned} \omega_4'^a(\phi,\psi;\ K) + \mathcal{G}_4'^a(\phi;\ K) &= \mathfrak{z}_4(\phi,\psi;\ K) + \mathcal{G}_4^a(\phi;\ K)\\ &= \mathfrak{z}_4(\phi,\psi;\ 0) + \mathcal{G}_4^a(\phi;\ 0) + \delta\mathfrak{f}_0'(\phi,\psi;\ K), \end{aligned}$$

where

$$\delta\mathfrak{f}_0'(\phi,\psi;\ K) = \mathfrak{z}_4(\phi,\psi;\ K) + \mathcal{G}_4^a(\phi;\ K) - \mathfrak{z}_4(\phi,\psi;\ 0) - \mathcal{G}_4^a(\phi;\ 0).$$

As the corresponding kernel $\delta f_0'(K)$ obeys

$$\tfrac{d}{ds}\delta f_0'(K+sK') = \tfrac{d}{ds}\Big[z_4(K+sK') - w_4^a(K+sK')\Big] + \tfrac{d}{ds}\delta f^{(j)}(K+sK'),$$

(XV.15) and $(I^\sim 2)$ give

$$\begin{aligned} \Big|\tfrac{d}{ds}\delta f_0'(K+sK')\big|_{s=0}\Big|^{\sim}_{3,\Sigma_j,\tilde\rho} &\le \Big[\tfrac{\rm const}{\alpha^4}\big(\alpha^2 + \lambda_0^{\upsilon/7}\big) + \tfrac{1}{64\alpha^4 \mathrm{B}^2}\Big]\\ &\quad \times \mathfrak{e}_j(\|K\|_{1,\Sigma_j})\,M^j\,\|K'\|_{1,\Sigma_j}. \end{aligned} \quad \text{(XV.17)}$$

By (XV.8),

$$\begin{aligned} \mathfrak{z}_4(\phi,\psi;\ 0) + \mathcal{G}_4^a(\phi;\ 0) &= \mathfrak{z}_4''(\phi,\psi + \hat B\phi + \hat\delta B(0)\phi;\ 0) + \mathcal{G}_4^a(\phi;\ 0)\\ &= \mathfrak{z}_4''(\phi,\psi + \hat B\phi;\ 0) + \mathcal{G}_4^a(\phi;\ 0) + \delta\mathfrak{f}_1'(\phi,\psi), \end{aligned}$$

where

$$\delta\mathfrak{f}'_1(\phi,\psi) = \mathfrak{z}''_4(\phi,\psi+\hat{B}\phi+\hat{\delta}B(0)\phi;0) - \mathfrak{z}''_4(\phi,\psi+\hat{B}\phi;0).$$

By Lemma XV.7.ii and Theorem XVII.6 of [FKTo3], with $w = w^a(0) - \omega^a_{1,1}(0)$, $C_\kappa = C^{(j)}_{u(0)}$, $D_\kappa = D_j(u;0)$, independent of κ, $B_\kappa = B + (\kappa_0+\kappa)\delta B(0)$, where $0 \le \kappa_0 \le 1$, $\gamma = \lambda_0^{\upsilon/7}$, $\mu = const$, $\Lambda = \frac{\lambda_0^{1-\upsilon}}{M^{j-1}}$, $c_B = \mathrm{const}$, $X = Y = \epsilon = 0$ and $Z = \frac{\lambda_0^{1-2\upsilon}\mathfrak{l}_j}{\alpha^6}$, the corresponding kernel $\delta f'_1$ is bounded by

$$\big|\delta f'_1\big|^{\sim}_{3,\Sigma_j,\tilde\rho} \le \mathrm{const}\,\lambda_0^{\upsilon/7}\,\frac{\lambda_0^{1-2\upsilon}\mathfrak{l}_j}{\alpha^6}\,\frac{1}{\alpha^4\mathrm{B}^2}\,\mathfrak{c}_j. \tag{XV.18}$$

Next, let

$$\delta f'_2 = \mathrm{shear}\Big(z''_4(0) - w^a_4(0) - \tfrac14\sum_{\ell=1}^{\infty}(-1)^\ell(12)^{\ell+1}\mathrm{Ant}\,L_\ell\big(w^a_4(0);C^{(j)}_{u(0)},D_j(u(0);0)\big),B\Big).$$

By (XV.14), Lemma XV.7.ii and Lemma XVII.5 of [FKTo3], with $c_B = \mathrm{const}$, $X = 0$, $X_B = 1$,

$$\big|\delta f'_2\big|_{3,\Sigma_j,\tilde\rho} \le \tfrac{\mathrm{const}}{\alpha^{10}}\,\mathfrak{l}_j\,\mathfrak{c}_j \le \tfrac{1}{\alpha^9}\,\mathfrak{l}_j\,\mathfrak{c}_j. \tag{XV.19}$$

Define

$$\delta f'_3 = \mathrm{shear}\Big(\tfrac14\sum_{\ell=1}^{\infty}(-1)^\ell(12)^{\ell+1}\mathrm{Ant}\,V_{\mathrm{pp}}\Big(L_\ell\big(w^a_4(0);C^{(j)}_{u(0)},C^{(\ge j+1)}_{u(0)}\big)^{\mathrm{pp}}\Big),B\Big)$$

and

$$\delta f'^{(j+1)}(K) = \delta f'_0(K)+\delta f'_1+\delta f'_2+\delta f'_3 \qquad \delta\mathfrak{f}'^{(j+1)}(\phi,\psi;K) = Gr\big(\phi,\psi;\delta f'^{(j+1)}(K)\big),$$

where $Gr(\phi,\psi,f)$ is the Grassmann function associated to the kernel f as in Definition XIV.14.ii. By Proposition VII.6, for $\ell \ge 1$,

$$\Big|V_{\mathrm{pp}}\Big(L_\ell\big(w^a_4(0);C^{(j)}_{u(0)},C^{(\ge j+1)}_{u(0)}\big)^{\mathrm{pp}}\Big)\Big|_{3,\Sigma_j} \le \big(\mathrm{const}\,\mathfrak{l}_j^{1/n_0}\mathfrak{c}_j\big)^\ell\,|w^a_4(0)|^{\ell+1}_{3,\Sigma_j}$$

so that

$$\Big|V_{\mathrm{pp}}\Big(L_\ell\big(w^a_4(0);C^{(j)}_{u(0)},C^{(\ge j+1)}_{u(0)}\big)^{\mathrm{pp}}\Big)\Big|_{3,\Sigma_j,\tilde\rho} \le \big(\mathrm{const}\,\lambda_0^{1-\upsilon}\mathfrak{l}_j^{1/n_0}\mathfrak{c}_j\big)^\ell\,|w^a_4(0)|^{\ell+1}_{3,\Sigma_j,\tilde\rho}.$$

The hypotheses of this proposition are fulfilled by parts (i) and (ii) of Lemma VIII.7. Observe that, by ($I^{\sim}1$),

$$\tfrac{M^{2j}}{\mathfrak{l}_j}\,\mathfrak{e}_j(\|K\|_{1,\Sigma_j})(64\alpha)^4\big(\tfrac{\mathfrak{l}_j\mathrm{B}}{M^j}\big)^2\,\tfrac{1}{\mathfrak{l}_j}\,|w_{0,4}(K)|_{3,\Sigma_j,\tilde\rho} \le \mathfrak{e}_j(\|K\|_{1,\Sigma_j})$$

so that

$$|w_{0,4}(K)|_{3,\Sigma_j,\tilde\rho} \le \tfrac{1}{(64\alpha)^4\mathrm{B}^2}\,\mathfrak{e}_j(\|K\|_{1,\Sigma_j}).$$

In particular,

$$|w^a_4(0)|_{3,\Sigma_j,\tilde\rho} \le \tfrac{1}{(64\alpha)^4\mathrm{B}^2}\,\mathfrak{c}_j.$$

Therefore, by Lemma XVII.5 of [FKTo3] and Corollary A.5.ii of [FKTo1],

$$\begin{aligned}\big|\delta f_3'\big|_{3,\Sigma_j,\tilde\rho} &\le \sum_{\ell=1}^{\infty}\big(\mathrm{const}\,\lambda_0^{1-\upsilon}\mathfrak{l}_j^{1/n_0}\mathfrak{c}_j\big)^{\ell}\,\big|w_4^a(0)\big|_{3,\Sigma_j,\tilde\rho}^{\ell+1}\\ &\le \sum_{\ell=1}^{\infty}\big(\mathrm{const}\,\lambda_0^{1-\upsilon}\mathfrak{l}_j^{1/n_0}\mathfrak{c}_j\big)^{\ell}\big(\tfrac{1}{\alpha^4}\mathfrak{c}_j\big)^{\ell+1}\\ &\le \mathrm{const}\,\tfrac{\lambda_0^{1-\upsilon}}{\alpha^8}\mathfrak{l}_j^{1/n_0}\,\mathfrak{c}_j^3\,\frac{1}{1-\mathrm{const}\,\frac{\lambda_0^{1-\upsilon}}{\alpha^4}\mathfrak{l}_j^{1/n_0}\,\mathfrak{c}_j^2}\\ &\le \mathrm{const}\,\tfrac{\lambda_0^{1-\upsilon}}{\alpha^8}\mathfrak{l}_j^{1/n_0}\,\mathfrak{c}_j. \end{aligned}\tag{XV.20}$$

Hence, by (XV.18), (XV.19) and (XV.20),

$$\big|\delta f'^{(j+1)}(0)\big|_{3,\Sigma_j,\tilde\rho} \le \Big\{\tfrac{\lambda_0^{1-2\upsilon}}{\alpha^{10}}\mathfrak{l}_j + \tfrac{\mathfrak{l}_j}{\alpha^9} + \tfrac{\mathrm{const}\,\lambda_0^{1-\upsilon}}{\alpha^8}\mathfrak{l}_j^{1/n_0}\Big\}\mathfrak{c}_j \le \tfrac{\mathfrak{l}_{j+1}^{1/n_0}}{\alpha^8}\mathfrak{c}_j \tag{XV.21}$$

and by (XV.17),

$$\big|\tfrac{d}{ds}\delta f'^{(j+1)}(K+sK')\big|_{s=0}\big|^{\sim}_{3,\Sigma_j,\tilde\rho} \le \tfrac{1}{\alpha^4\mathrm{B}^2}\,\mathfrak{e}_j(\|K\|_{1,\Sigma_j})\,M^j\,\|K'\|_{1,\Sigma_j}. \tag{XV.22}$$

Observe that $D_j(u(0);0) = C_{u(0)}^{(\ge j+1)}$ and, by Lemma XIV.3,

$$\begin{aligned}L_\ell\big(w_4^a(0);C_{u(0)}^{(j)},C_{u(0)}^{(\ge j+1)}\big) &= V_{\mathrm{pp}}\Big(L_\ell\big(w_4^a(0);C_{u(0)}^{(j)},C_{u(0)}^{(\ge j+1)}\big)^{\mathrm{pp}}\Big)\\ &\quad + V_{\mathrm{ph}}\Big(L_\ell\big(w_4^a(0);C_{u(0)}^{(j)},C_{u(0)}^{(\ge j+1)}\big)^{\mathrm{ph}}\Big)\end{aligned}$$

so that the kernel of $\mathfrak{z}_4''(\phi,\psi+\hat B\phi;0)$ is

$$\begin{aligned}&\mathrm{shear}\Big(w_4^a(0)+\tfrac14\sum_{\ell=1}^{\infty}(-1)^{\ell}(12)^{\ell+1}\mathrm{Ant}\,V_{\mathrm{ph}}\Big(L_\ell\big(w_4^a(0);C_{u(0)}^{(j)},C_{u(0)}^{(\ge j+1)}\big)^{\mathrm{ph}}\Big),B\Big)\\ &\quad+\delta f_2'+\delta f_3'.\end{aligned}\tag{XV.23}$$

To verify $(O^{\sim}2)$, set $v'^{(i)} = v^{(i)}$ for all $2\le i\le j-1$, $v'^{(j)} = \delta f^{(j)}(0)$ and define $f'^{(i)}$, $2\le i\le j$, by

$$f'^{(i)} = \mathrm{shear}\big(v'^{(i)},C_{u(0)}^{[i,j]}(k)A(k)\big) = \begin{cases}\mathrm{shear}\big(f^{(i)},B\big) & \text{if } 2\le i\le j-1\\ \mathrm{shear}\big(\delta f^{(j)}(0),B\big) & \text{if } i=j\end{cases},$$

Observe that $v'^{(j)}$ vanishes unless all of its ϕ momenta are in the support of $\nu^{(<j)}$, since the same is true for $\delta f^{(j)}(0)$ by $(I^{\sim}1)$ and $(I^{\sim}2)$. Since $B(k)$ is supported on the j^{th} neighbourhood $f'^{(i)}_{\Sigma_j} = \mathrm{shear}\big(f^{(i)}_{\Sigma_j},B\big)$ for all $2\le i\le j-1$. Consequently, observing that the pure ψ parts of $f^{(i)}$ and $f'^{(i)}$ coincide and that only the pure ψ parts contribute to internal ladder vertices,

$$\mathrm{shear}\Big(\mathrm{Ant}\,\Big(V_{\mathrm{ph}}\big(\mathcal{L}^{(j)}(\vec p,\vec f)\big)\Big)_{\Sigma_j},B\Big) = \mathrm{Ant}\,\Big(V_{\mathrm{ph}}\big(\mathcal{L}^{(j)}(\vec p,\vec f')\big)\Big)_{\Sigma_j}.$$

By $(I^{\sim}2)$, the kernel of $\omega_4^a(\phi,\psi;0)+\mathcal{G}_4^a(\phi;0)$ is $\delta f^{(j)}(0)+\sum_{i=2}^{j-1} f_{\Sigma_j}^{(i)}+\frac{1}{8}\mathrm{Ant}\Big(V_{\rm ph}\big(\mathcal{L}^{(j)}(\vec p,\vec f)\big)\Big)_{\Sigma_j}$ so that the kernel of $\omega_4^a(\phi,\psi+\hat B\phi;0)+\mathcal{G}_4^a(\phi;0)$ is $\sum_{i=2}^{j} f'^{(i)}_{\Sigma_j}+\frac{1}{8}\mathrm{Ant}\Big(V_{\rm ph}\big(\mathcal{L}^{(j)}(\vec p,\vec f')\big)\Big)_{\Sigma_j}$. Therefore, by the Definition VII.7 of iterated particle–hole ladders,

$$\begin{aligned}
&\omega_4'^a(\phi,\psi;K)+\mathcal{G}_4'^a(\phi;K)\\
&\quad= Gr\Big(\phi,\psi+\hat B\phi;\, w_4^a(0)+\tfrac14\sum_{\ell=1}^{\infty}(-1)^\ell(12)^{\ell+1}\mathrm{Ant}\\
&\qquad\times V_{\rm ph}\Big(L_\ell\big(w_4^a(0);C_{u(0)}^{(j)},C_{u(0)}^{(\ge j+1)}\big)^{\rm ph}\Big)\Big)+\mathcal{G}_4^a(\phi;0)+\delta\mathfrak{f}'^{(j+1)}(\phi,\psi;K)\\
&\quad= Gr\Big(\phi,\psi;\sum_{i=2}^{j} f'^{(i)}_{\Sigma_j}+\delta f'^{(j+1)}(K)+\tfrac18\mathrm{Ant}\Big(V_{\rm ph}\big(\mathcal{L}^{(j)}(\vec p,\vec f')\big)\Big)_{\Sigma_j}\Big)\\
&\qquad+Gr\Big(\phi,\psi+\hat B\phi;\,\tfrac14\sum_{\ell=1}^{\infty}(-1)^\ell(12)^{\ell+1}\mathrm{Ant}\,V_{\rm ph}\Big(L_\ell\big(w_4^a(0);C_{u(0)}^{(j)},C_{u(0)}^{(\ge j+1)}\big)^{\rm ph}\Big)\Big)\\
&\quad= Gr\Big(\phi,\psi;\sum_{i=2}^{j} f'^{(i)}_{\Sigma_j}+\delta f'^{(j+1)}(K)+\tfrac18\mathrm{Ant}\Big(V_{\rm ph}\big(\mathcal{L}^{(j+1)}(\vec p,\vec f')\big)\Big)\Big).
\end{aligned}$$

The estimates on $\big|\delta f'^{(j+1)}(K)\big|_{3,\Sigma_j,\tilde\rho}$ required for $(O^{\sim}2)$ were proven in (XV.21) and (XV.22). By $(I^{\sim}2)$,

$$\big|v'^{(j)}\big|_{3,\Sigma_j,\tilde\rho}=\big|\delta f^{(j)}(0)\big|_{3,\Sigma_j,\tilde\rho}\le\tfrac{\mathfrak{l}_j^{1/n_0}}{\alpha^7}\mathfrak{c}_j$$

and $\big|v'^{(i)}\big|_{3,\Sigma_i,\tilde\rho}=\big|v^{(i)}\big|_{3,\Sigma_i,\tilde\rho}\le\tfrac{\mathfrak{l}_i^{1/n_0}}{\alpha^7}\mathfrak{c}_i$ for $2\le i\le j-1$.

Verification of $(O^{\sim}3)$. Let

$$2\check G_2(k;K)=C_{u(0)}^{(<j)}(k)+\tfrac{1}{(ik_0-e(\mathbf{k}))^2}\sum_{i=2}^{j-1}\Big\{\delta q^{(i)}(k;K)+\sum_{\ell=i}^{j}q^{(i,\ell)}(k)\Big\}$$

be the decomposition of $(I^{\sim}3)$ and set

$$\begin{aligned}
q'^{(i,\ell)}(k)&=q^{(i,\ell)}(k),\qquad 2\le i\le j-1,\ i\le\ell\le j,\\
q'^{(j,j)}(k)&=2\check z_{2,0}(k;0)-E^{(j)}(k;0),\\
\delta q'^{(i)}(k;K)&=\delta q^{(i)}(k;K),\qquad 2\le i\le j-1,\\
\delta q'^{(j)}(k;K)&=2\check z_{2,0}(k;K)-2\check z_{2,0}(k;0)-E^{(j)}(k;K)+E^{(j)}(k;0).
\end{aligned}$$

By Lemmas XV.5 and XV.6, both conclusions of Lemma IX.5.i of [FKTo2] and the fact that the transpose of J is $-J$,

$$2\check G_2'(k;K)=C_{u(0)}^{(\le j)}(k)+\tfrac{1}{(ik_0-e(\mathbf{k}))^2}\sum_{i=2}^{j}\Big\{\delta q'^{(i)}(k;K)+\sum_{\ell=i}^{j}q'^{(i,\ell)}(k)\Big\}.$$

By (XV.13), Lemma XII.12 of [FKTo3], Lemma XV.7.iii and (XV.16),

$$\begin{aligned}\big\| q^{(j,j)}(k) \big\|^{\tilde{}} &\le 4\tfrac{\lambda_0^{1-9\upsilon/7}}{\alpha^7}\tfrac{\mathfrak{l}_j}{M^j}\mathfrak{c}_j + \text{const}\,\tfrac{\lambda_0^{1-8\upsilon/7}}{\alpha^6}\tfrac{\mathfrak{l}_j}{M^j}\mathfrak{c}_j \le \lambda_0^{1-2\upsilon}\tfrac{\mathfrak{l}_j}{M^j}\mathfrak{c}_j,\\ \big\| \tfrac{d}{ds}\delta q'^{(j)}(K+sK')\big|_{s=0} \big\|^{\tilde{}} &\le \Big\{ \tfrac{\text{const}\,\lambda_0^{1-9\upsilon/7}}{\alpha^2}\mathfrak{e}_j\big(\|K\|_{1,\Sigma_j}\big) + const\ \mathfrak{e}_{j+\frac12}\big(\|K\|_{1,\Sigma_j}\big)\Big\}\|K'\|_{1,\Sigma_j}\\ &\le \sqrt{M^{\aleph'-\aleph}}\,\mathfrak{e}_{j+\frac12}\big(\|K\|_{1,\Sigma_j}\big)\,\|K'\|_{1,\Sigma_j},\end{aligned}$$

if M is large enough.

Recall that $\omega_{1,1}^a(\phi,\psi)$ vanishes unless $\hat{\nu}^{(<j)}\phi$ is nonzero. Hence, by conservation of momentum, $\big(w_{1,1}^a\big)_{\Sigma_{j+1}}$ vanishes and, by (XV.13) and Proposition XIX.4.ii of [FKTo4],

$$\begin{aligned}\big|\big(w'^{a\sim}_{1,1}\big)_{\Sigma_{j+1}}\big|^{\tilde{}}_{1,\Sigma_{j+1},\tilde\rho} = \big|\big(z^{\sim}_{1,1}\big)_{\Sigma_{j+1}}\big|^{\tilde{}}_{1,\Sigma_{j+1},\tilde\rho} &\le const\ \mathfrak{c}_j\big|z^{\sim}_{1,1}\big|^{\tilde{}}_{1,\Sigma_j,\tilde\rho}\\ &\le \tfrac{\text{const}}{\alpha^8}\tfrac{\mathfrak{l}_j}{M^j}\mathfrak{e}_j\big(\|K\|_{1,\Sigma_j}\big)\\ &\le \tfrac{1}{\alpha^7}\tfrac{\mathfrak{l}_j}{M^j}\mathfrak{e}_j\big(\|K\|_{1,\Sigma_j}\big)\end{aligned}$$

and, by (XV.15),

$$\begin{aligned}\big|\tfrac{d}{ds}w_{1,1'^{a\sim}}(K+sK')_{\Sigma_{j+1}}\big|_{s=0}\big|^{\tilde{}}_{1,\Sigma_{j+1},\tilde\rho} &= \big|\tfrac{d}{ds}z^{\sim}_{1,1}(K+sK')_{\Sigma_{j+1}}\big|_{s=0}\big|^{\tilde{}}_{1,\Sigma_{j+1},\tilde\rho}\\ &\le const\ \mathfrak{c}_j\big|\tfrac{d}{ds}z^{\sim}_{1,1}(K+sK')\big|_{s=0}\big|^{\tilde{}}_{1,\Sigma_j,\tilde\rho}\\ &\le \tfrac{\text{const}}{\alpha^2\mathrm{B}}\big(\tfrac{1}{\alpha^2}+\lambda_0^{\upsilon/7}\big)\,\mathfrak{e}_j(\|K\|_{1,\Sigma_j})\,\|K'\|_{1,\Sigma_j}\\ &\le \big(\tfrac{1}{\alpha}+\lambda_0^{\upsilon/8}\big)\tfrac{1}{\alpha^2\mathrm{B}}\,\mathfrak{e}_j(\|K\|_{1,\Sigma_j})\,\|K'\|_{1,\Sigma_j}.\end{aligned}$$

Verification of ($O^{\sim}4$). Apply Remark B.5 of [FKTo2]. □

3. Sector Refinement, ReWick Ordering and Renormalization.

Theorem XV.8. *If* $(\mathcal{W},\mathcal{G},u,\vec{p}) \in \tilde{\mathcal{D}}^{(j)}_{\text{out}}$ *then* $\mathcal{O}_j(\mathcal{W},\mathcal{G},u,\vec{p}) \in \tilde{\mathcal{D}}^{(j+1)}_{\text{in}}$.

The rest of this subsection is devoted to the proof of this theorem. Let $(\mathcal{W}',\mathcal{G}',u',\vec{p}\,') = \mathcal{O}_j(\mathcal{W},\mathcal{G},u,\vec{p})$.

Lemma XV.9.

$$\begin{aligned}\Big\|A(k)^2\big[C^{(\le j)}_{u'(0)}(k) - C^{(\le j)}_{u(0)}(k)\big]\Big\|^{\tilde{}} &\le \text{const}\ \tfrac{\lambda_0^{1-\upsilon}}{\alpha^5}\tfrac{\mathfrak{l}_{j+1}}{M^{j+1}}\mathfrak{c}_{j,j+1},\\ \Big\|A(k)\big[C^{(\le j)}_{u'(0)}(k) - C^{(\le j)}_{u(0)}(k)\big]\Big\|^{\tilde{}} &\le \text{const}\ \tfrac{\lambda_0^{1-\upsilon}}{\alpha^5}\mathfrak{l}_j\mathfrak{c}_{j+1},\\ \Big\|A(k)\big[C^{[i,j]}_{u'(0)}(k) - C^{[i,j]}_{u(0)}(k)\big]\Big\|^{\tilde{}} &\le \text{const}\ \tfrac{\lambda_0^{1-\upsilon}}{\alpha^5}\mathfrak{l}_j\mathfrak{c}_{j+1}\quad \textit{for all } 2\le i\le j,\\ \big\|A(k)C^{(\le j)}_{u(0)}(k)\big\|^{\tilde{}},\ \big\|A(k)C^{(\le j)}_{u'(0)}(k)\big\|^{\tilde{}} &\le \text{const}\ \mathfrak{c}_{j+1},\\ \big\|A(k)C^{[i,j]}_{u(0)}(k)\big\|^{\tilde{}},\ \big\|A(k)C^{[i,j]}_{u'(0)}(k)\big\|^{\tilde{}} &\le \text{const}\ \mathfrak{c}_{j+1}\qquad \textit{for all } 2\le i\le j.\end{aligned}$$

Proof. Recall from Definition VIII.1 that $\check{u}'(k;0) - \check{u}(k;0) = \check{p}^{(j)}(k)$. Hence

$$A(k)^2\big[C^{(\le j)}_{u'(0)}(k) - C^{(\le j)}_{u(0)}(k)\big] = \frac{[ik_0 - e(\mathbf{k})]^2[\check{u}'(k;0) - \check{u}(k;0)]\nu^{(\le j)}(k)}{[ik_0 - e(\mathbf{k}) - \check{u}(k;0)][ik_0 - e(\mathbf{k}) - \check{u}'(k;0)]}$$
$$= \frac{\check{p}^{(j)}(k)\nu^{(\le j)}(k)}{\big[1 - \frac{\check{u}(k;0)}{ik_0 - e(\mathbf{k})}\big]\big[1 - \frac{\check{u}'(k;0)}{ik_0 - e(\mathbf{k})}\big]}.$$

By (XV.4), (XV.5) and (IX.21),

$$\big\|\nu^{(\le j)}(k)\big\|^{\tilde{}} \le const\ \mathfrak{c}_{j+\frac{1}{2}},$$
$$\|\check{u}(0)\|^{\tilde{}},\ \|\check{u}'(0)\|^{\tilde{}} \le const\ \tfrac{1}{M^j}\mathfrak{c}_j,$$
$$\big\|\check{p}^{(j)}(k)\big\|^{\tilde{}} \le 2\tfrac{\lambda_0^{1-\upsilon}}{\alpha^5}\ \tfrac{\mathfrak{l}_j}{M^j}\mathfrak{c}_j.$$

Since $\Big|\frac{\check{u}(k;0)}{ik_0 - e(\mathbf{k})}\Big| \le \frac{1}{2}$ and $\Big|\frac{u'(k;0)}{ik_0 - e(\mathbf{k})}\Big| \le \frac{1}{2}$ on the support of $\nu^{(\le j)}$, we have, using the second bound of (XV.4) to control the denominator,

$$\left\|\frac{\check{p}^{(j)}(k)\nu^{(\le j)}(k)}{\big[1 - \frac{\check{u}(k;0)}{ik_0 - e(\mathbf{k})}\big]\big[1 - \frac{\check{u}'(k;0)}{ik_0 - e(\mathbf{k})}\big]}\right\|^{\tilde{}} \le const\ \frac{\lambda_0^{1-\upsilon}}{\alpha^5}\ \frac{\mathfrak{l}_j}{M^j}\mathfrak{c}_{j+\frac{1}{2}} \le \text{const}\ \frac{\lambda_0^{1-\upsilon}}{\alpha^5}\ \frac{\mathfrak{l}_{j+1}}{M^{j+1}}\mathfrak{c}_{j,j+1}.$$

The proof of the second and third bounds are virtually identical, using

$$\big\|\nu^{[i,j]}(k)\big\|^{\tilde{}} \le const\ \mathfrak{c}_{j+\frac{1}{2}},$$

and with one additional use of the second bound of (XV.4). The proof of the final two bounds uses

$$(ik_0 - e(\mathbf{k}))C^I_{u(0)}(k) = (ik_0 - e(\mathbf{k}))\frac{\nu^I(k)}{ik_0 - e(\mathbf{k}) - \check{u}(k;0)} = \frac{\nu^I(k)}{1 - \frac{\check{u}(k;0)}{ik_0 - e(\mathbf{k})}}$$

with $I = [i,j]$ and $I = (\ge j)$, and (XV.5) and the corresponding properties with $u(0)$ replaced by $u'(0)$. □

Proof of Theorem XV.8. Let

$$w = \sum_{m,n}\sum_{s_1,\cdots,s_n\in\Sigma_j}\int d\eta_1\cdots d\eta_m\, d\xi_1\cdots d\xi_n\ w_{m,n}(\eta_1,\cdots,\eta_m\,(\xi_1,s_1),\cdots,(\xi_n,s_n);\,K)$$
$$\times\,\phi(\eta_1)\cdots\phi(\eta_m)\ \psi((\xi_1,s_1))\cdots\psi((\xi_n,s_n))$$

be the Σ_j–sectorized representative of $\mathcal{W}$ specified in (O1). Choose the Σ_{j+1}–sectorized representative w' of $\mathcal{W}'$ as in (IX.28), where w'', the Σ_j–sectorized representative, was defined in (IX.18) and $\tilde{w}$ was defined in (IX.16). Recall that, by (IX.19),

$$\mathcal{G}'(\phi;K') = \mathcal{G}(\phi;K(K')) + \tilde{w}(\phi,0;K') - \tilde{w}(0,0;K'). \tag{XV.24}$$

Preparation for the verification of ($I^\sim 1$), ($I^\sim 2$) and ($I^\sim 3$). Recall from (IX.29) that

$$\tilde{w}(\phi,\psi;K') = {:}w(\phi,\psi;K(K')){:}_{\psi,-E_{\Sigma_j}(K';q_0)},$$

where $E_{\Sigma_j}(K';q)$ was defined just before (IX.16). We proved in Lemma B.1.iii that $|E(K';q_0)| \le \frac{\lambda_0^{1-\upsilon}\mathfrak{l}_j}{|ik_0-e(\mathbf{k})|}$. Hence, by Proposition XVI.8.ii of [FKTo3], $\sqrt{2\lambda_0^{1-\upsilon}\mathfrak{l}_j \mathrm{B}_3 \frac{\mathfrak{l}_j}{M^j}} \le \sqrt{\lambda_0^{1-\upsilon}\mathfrak{l}_j}\sqrt{\frac{\mathfrak{l}_j \mathrm{B}}{M^j}}$ is an integral bound for E_{Σ_j} for the configuration $|\cdot|^{\sim}_{p,\Sigma_j,\tilde{\rho}}$ of seminorms. Hence by Corollary II.32.ii of [FKTr1],

$$N_j^\sim\Big(\tilde{w}^a(K') - w^a\big(K(K')\big), \tfrac{\alpha}{2}, X\Big) \le \tfrac{8\lambda_0^{1-\upsilon}}{\alpha^2}\mathfrak{l}_j\, N_j^\sim\big(w^a(K(K')),\alpha, X\big) \tag{XV.25}$$

for all $X \in \mathfrak{N}_{d+1}$. In particular

$$N_j^\sim\big(\tilde{w}^a(K'), \tfrac{\alpha}{2}, X\big) \le \tfrac{3}{2} N_j^\sim\big(w^a(K(K')),\alpha, X\big). \tag{XV.26}$$

As in (IX.35),

$$\begin{aligned} N_j^\sim\Big(\tfrac{d}{ds}\tilde{w}^a(K'+sK'')\big|_{s=0} - \tfrac{d}{ds}w^a\big(K(K'+sK'')\big)\big|_{s=0}, \tfrac{\alpha}{2}, \|K'\|_{1,\Sigma_{j+1}}\Big)&\\ \le \tfrac{\mathrm{const}}{(\alpha-1)^2} M^j \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\|K''\|_{1,\Sigma_{j+1}}& \end{aligned} \tag{XV.27}$$

and

$$\begin{aligned} N_j^\sim\Big(\tfrac{d}{ds}\tilde{w}^a(K'+sK'')\big|_{s=0}, \tfrac{\alpha}{2}, \|K'\|_{1,\Sigma_{j+1}}\Big)&\\ \le const\, M^{j+\aleph}\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\|K''\|_{1,\Sigma_{j+1}}.& \end{aligned} \tag{XV.28}$$

Verification of ($I^\sim 3$). Let $\tilde{w}_{2,0}$ be kernel of the part of $\tilde{w}$ that is of degree two in ϕ and degree zero in ψ. By (XV.24) and ($O^\sim 3$),

$$\begin{aligned} 2\check{G}_2'(k;K') &= 2\check{G}_2(k;K(K')) + 2(\tilde{w}_{2,0})\check{}(k;K')\\ &= C_{u(0)}^{(\le j)}(k) + \tfrac{1}{(ik_0-e(\mathbf{k}))^2}\sum_{i=2}^{j}\Big\{\delta q^{(i)}(k;K(K')) + \sum_{\ell=i}^{j} q^{(i,\ell)}(k)\Big\}\\ &\quad + 2(\tilde{w}_{2,0})\check{}(k;K')\\ &= C_{u'(0)}^{(<j+1)}(k) + \tfrac{1}{(ik_0-e(\mathbf{k}))^2}\sum_{i=2}^{j}\Big\{\delta q'^{(i)}(k;K') + \sum_{\ell=i}^{j+1} q'^{(i,\ell)}(k)\Big\} \end{aligned}$$

with

$$\begin{aligned} q'^{(i,\ell)}(k) &= q^{(i,\ell)}(k) && 2\le i\le j,\ i\le \ell\le j,\\ q'^{(i,j+1)}(k) &= \delta q^{(i)}(k;K(0)) && 2\le i\le j-1,\\ \delta q'^{(i)}(k;K') &= \delta q^{(i)}(k;K(K')) - \delta q^{(i)}(k;K(0)) && 2\le i\le j-1,\\ q'^{(j,j+1)}(k) &= \delta q^{(j)}(k;K(0)) + 2(\tilde{w}^a_{2,0})\check{}(k;0) + [ik_0-e(\mathbf{k})]^2\big[C_{u(0)}^{(\le j)}(k) - C_{u'(0)}^{(\le j)}(k)\big],\\ \delta q'^{(j)}(k;K') &= \delta q^{(j)}(k;K(K')) - \delta q^{(j)}(k;K(0)) + 2(\tilde{w}^a_{2,0})\check{}(k;K') - 2(\tilde{w}^a_{2,0})\check{}(k;0). \end{aligned}$$

By $(O^{\sim}3)$, Lemma IX.8.i and Remark A.6 of [FKTo1],

$$\begin{aligned}
&\left\|\delta q^{(i)}(k;\,K(0))\right\|^{\sim} \\
&\quad= \left\|\delta q^{(i)}(k;\,\delta K(0))\right\|^{\sim} \\
&\quad\le M^{\aleph'(j-i)}\mathfrak{e}_{i+\frac12,j+\frac12}\big(\|\delta K(0)\|_{1,\Sigma_j}\big)\,\|\delta K(0)\|_{1,\Sigma_j}\begin{cases}1 & \text{if } i<j\\ \sqrt{M^{\aleph'-\aleph}} & \text{if } i=j\end{cases} \\
&\quad\le \mathit{const}\,\sqrt{M^{\aleph'-\aleph}}M^{\aleph'(j-i)}\,\frac{\lambda_0^{1-\upsilon}}{\alpha^6}\,\frac{\mathfrak{l}_j}{M^j}\,\mathfrak{c}_{i+\frac12,j+\frac12} \\
&\quad\le M^{\aleph'(j+1-i)}\,\frac{\lambda_0^{1-\upsilon}}{\alpha^5}\,\frac{\mathfrak{l}_{j+1}}{M^{j+1}}\,\mathfrak{c}_{i,j+1}
\end{aligned}$$

if α is large enough. This implies the desired bound on $q'^{(i,j+1)}$ for $i\le j-1$. By Lemma XII.12 of [FKTo3], (XV.25) (recall that $w^a_{2,0}=0$), $(O^{\sim}1)$ and Lemma IX.8.iii,

$$\begin{aligned}
\left\|(\tilde w^a_{2,0})\check{}\,(k;0)\right\|^{\sim} \le 2\,\big|(\tilde w^a_{2,0})\check{}\,(k;0)\big|^{\sim}_{1,\Sigma_j} &\le \frac{\mathrm{const}\,\lambda_0^{1-9\upsilon/7}}{\alpha^2M^j}\,\frac{8\lambda_0^{1-\upsilon}}{\alpha^2}\,\mathfrak{l}_j\,\mathfrak{e}_j\big(\|\delta K(0)\|_{1,\Sigma_j}\big) \\
&\le \frac{\lambda_0^{2-3\upsilon}}{\alpha^4}\,\frac{\mathfrak{l}_j}{M^j}\,\mathfrak{c}_{j,j+1}.
\end{aligned}$$

So, by Lemma XV.9,

$$\left\|q'^{(j,j+1)}\right\|^{\sim}\le \lambda_0^{1-\upsilon}\,\frac{\mathfrak{l}_{j+1}}{M^{j+1}}\,\mathfrak{c}_{j,j+1}\left[\frac{M^{\aleph'}}{\alpha^5}+2\frac{\lambda_0^{1-2\upsilon}}{\alpha^4}M^{\aleph}+\frac{\mathrm{const}}{\alpha^5}\right]\le \frac{\lambda_0^{1-\upsilon}}{\alpha^4}\,\frac{\mathfrak{l}_{j+1}}{M^{j+1}}\,\mathfrak{c}_{j,j+1}.$$

By (XV.28),

$$\begin{aligned}
\left\|\tfrac{d}{ds}(\tilde w^a_{2,0})\check{}\,(K'+sK'')\big|_{s=0}\right\|^{\sim} &\le \frac{\mathrm{const}\,\lambda_0^{1-9\upsilon/7}}{\alpha^2M^j}\,M^{j+\aleph}\,\mathfrak{e}_{j+1}\big(\|K'\|_{1,\Sigma_{j+1}}\big)\|K''\|_{1,\Sigma_{j+1}} \\
&\le \frac{\lambda_0^{1-2\upsilon}}{\alpha^2}\,\mathfrak{e}_{j+\frac12,j+1}\big(\|K'\|_{1,\Sigma_{j+1}}\big)\|K''\|_{1,\Sigma_{j+1}} \qquad \text{(XV.29)}
\end{aligned}$$

and, by $(O^{\sim}3)$ and parts (ii) and (iii) of Lemma IX.8,

$$\begin{aligned}
&\left\|\tfrac{d}{ds}\delta q^{(i)}\big(K(K'+sK'')\big)\big|_{s=0}\right\|^{\sim} = \left\|\tfrac{d}{ds}\delta q^{(i)}\Big(K(K')+s\tfrac{d}{dx}K(K'+xK'')\big|_{x=0}\Big)\Big|_{s=0}\right\|^{\sim} \\
&\quad\le M^{\aleph'(j-i)}\mathfrak{e}_{i+\frac12,j+\frac12}\big(\|K(K')\|_{1,\Sigma_j}\big) \\
&\qquad\times\left\|\tfrac{d}{dx}K(K'+xK'')\big|_{x=0}\right\|_{1,\Sigma_j}\begin{cases}1 & \text{if } i<j\\ \sqrt{M^{\aleph'-\aleph}} & \text{if } i=j\end{cases} \\
&\quad\le \mathit{const}\,M^{\aleph'(j-i)}\mathfrak{e}_{i+\frac12,j+\frac32}\big(\|K'\|_{1,\Sigma_{j+1}}\big)\,M^{\aleph}\mathfrak{e}_{0,j}\big(\|K'\|_{1,\Sigma_{j+1}}\big)\|K''\|_{1,\Sigma_{j+1}}\sqrt{M^{\aleph'-\aleph}} \\
&\quad\le \tfrac12\,M^{\aleph'(j+1-i)}\mathfrak{e}_{i+\frac12,j+\frac32}\big(\|K'\|_{1,\Sigma_{j+1}}\big)\,\|K''\|_{1,\Sigma_{j+1}}
\end{aligned}$$

if M is large enough. This and, when $i=j$, (XV.29) give the desired bound on $\frac{d}{ds}\delta q'^{(i)}(K'+sK'')$.

As $w'^a_{1,1}(K')=\tilde w^a_{1,1}(K')_{\Sigma_{j+1}}$, we have, by Proposition XIX.4.ii of [FKTo4], (XV.25), $(O^{\sim}1)$, $(O^{\sim}3)$ and Lemma IX.8.iii,

$$\begin{aligned}
\big|w'^{a\sim}_{1,1}(K')\big|^{\sim}_{1,\Sigma_{j+1},\tilde\rho} &\le \mathit{const}\,\mathfrak{c}_j\,\big|\tilde w^{a\sim}_{1,1}(K')-w^{a\sim}_{1,1}\big(K(K')\big)\big|^{\sim}_{1,\Sigma_j,\tilde\rho} \\
&\qquad+\big|w^{a\sim}_{1,1}\big(K(K')\big)_{\Sigma_{j+1}}\big|^{\sim}_{1,\Sigma_{j+1},\tilde\rho} \\
&\le \mathit{const}\,\mathfrak{c}_j\,\frac{1}{M^j\mathrm{B}(\alpha/2)^2}\,\frac{8\lambda_0^{1-\upsilon}}{\alpha^2}\,\mathfrak{l}_j\,N_j^{\sim}\big(w^a(K(K')),\alpha,\ \|K(K')\|_{1,\Sigma_j}\big)
\end{aligned}$$

$$
\begin{aligned}
&+\big| w^{a\sim}_{1,1}(K(K'))_{\Sigma_{j+1}}\big|^{\sim}_{1,\Sigma_{j+1},\tilde\rho}\\
&\le const\ \mathfrak{c}_j\ \tfrac{\lambda_0^{1-\upsilon}}{\alpha^4}\tfrac{\mathfrak{l}_j}{M^j}\mathfrak{e}_j(\|K(K')\|_{1,\Sigma_j}) + \tfrac{1}{\alpha^7}\tfrac{\mathfrak{l}_j}{M^j}\mathfrak{e}_j(\|K(K')\|_{1,\Sigma_j})\\
&\le const\ \tfrac{\mathfrak{l}_j}{M^j}\Big[\tfrac{\lambda_0^{1-\upsilon}}{\alpha^4}+\tfrac{1}{\alpha^7}\Big]\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\\
&\le \tfrac{1}{\alpha^6}\tfrac{\mathfrak{l}_{j+1}}{M^{j+1}}\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}}).
\end{aligned}\tag{XV.30}
$$

The derivative is bounded similarly, using (XV.27), ($O^{\sim}3$), and parts (ii) and (iii) of Lemma IX.8,

$$
\begin{aligned}
&\big|\tfrac{d}{ds}w'^{a\sim}_{1,1}(K'+sK'')\big|_{s=0}\big|^{\sim}_{1,\Sigma_{j+1},\tilde\rho}\\
&\quad\le const\ \mathfrak{c}_j\ \big|\tfrac{d}{ds}\big\{\tilde w^{a\sim}_{1,1}(K'+sK'') - w^{a\sim}_{1,1}\big(K(K'+sK'')\big)\big\}\big|_{s=0}\big|^{\sim}_{1,\Sigma_j,\tilde\rho}\\
&\qquad+\big|\tfrac{d}{ds}w^{a\sim}_{1,1}\big(K(K'+sK'')\big)_{\Sigma_{j+1}}\big|_{s=0}\big|^{\sim}_{1,\Sigma_{j+1},\tilde\rho}\\
&\quad\le \tfrac{\rm const}{(\alpha-1)^2\alpha^2{\rm B}}\ \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\|K''\|_{1,\Sigma_{j+1}}\\
&\qquad+\big(\tfrac{1}{\alpha}+\lambda_0^{\upsilon/8}\big)\tfrac{1}{\alpha^2{\rm B}}\mathfrak{e}_j(\|K(K')\|_{1,\Sigma_j})\big\|\tfrac{d}{dx}K(K'+xK'')\big|_{x=0}\big\|_{1,\Sigma_j}\\
&\quad\le \tfrac{1}{\alpha^2{\rm B}}\Big[\tfrac{\rm const}{(\alpha-1)^2}+const\ M^{\aleph}\big(\tfrac{1}{\alpha}+\lambda_0^{\upsilon/8}\big)\Big]\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\|K''\|_{1,\Sigma_{j+1}}.
\end{aligned}\tag{XV.31}
$$

Verification of ($I^{\sim}2$). Observe that by (XV.25), ($O^{\sim}1$) and Lemma IX.8.iii,

$$
\begin{aligned}
&\tfrac{M^{2j}}{\mathfrak{l}_j}\ \tfrac{\alpha^4}{16}\left(\tfrac{\mathfrak{l}_j{\rm B}}{M^j}\right)^2\tfrac{1}{\mathfrak{l}_j}\ \big|\tilde w^{a\sim}_4(K') - w^{a\sim}_4\big(K(K')\big)\big|^{\sim}_{3,\Sigma_j,\tilde\rho}\\
&\quad\le \tfrac{8\lambda_0^{1-\upsilon}}{\alpha^2}\mathfrak{l}_j\ N^{\sim}_j\big(w^a(K(K')),\alpha,\ \|K(K')\|_{1,\Sigma_j}\big)\\
&\quad\le \tfrac{8\lambda_0^{1-\upsilon}}{\alpha^2}\mathfrak{l}_j\ \mathfrak{e}_j(\|K(K')\|_{1,\Sigma_j})\\
&\quad\le const\,\tfrac{\lambda_0^{1-\upsilon}}{\alpha^2}\mathfrak{l}_j\ \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})
\end{aligned}
$$

so that, by Proposition XIX.4.ii of [FKTo4] and Corollary A.5.ii of [FKTo1],

$$
\Big|\Big(\tilde w^{a\sim}_4(K') - w^{a\sim}_4\big(K(K')\big)\Big)_{\Sigma_{j+1}}\Big|^{\sim}_{3,\Sigma_{j+1},\tilde\rho} \le const\,\tfrac{\lambda_0^{1-\upsilon}}{\alpha^6}\mathfrak{l}_j\ \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}}).\tag{XV.32}
$$

Set, for all $2\le i\le j$, $v'^{(i)} = v^{(i)}$ and define

$$
f'^{(i)} = {\rm shear}\Big(v'^{(i)}, C^{[i,j+1)}_{u'(0)}(k)A(k)\Big) = {\rm shear}\Big(f^{(i)}, \big[C^{[i,j]}_{u'(0)}(k) - C^{[i,j]}_{u(0)}(k)\big]A(k)\Big).
$$

Also set

$$
\begin{aligned}
\delta f'^{(j+1)}(K') = \delta f^{(j+1)}(K(K'))_{\Sigma_{j+1}} &+ \sum_{i=2}^{j}\big[f^{(i)}_{\Sigma_{j+1}} - f'^{(i)}_{\Sigma_{j+1}}\big]\\
&+\big(\tilde w^a_4(K') - w^a_4(K(K'))\big)_{\Sigma_{j+1}}\\
&+\tfrac18{\rm Ant}\Big(V_{\rm ph}\big(\mathcal{L}^{(j+1)}(\vec p\,',\vec f)\big)\Big)_{\Sigma_{j+1}}\\
&-\tfrac18{\rm Ant}\Big(V_{\rm ph}\big(\mathcal{L}^{(j+1)}(\vec p\,',\vec f')\big)\Big)_{\Sigma_{j+1}}.
\end{aligned}
$$

These definitions have been chosen so that

$$\begin{aligned} w_4'^a(K') + G_4'^a(K') &= \tilde w_4^a(K')_{\Sigma_{j+1}} + G_4^a(K(K')) \\ &= \delta f'^{(j+1)}(K') + \sum_{i=2}^{j} f'^{(i)}_{\Sigma_{j+1}} + \tfrac18 \mathrm{Ant}\Big(V_{\rm ph}\big(\mathcal{L}^{(j+1)}(\vec p\,', \vec f\,')\big)\Big)_{\Sigma_{j+1}}. \end{aligned}$$

Furthermore the required bounds

$$\big|v'^{(i)}\big|^{\sim}_{3,\Sigma_i,\tilde\rho} \le \frac{\mathfrak{l}_i^{1/n_0}}{\alpha^7}\mathfrak{c}_i \qquad\qquad \text{for all } 2\le i\le j$$

are trivially satisfied, so it remains only to bound $\big|\delta f'^{(j+1)}(K')\big|^{\sim}_{3,\Sigma_{j+1},\tilde\rho}$.

Since $\check u'(k;0) - \check u(k;0) = \check p_j(k)$, $C^{[i,j]}_{u'(0)}(k) - C^{[i,j]}_{u(0)}(k)$ is supported in the $j^{\rm th}$ neighbourhood and

$$f'^{(i)}_{\Sigma_j} = \mathrm{shear}\Big(f^{(i)}_{\Sigma_j}, \big[C^{[i,j]}_{u'(0)}(k) - C^{[i,j]}_{u(0)}(k)\big]A(k)\Big).$$

Hence by Lemma XVII.5 of [FKTo3], with $X=0$ and $X_B = \mathrm{const}\,\frac{\lambda_0^{1-\upsilon}}{\alpha^5}\mathfrak{l}_j$ and Lemma XV.9, followed by Proposition XIX.4.ii of [FKTo4],

$$\begin{aligned} \big|f'^{(i)}_{\Sigma_j} - f^{(i)}_{\Sigma_j}\big|^{\sim}_{3,\Sigma_j,\tilde\rho} &\le \mathrm{const}\,\frac{\lambda_0^{1-\upsilon}}{\alpha^5}\mathfrak{l}_j\,\mathfrak{c}_j\big|f^{(i)}_{\Sigma_j}\big|^{\sim}_{3,\Sigma_j,\tilde\rho} \\ &\le \mathrm{const}\,\frac{\lambda_0^{1-\upsilon}}{\alpha^5}\mathfrak{l}_j\,\mathfrak{c}_j\big|f^{(i)}\big|^{\sim}_{3,\Sigma_i,\tilde\rho}. \end{aligned} \tag{XV.33}$$

By Lemma XVII.5 of [FKTo3], with j replaced by i, $X=0$, $X_B = \mathrm{const}\,\mathfrak{c}_{j+1}$ and Lemma XV.9, followed by ($O^\sim 2$),

$$\begin{aligned} \big|f^{(i)}\big|^{\sim}_{3,\Sigma_i,\tilde\rho} &\le \mathrm{const}\,\mathfrak{c}_{j+1}\big|v^{(i)}\big|^{\sim}_{3,\Sigma_i,\tilde\rho} \le \mathrm{const}\,\frac{\mathfrak{l}_i^{1/n_0}}{\alpha^7}\mathfrak{c}_{j+1} \\ \big|f'^{(i)}\big|^{\sim}_{3,\Sigma_i,\tilde\rho} &\le \mathrm{const}\,\mathfrak{c}_{j+1}\big|v^{(i)}\big|^{\sim}_{3,\Sigma_i,\tilde\rho} \le \mathrm{const}\,\frac{\mathfrak{l}_i^{1/n_0}}{\alpha^7}\mathfrak{c}_{j+1} \end{aligned} \tag{XV.34}$$

so that, by Proposition XIX.4.ii of [FKTo4] and Corollary A.5 of [FKTo1],

$$\Big|\sum_{i=2}^{j}\big[f^{(i)}_{\Sigma_{j+1}} - f'^{(i)}_{\Sigma_{j+1}}\big]\Big|^{\sim}_{3,\Sigma_{j+1},\tilde\rho} \le \mathrm{const}\,\sum_{i=2}^{j}\frac{\lambda_0^{1-\upsilon}}{\alpha^{12}}\mathfrak{l}_j\mathfrak{l}_i^{1/n_0}\mathfrak{c}_{j+1} \le \frac{\lambda_0^{1-\upsilon}}{\alpha^{11}}\mathfrak{l}_j\mathfrak{c}_{j+1}. \tag{XV.35}$$

Also by Lemma XV.9 and Corollary XIV.15.ii, with j replaced by $j+1$, $\rho = \lambda_0^{1-9\upsilon/7}$, $\varepsilon = \frac{\aleph}{n_0}$, $B' = [C^{(\le j)}_{u'(0)}(k) - C^{(\le j)}_{u(0)}(k)]A(k)$, $B = C^{(\le j)}_{u(0)}(k)A(k) + sB'$, for $0\le s\le 1$, $c_B = \mathrm{const}$ and $c' = \frac{\lambda_0^{1-\upsilon}}{\alpha^5}\mathfrak{l}_j$

$$\begin{aligned} &\Big|\mathrm{Ant}\,V_{\rm ph}\big(\mathcal{L}^{(j+1)}(\vec p\,', \vec f\,)_{\Sigma_{j+1}}\big) - \mathrm{Ant}\,V_{\rm ph}\big(\mathcal{L}^{(j+1)}(\vec p\,', \vec f\,')_{\Sigma_{j+1}}\big)\Big|^{\sim}_{3,\Sigma_{j+1},\tilde\rho} \\ &\quad\le \mathrm{const}\,\frac{\lambda_0^{2-18\upsilon/7}}{\alpha^5}\mathfrak{l}_j\mathfrak{c}_{j+1}. \end{aligned} \tag{XV.36}$$

By Definition VII.7, $\mathcal{L}^{(j+1)}(\vec p, \vec f\,)$ depends only on $p^{(2)},\cdots,p^{(j-1)}$ and $f^{(2)},\cdots,f^{(j)}$. In particular $\mathcal{L}^{(j+1)}(\vec p\,', \vec f\,) = \mathcal{L}^{(j+1)}(\vec p, \vec f\,)$.

To bound $\left|\delta f'^{(j+1)}(0)\right|^{\sim}_{3,\Sigma_{j+1},\tilde\rho}$, we first use Proposition XIX.4.ii of [FKTo4], (XV.32), (XV.35) and (XV.36) to get the first line, then $(O^{\sim}2)$ to get the second line and Lemma IX.8 and Corollary A.5.ii of [FKTo1] to get the third line,

$$\begin{aligned}
\left|\delta f'^{(j+1)}(0)\right|^{\sim}_{3,\Sigma_{j+1},\tilde\rho} &\le \mathit{const}\ \mathfrak{c}_j \left|\delta f^{(j+1)}(K(0))\right|^{\sim}_{3,\Sigma_j,\tilde\rho} + \mathit{const}\ \tfrac{\lambda_0^{1-\upsilon}}{\alpha^5}\mathfrak{l}_j\ \mathfrak{c}_{j+1}\\
&\le \mathit{const}\ \mathfrak{c}_j\left\{\tfrac{\mathfrak{l}_{j+1}^{1/n_0}}{\alpha^8} + \tfrac{1}{\mathrm{B}^2\alpha^4}M^j\|K(0)\|_{1,\Sigma_j}\right\}\mathfrak{e}_j(\|K(0)\|_{1,\Sigma_j}) + \mathit{const}\tfrac{\lambda_0^{1-\upsilon}}{\alpha^5}\mathfrak{l}_j\ \mathfrak{c}_{j+1}\\
&\le \mathit{const}\left\{\tfrac{\mathfrak{l}_{j+1}^{1/n_0}}{\alpha^8} + \tfrac{\lambda_0^{1-\upsilon}}{\alpha^5}\mathfrak{l}_j\right\}\mathfrak{c}_{j+1}\\
&\le \tfrac{\mathfrak{l}_{j+1}^{1/n_0}}{\alpha^7}\mathfrak{c}_{j+1},
\end{aligned}\tag{XV.37}$$

since const $\lambda_0^{1-\upsilon}\mathfrak{l}_j \le \frac{\mathfrak{l}_{j+1}^{1/n_0}}{2\alpha^2}$. To bound the derivative of $\delta f'^{(j+1)}$, we first use Proposition XIX.4.ii of [FKTo4] to get the first line, then $(O^{\sim}2)$ and (XV.27) to get the second line and Lemma IX.8 and Corollary A.5.ii of [FKTo1] to get the third line,

$$\begin{aligned}
&\left|\tfrac{d}{ds}\delta f'^{(j+1)}(K'+sK'')\big|_{s=0}\right|^{\sim}_{3,\Sigma_{j+1},\tilde\rho}\\
&\quad\le \mathit{const}\ \mathfrak{c}_j\left|\tfrac{d}{ds}\delta f^{(j+1)}(K(K'+sK''))\big|_{s=0}\right|^{\sim}_{3,\Sigma_j,\tilde\rho}\\
&\qquad+\mathit{const}\ \mathfrak{c}_j\left|\tfrac{d}{ds}\tilde w_4^{a\sim}(K'+sK'')\big|_{s=0} - \tfrac{d}{ds}w_4^{a\sim}(K(K'+sK''))\big|_{s=0}\right|^{\sim}_{3,\Sigma_j,\tilde\rho}\\
&\quad\le \mathit{const}\tfrac{1}{\alpha^4\mathrm{B}^2}\mathfrak{e}_j(\|K(K')\|_{1,\Sigma_j})M^j\left\|\tfrac{d}{dx}K(K'+xK'')\big|_{x=0}\right\|_{1,\Sigma_j}\\
&\qquad+\tfrac{\mathrm{const}}{(\alpha-1)^2\alpha^4\mathrm{B}^2}M^j\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\|K''\|_{1,\Sigma_{j+1}}\\
&\quad\le \tfrac{1}{\alpha^4\mathrm{B}^2}\left\{\mathit{const}\tfrac{1}{M^{1-\aleph}} + \tfrac{\mathrm{const}}{(\alpha-1)^2}\right\}\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})M^{j+1}\|K''\|_{1,\Sigma_{j+1}}\\
&\quad\le \tfrac{1}{64\alpha^4\mathrm{B}^2}\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})M^{j+1}\|K''\|_{1,\Sigma_{j+1}}
\end{aligned}\tag{XV.38}$$

if M and α are large enough.

Verification of $(I^{\sim}1)$. Let $\tilde\omega_n$ and ω'_n be the parts of $\tilde w$ and w', respectively, that are of degree n in ϕ and ψ combined. By (XV.37), (XV.38), (XV.34), Proposition XIX.4.ii of [FKTo4] and Corollary XIV.15.i (with $\rho = \lambda_0^{1-9\upsilon/7}$, $\varepsilon = \frac{\aleph}{n_0}$, $B = C_{u'(0)}^{(\le j)}(k)A(k)$, $c_B = \mathrm{const}$)

$$\begin{aligned}
\left|w_{4'a\sim}(K')\right|^{\sim}_{3,\Sigma_{j+1},\tilde\rho} &\le \left|\delta f'^{(j+1)}(K')\right|^{\sim}_{3,\Sigma_{j+1},\tilde\rho} + \sum_{i=2}^{j}\left|f'^{(i)}_{\Sigma_{j+1}}\right|^{\sim}_{3,\Sigma_{j+1},\tilde\rho}\\
&\quad+\tfrac18\left|\mathrm{Ant}\left(V_{\mathrm{ph}}\big(\mathcal{L}^{(j+1)}(\vec p\,',\vec f')\big)\right)_{\Sigma_{j+1}}\right|^{\sim}_{3,\Sigma_{j+1},\tilde\rho}\\
&\le \tfrac{1}{\alpha^4\mathrm{B}^2}\big(\tfrac{\mathit{const}}{M^{1-\aleph}} + \tfrac{\mathrm{const}}{(\alpha-1)^2}\big)M^{j+1}\|K'\|_{1,\Sigma_{j+1}}\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\\
&\quad+\mathrm{const}\sum_{i=2}^{j+1}\tfrac{\mathfrak{l}_i^{1/n_0}}{\alpha^7}\mathfrak{c}_{j+1} + \mathrm{const}\,\lambda_0^{1-2\upsilon}\mathfrak{c}_{j+1}\\
&\le \tfrac{1}{\alpha^4\mathrm{B}^2}\left\{\tfrac{\mathrm{const}}{\alpha^3} + \tfrac{\mathit{const}}{M^{1-\aleph}} + \tfrac{\mathrm{const}}{(\alpha-1)^2}\right\}\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}}).
\end{aligned}$$

Consequently, by Proposition XIX.1 of [FKTo4],

$$\left|w_4'^{a\sim}(K')\right|^{\sim}_{1,\Sigma_{j+1},\tilde\rho} \le \tfrac{1}{\alpha^4 \mathrm{B}^2 \mathfrak{l}_{j+1}} \left\{\tfrac{\mathrm{const}}{\alpha^3} + \tfrac{const}{M^{1-\aleph}} + \tfrac{\mathrm{const}}{(\alpha-1)^2}\right\} \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}}).$$

Therefore, by Corollary A.5 of [FKTo1],

$$\begin{aligned}
&N^{\sim}_{j+1}(\omega_4'^{a}(K'), 64\alpha,\ \|K'\|_{1,\Sigma_{j+1}}) \\
&\quad \le 2^{24}\,\alpha^4\,\mathrm{B}^2\ \mathfrak{l}_{j+1}\,\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\Big(\left|w_4'^{a}(K')\right|^{\sim}_{1,\Sigma_{j+1},\tilde\rho} + \left|w_4'^{a\sim}(K')\right|^{\sim}_{2,\Sigma_{j+1},\tilde\rho} \\
&\qquad + \tfrac{1}{\mathfrak{l}_{j+1}}\left|w_4'^{a\sim}(K')\right|^{\sim}_{3,\Sigma_{j+1},\tilde\rho} + \tfrac{1}{\mathfrak{l}_{j+1}}\left|w_4'^{a\sim}(K')\right|^{\sim}_{4,\Sigma_{j+1},\tilde\rho}\Big) \\
&\quad \le 2^{25}\,\alpha^4\,\mathrm{B}^2\ \mathfrak{l}_{j+1}\,\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\Big(\left|w_4'^{a\sim}(K')\right|^{\sim}_{1,\Sigma_{j+1},\tilde\rho} + \tfrac{1}{\mathfrak{l}_{j+1}}\left|w_4'^{a\sim}(K')\right|^{\sim}_{3,\Sigma_{j+1},\tilde\rho}\Big) \\
&\quad \le \left\{\tfrac{\mathrm{const}}{\alpha^3} + \tfrac{const}{M^{1-\aleph}} + \tfrac{\mathrm{const}}{(\alpha-1)^2}\right\}\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}}) \\
&\quad \le \tfrac{1}{3}\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})
\end{aligned} \tag{XV.39}$$

if M and α are large enough. We have used that $|\ \cdot\ |^{\sim}_{2,\Sigma_{j+1},\tilde\rho} \le |\ \cdot\ |^{\sim}_{1,\Sigma_{j+1},\tilde\rho}$ and $|\cdot|^{\sim}_{4,\Sigma_{j+1},\tilde\rho} \le |\cdot|^{\sim}_{3,\Sigma_{j+1},\tilde\rho}$. Similarly, since $\omega'^{a}_{2,0}(K') = \omega'^{a}_{0,2}(K') = 0$, (XV.30) implies

$$\begin{aligned}
N^{\sim}_{j+1}(\omega_2'^{a}(K'), 64\alpha, \|K'\|_{1,\Sigma_{j+1}}) &\le 2^{12}\alpha^2 \mathrm{B} M^{j+1}\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\left|w'^{a\sim}_{1,1}(K')\right|^{\sim}_{1,\Sigma_{j+1},\tilde\rho} \\
&\le \tfrac{\mathrm{const}}{\alpha^4}\mathfrak{l}_{j+1}\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})^2 \\
&\le \tfrac{1}{3}\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}}).
\end{aligned} \tag{XV.40}$$

By (IX.18), (XV.26), $(O^{\sim}1)$ and Lemma IX.8.iii,

$$\begin{aligned}
N^{\sim}_{j}(w''^{a}(K'), \tfrac{\alpha}{2}, 0) &\le \tfrac{3}{2}\, N^{\sim}_{j}(w^{a}(K(K')), \alpha, 0) \\
&\le \tfrac{3}{2}\, N^{\sim}_{j}(w^{a}(K(K')), \alpha,\ \|K(K')\|_{1,\Sigma_j}) \\
&\le \tfrac{3}{2}\ \mathfrak{e}_{j}(\|K(K')\|_{1,\Sigma_j}) \\
&\le const\ \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})
\end{aligned}$$

so that, by Corollary XIX.9 of [FKTo4] and Corollary A.5 of [FKTo1],

$$\begin{aligned}
&N^{\sim}_{j+1}(w'^{a}(K') - \omega_4'^{a}(K') - \omega_2'^{a}(K'), 64\alpha, \|K'\|_{1,\Sigma_{j+1}}) \\
&\quad \le \tfrac{1}{M^{(1-\aleph)/8}}\ \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\, N^{\sim}_{j}\big(w''^{a}(K') - \omega_4''^{a}(K') - \omega_2''^{a}(K'),\ \tfrac{\alpha}{2},\ \|K'\|_{1,\Sigma_{j+1}}\big) \\
&\quad \le \tfrac{1}{M^{(1-\aleph)/8}}\ \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})^2\, N^{\sim}_{j}\big(w''^{a}(K'),\ \tfrac{\alpha}{2}, 0\big) \\
&\quad \le const\ \tfrac{1}{M^{(1-\aleph)/8}}\ \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})^3 \\
&\quad \le \tfrac{1}{3}\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}}).
\end{aligned} \tag{XV.41}$$

Combining (XV.39), (XV.40) and (XV.41), we get

$$N^{\sim}_{j+1}(w'(K'), 64\alpha, \|K'\|_{1,\Sigma_{j+1}}) \le \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}}).$$

By (IX.18) and (XV.28),

$$N^{\sim}_{j}\Big(\tfrac{d}{ds}w''^{a}(K' + sK'')\big|_{s=0},\ \tfrac{\alpha}{2},\ \|K'\|_{1,\Sigma_{j+1}}\Big) \le const\ M^{j+\aleph}\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\|K''\|_{1,\Sigma_{j+1}}.$$

Therefore by Corollary XIX.9 of [FKTo4] and Corollary A.5 of [FKTo1],

$$\begin{aligned}
&N^{\sim}_{j+1}\Big(\tfrac{d}{ds}w'^{a}(K'+sK'')\big|_{s=0}-\tfrac{d}{ds}\omega'^{a}_{2}(K'+sK'')\big|_{s=0},64\alpha,\|K'\|_{1,\Sigma_{j+1}}\Big)\\
&\le const\ \mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\,N^{\sim}_{j}\Big(\tfrac{d}{ds}w''(K'+sK'')\big|_{s=0},\tfrac{\alpha}{2},\|K'\|_{1,\Sigma_{j+1}}\Big)\\
&\le const\ M^{j+\aleph}\,\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})^2\|K''\|_{1,\Sigma_{j+1}}\\
&\le \tfrac{1}{2}M^{j+1}\,\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\|K''\|_{1,\Sigma_{j+1}}.
\end{aligned}$$

By (XV.31),

$$\begin{aligned}
&N^{\sim}_{j+1}\Big(\tfrac{d}{ds}\omega'^{a}_{2}(K'+sK'')\big|_{s=0},64\alpha,\|K'\|_{1,\Sigma_{j+1}}\Big)\\
&\le 64^2M^{j+1}\Big[\tfrac{\mathrm{const}}{(\alpha-1)^2}+const\ M^{\aleph}\big(\tfrac{1}{\alpha}+\lambda_0^{\upsilon/8}\big)\Big]\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})^2\|K''\|_{1,\Sigma_{j+1}}\\
&\le \tfrac{1}{2}M^{j+1}\,\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\|K''\|_{1,\Sigma_{j+1}}
\end{aligned}$$

so

$$N^{\sim}_{j+1}\Big(\tfrac{d}{ds}w'^{a}(K'+sK'')\big|_{s=0},64\alpha,\|K'\|_{1,\Sigma_{j+1}}\Big)\le M^{j+1}\,\mathfrak{e}_{j+1}(\|K'\|_{1,\Sigma_{j+1}})\|K''\|_{1,\Sigma_{j+1}}$$

as desired. That $w'^{a}(\phi,\psi;K)-w'^{a}(0,\psi;K)$ vanishes unless $\hat{\nu}^{(<j+1)}\phi$ is nonzero is inherited from the same property of w^a.

Verification of ($I^{\sim}4$). Bar/unbar invariance and k_0–reversal reality are inherited from the corresponding quantities in $\tilde{\mathcal{D}}^{(j)}_{\mathrm{out}}$ by Remarks B.2 and B.5 of [FKTo2]. □

Proof of Theorem XII.1. Initialization at $j=j_0$. As at the beginning of §X, let $\tilde{w}(\phi,\psi;K)$ be the Σ_{j_0}–sectorized representative for $\tilde{\Omega}_{C^{(\le j_0)}_{u(K)}}\big(\mathcal{V}(\psi)\big)(\phi,\psi)-\frac{1}{2}\phi\, JC^{(\le j_0)}_{u_{j_0}(K)}J\phi$ chosen in Theorems VI.12 and XIII.17 and set

$$w(\phi,\psi;K)=\tilde{w}(\phi,\psi;K)-\tilde{w}(\phi,0;K).$$

Set

- $v^{(2)}=\cdots=v^{(j_0)}=0$,
- $f^{(2)}=\cdots=f^{(j_0)}=0$,
- $q^{(i,\ell)}=0$ for all $2\le i\le\ell\le j_0$ except $i=\ell=j_0$,
- $\delta q^{(2)}=\cdots=\delta q^{(j_0-1)}=0$,

and define

$$\begin{aligned}
\delta f^{(j_0+1)}(K)&=\text{the kernel of the part of }\tilde{w}(\phi,\psi;K)\text{ that is quartic in }(\phi,\psi)\ ,\\
q^{(j_0,j_0)}&=2\tilde{w}^{a}_{2,0}(k;0),\\
\delta q^{(j_0)}(K)&=2\tilde{w}^{a}_{2,0}(k;K)-2\tilde{w}^{a}_{2,0}(k;0)+\big[C^{(\le j_0)}_{u_{j_0}(K)}(k)-C^{(\le j_0)}_{u_{j_0}(0)}(k)\big]\big(ik_0-e(\mathbf{k})\big)^2\\
&=2\tilde{w}^{a}_{2,0}(k;K)-2\tilde{w}^{a}_{2,0}(k;0)-\tfrac{U(\mathbf{k})-\nu^{(>j_0)}(k)}{1+\check{K}(\mathbf{k})/(ik_0-e(\mathbf{k}))}\check{K}(\mathbf{k}).
\end{aligned}$$

By Theorem XIII.17, with w^a replaced by $\tilde w^a$,

$$N^{\sim}_{j_0}\big(\tilde w^a(K),\alpha,\,\|K\|_{1,\Sigma_{j_0}}\big)\le \text{const}\ \alpha^4\lambda_0^{\upsilon}\,\mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big),$$
$$N^{\sim}_{j_0}\big(\tfrac{d}{ds}\tilde w^a(K+sK')\big|_{s=0},\alpha,\,\|K\|_{1,\Sigma_{j_0}}\big)\le \text{const}\ \alpha^4\lambda_0^{\upsilon}\,\mathfrak{e}_{j_0}\big(\|K\|_{1,\Sigma_{j_0}}\big).\,\|K'\|_{1,\Sigma_{j_0}}.$$

If λ_0 is sufficiently small, depending on M, and $\alpha<\frac{1}{\lambda_0^{\upsilon/10}}$ these bounds and Lemma XV.7.i (with $C^{(\le j_0)}$ replacing $C^{(j)}$) imply the bounds on

- $w^a(K)$ and $\frac{d}{ds}w^a(K+sK')$ imposed by $(O^{\sim}1)$,
- $\big|\delta f^{(j+1)}(0)\big|^{\sim}_{3,\Sigma_j,\tilde\rho}$ and $\big|\frac{d}{ds}\delta f^{(j+1)}(K+sK')\big|_{s=0}\big|^{\sim}_{3,\Sigma_j,\tilde\rho}$ imposed by $(O^{\sim}2)$,
- $\big\|q^{(j_0,j_0)}(k)\big\|^{\sim}$ and $\big\|\frac{d}{ds}\delta q^{(j_0)}(K+sK')\big|_{s=0}\big\|^{\sim}$ imposed by $(O^{\sim}3)$,
- $\big|w^{a\sim}_{1,1}(K)_{\Sigma_{j_0+1}}\big|^{\sim}_{1,\Sigma_{j_0+1},\tilde\rho}$ and $\big|\frac{d}{ds}w^{a\sim}_{1,1}(K+sK')_{\Sigma_{j_0+1}}\big|_{s=0}\big|^{\sim}_{1,\Sigma_{j_0+1},\tilde\rho}$ imposed by $(O^{\sim}3)$.

The support properties required by $(O^{\sim}1)$ and $(O^{\sim}3)$ follow from the conclusion in Theorem XIII.17 that $\tilde w^a(\phi,\psi,K)-\tilde w^a(0,\psi,K)$ vanishes unless $\hat\nu^{(\le j_0)}\phi$ is nonzero. Finally $(O^{\sim}4)$ follows from Remark B.5 of [FKTo2] and the requirement, stated in the Introduction, §XI, to this part, that V satisfy the reality condition (I.1) and be bar/unbar exchange invariant.

Recursive step $j-1\to j$. In the proof of Theorem VIII.5 in §X, we constructed $\mathcal{G}^{\rm rg}$ so that $\big(\mathcal{W}_j,\mathcal{G}^{\rm rg}_j,u_j,(p^{(2)},\cdots,p^{(j-1)})\big)\in\mathcal{D}^{(j)}_{\rm out}$. By Theorems XV.4 and XV.8, we have, in addition, that $\big(\mathcal{W}_j,\mathcal{G}^{\rm rg}_j,u_j,(p^{(2)},\cdots,p^{(j-1)})\big)\in\tilde{\mathcal{D}}^{(j)}_{\rm out}$. Let

$$2\check G^{\rm rg}_{j,2}(k)=C^{(\le j)}_{u(0)}(k)+\tfrac{1}{[ik_0-e(\mathbf{k})]^2}\sum_{i=2}^{j}\sum_{\ell=i}^{j}q^{(i,\ell)}(k)$$

be the decomposition of $(O^{\sim}3)$, but with $K=0$. By $(O^{\sim}3)$,

$$\sup_k\big|\mathrm{D}^{\delta}q^{(i,\ell)}(k)\big|\le\delta!\lambda_0^{1-2\upsilon}\tfrac{\mathfrak{l}_\ell}{M^\ell}M^{\aleph'(\ell-i)}M^{\delta_0 i}M^{|\boldsymbol{\delta}|\ell}.$$

By $(O^{\sim}4)$, $q^{(i,\ell)}(-k_0,\mathbf{k})=\overline{q^{(i,\ell)}(k_0,\mathbf{k})}$ for all of the required i,ℓ. □

Proof of Theorem I.7. By $(O^{\sim}2)$, the kernel of the quartic part of $\mathcal{G}^{\rm rg}_j(0)$, amputated by $ik_0-e(\mathbf{k})$ rather than $\frac{1}{\check G_2(k)}=ik_0-e(\mathbf{k})-\Sigma(k)$, is

$$P_\phi\Big[\delta f^{(j+1)}(0)+\sum_{i=2}^{j}\big(f^{(i)}_j\big)_{\Sigma_j}+\tfrac18\mathrm{Ant}\Big(V_{\rm ph}\big(\mathcal{L}^{(j+1)}(\vec p,\vec f_j)\big)\Big)\Big],$$

where the $i^{\rm th}$ component of $\vec f_j$ is $f^{(i)}_j=\mathrm{shear}\big(v^{(i)},C^{[i,j]}_{u_j}(k)A(k)\big)$, $u_j=\sum_{i=2}^{j-1}p^{(i)}_{\Sigma_j}$ and P_ϕ is projection onto the pure ϕ part. Recall from Definition VII.7 that the ladders $\mathcal{L}^{(n)}(\vec p,\vec f)$ were defined inductively by

$$\begin{aligned}\mathcal{L}^{(0)}(\vec p,\vec f)&=0,\\ \mathcal{L}^{(n+1)}(\vec p,\vec f)&=\mathcal{L}^{(n)}(\vec p,\vec f)_{\Sigma_n}+2\sum_{\ell=1}^{\infty}(-1)^\ell(12)^{\ell+1}L_\ell\big(w_n;C^{(n)}_{u_n},C^{(\ge n+1)}_{u_n}\big)^{\rm ph},\end{aligned}\qquad\text{(XV.42)}$$

where $w_n = \sum_{i=2}^n f_{\Sigma_n}^{(i)} + \frac{1}{8}\mathrm{Ant}\Big(V_{\rm ph}\big(\mathcal{L}^{(n)}(\vec{p}, \vec{f}\,)\big)\Big)_{\Sigma_n}$. By the construction of Theorem VIII.5, the quartic part of $\mathcal{G}$, again amputated by $ik_0 - e(\mathbf{k})$ rather than $ik_0 - e(\mathbf{k}) - \Sigma(k)$, is

$$\sum_{i=2}^{\infty} P_\phi f^{(i)} + \lim_{j\to\infty} \tfrac{1}{8}\mathrm{Ant}\Big(V_{\rm ph}\big(\mathcal{L}^{(j+1)}(\vec{p}, \vec{f}\,)\big)\Big)$$

with $f^{(i)} = \mathrm{shear}\big(v^{(i)}, C_P^{(\ge i)}(k)A(k)\big)$ and $P(k) = \sum_{i=2}^\infty \check{p}^i(k)$ as in Lemma XII.2. The sum makes sense even though different terms have different sectorization scales because ϕ arguments are not involved in sectorization. We shall prove convergence shortly.

The quartic part of $\mathcal{G}$, correctly amputated by $ik_0 - e(\mathbf{k}) - \Sigma(k)$, is

$$\sum_{i=2}^{\infty} P_\phi F^{(i)} + \lim_{j\to\infty} \tfrac{1}{8}\mathrm{Ant}\Big(V_{\rm ph}\big(\mathcal{L}^{(j+1)}(\vec{p}, \vec{F})\big)\Big),$$

where

$$\begin{aligned} F^{(i)} &= \mathrm{sct}\big(f^{(i)}, \tfrac{ik_0-e(\mathbf{k})-\Sigma(k)}{ik_0-e(\mathbf{k})}\big) \\ &= \mathrm{sct}\Big(\mathrm{shear}\big(v^{(i)}, \nu^{(\ge i)}\tfrac{ik_0-e(\mathbf{k})}{ik_0-e(\mathbf{k})-P(k)}\big)\ ,\ \tfrac{ik_0-e(\mathbf{k})-P(k)}{ik_0-e(\mathbf{k})}A_2(k)\Big) \\ &= \mathrm{sct}\Big(\mathrm{shear}\Big\{\mathrm{sct}\big(v^{(i)}, \tfrac{ik_0-e(\mathbf{k})-P(k)}{ik_0-e(\mathbf{k})}\big)\ ,\ \nu^{(\ge i)}\Big\}\ ,\ A_2(k)\Big) \\ &= \mathrm{sct}\Big(\mathrm{shear}\Big\{\mathrm{sct}\big(v^{(i)}, A_1(k)\big)\ ,\ \nu^{(\ge i)}\Big\}\ ,\ A_2(k)\Big) \end{aligned}$$

since, by ($O^\sim 2$), $v^{(i)}$ vanishes unless its ϕ momenta are in the support of $\nu^{(<i)}$ and the factor $\nu^{(\le i)}$ in the definition of A_1 (in Lemma XII.7) is identically one on that support. Set

$$\tilde{v}^{(i)} = \mathrm{shear}\Big(\mathrm{sct}\big(v^{(i)}, A_1(k)\big)\ ,\ \nu^{(\ge i)}\Big).$$

In this notation, the quartic part of $\mathcal{G}$, correctly amputated by $ik_0 - e(\mathbf{k}) - \Sigma(k)$, is

$$\mathrm{sct}\Big(\sum_{i=2}^{\infty} P_\phi\Big[\tilde{v}^{(i)} + \lim_{j\to\infty} \tfrac{1}{8}\mathrm{Ant}\Big(V_{\rm ph}\big(\mathcal{L}^{(j+1)}(\vec{p}, \vec{\tilde{v}})\big)\Big)\Big]\ ,\ A_2(k)\Big).$$

We have proven in Lemma XII.7.ii that $A_2(k)$ is $C^{1/2}$. So it suffices to prove that $\sum_{i=2}^{\infty} P_\phi \tilde{v}^{(i)}$ and $\lim_{j\to\infty} \frac{1}{8}\mathrm{Ant}\Big(V_{\rm ph}\big(\mathcal{L}^{(j+1)}(\vec{p}, \vec{\tilde{v}})\big)\Big)$ have the continuity properties specified in the statement of the theorem.

By ($O^\sim 2$), Lemma XII.7.i and Remark XIV.20.ii,

$$\big|\tilde{v}^{(i)}\big|_{3,\Sigma_i}^{\sim} \le \mathrm{const}\,\lambda_0^{1-11\upsilon/7}\mathfrak{l}_i^{1/n_0}\,\mathfrak{c}_i. \tag{XV.43}$$

Define, for $f \in \check{\mathcal{F}}_{4,\Sigma}$,

$$|||f|||^{\sim} = \sup\Big\{\ \big|g\big((k_1,\sigma_1,1),(k_2,\sigma_2,0),(k_3,\sigma_3,1),(k_4,\sigma_4,0)\big)\big|\ \Big|\ \sigma_i \in \{\uparrow,\downarrow\},\ k_1+k_3=k_2+k_4\ \Big\},$$

where g is the function on $\check{\mathcal{B}}_4$ such that

$$f\big|_{(0,\cdots,0)}(\check{\eta}_1,\cdots,\check{\eta}_n) = (2\pi)^3\delta(\check{\eta}_1+\cdots+\check{\eta}_n)\,g(\check{\eta}_1,\cdots,\check{\eta}_n).$$

By Lemma XII.12 of [FKTo3],

$$|||\tilde{v}^{(i)}|||^{\tilde{}} \le \text{const}\, \lambda_0^{1-11\upsilon/7} \mathfrak{l}_i^{1/n_0}$$

and, if D is dd–operator of type (4, 0) with $|\delta(\mathrm{D})| = 1$, in the sense of Definition XIII.5,

$$|||\mathrm{D}\tilde{v}^{(i)}|||^{\tilde{}} \le \text{const}\, \lambda_0^{1-11\upsilon/7} \mathfrak{l}_i^{1/n_0} M^i.$$

By Lemma C.1, with $\alpha = \frac{\aleph}{n_0}$ and $\beta = 1 - \frac{\aleph}{n_0}$, $\sum\limits_{i=2}^{\infty} P_\phi \tilde{v}^{(i)}$ is $C^{\aleph/n_0}$.

The remaining estimates use cancellations between scales. For this reason, we want to replace the j–dependent functions $\check{u}_j = \sum_{i=2}^{j-1} \check{p}^{(i)}(k)$ in the recursive definition (XV.42) of iterated particle hole ladders by one j–independent function $P(k) = \sum_{i=2}^{\infty} \check{p}^{(i)}(k)$. As in Definition D.1, we define, recursively on $0 \le n < \infty$, the compound particle hole ladder up to scale n as

$$\begin{aligned} \mathcal{L}_P^{(0)}(\vec{f}\,) &= 0, \\ \mathcal{L}_P^{(n+1)}(\vec{f}\,) &= \mathcal{L}_P^{(n)}(\vec{f}\,)_{\Sigma_n} + 2 \sum_{\ell=1}^{\infty} (-1)^\ell (12)^{\ell+1} L_\ell\big(w_{P,n}; C_P^{(n)}, C_P^{(\ge n+1)}\big)^{\mathrm{ph}}, \end{aligned} \tag{XV.44}$$

where $w_{P,n} = \sum_{i=2}^n f_{\Sigma_n}^{(i)} + \frac{1}{8}\mathrm{Ant}\Big(V_{\mathrm{ph}}\big(\mathcal{L}_P^{(n)}(\vec{f}\,)\big)\Big)_{\Sigma_n}$. For $j \ge 1$ set

$$\begin{aligned} \tilde{w}_j &= \sum_{i=2}^{j} \tilde{v}_{\Sigma_j}^{(i)} + \tfrac{1}{8}\mathrm{Ant}\Big(V_{\mathrm{ph}}\big(\mathcal{L}^{(j)}(\vec{p}, \vec{\tilde{v}})\big)\Big)_{\Sigma_j}, \\ \tilde{v}'^{(j+1)} &= \tilde{v}^{(j+1)} - \tfrac{1}{4} \sum_{\ell=1}^{\infty} (-1)^\ell (12)^{\ell+1} \mathrm{Ant} \\ &\qquad \times V_{\mathrm{ph}}\Big(L_\ell(\tilde{w}_j; C_P^{(j)}, C_P^{(\ge j+1)})^{\mathrm{ph}} - L_\ell(\tilde{w}_j; C_{u_j}^{(j)}, C_{u_j}^{(\ge j+1)})^{\mathrm{ph}}\Big)_{\Sigma_{j+1}}. \end{aligned}$$

By (D.3), with the replacements $F^{(i)} \to \tilde{v}^{(i)}$, $F'^{(i)} \to \tilde{v}'^{(i)}$, $w_j \to \tilde{w}_j$, $v_j \to u_j$ and $v \to P$,

$$\tilde{w}_j = \sum_{i=2}^{j} \tilde{v}'^{(i)}_{\Sigma_j} + \tfrac{1}{8}\mathrm{Ant}\Big(V_{\mathrm{ph}}\big(\mathcal{L}_P^{(j)}(\vec{\tilde{v}}\,')\big)\Big)_{\Sigma_j}$$

for all $j \ge 2$. Set

$$\delta\mathcal{L}^{(n)} = 2 \sum_{\ell=1}^{\infty} (-1)^\ell (12)^{\ell+1} \Big(L_\ell\big(\tilde{w}_n; C_{u_n}^{(n)}, C_{u_n}^{(\ge n+1)}\big) - L_\ell\big(\tilde{w}_n; C_P^{(n)}, C_P^{(\ge n+1)}\big)\Big)^{\mathrm{ph}}.$$

Subtracting (XV.44) from (XV.42)

$$\mathcal{L}^{(n+1)}(\vec{p}, \vec{\tilde{v}}) - \mathcal{L}_P^{(n+1)}(\vec{\tilde{v}}\,') = \mathcal{L}^{(n)}(\vec{p}, \vec{\tilde{v}})_{\Sigma_n} - \mathcal{L}_P^{(n)}(\vec{\tilde{v}}\,')_{\Sigma_n} + \delta\mathcal{L}^{(n)}.$$

Since $\mathcal{L}^{(1)}(\vec{p}, \vec{\tilde{v}}) = \mathcal{L}_v^{(1)}(\vec{\tilde{v}}\,') = 0$,

$$\mathcal{L}^{(n+1)}(\vec{p}, \vec{\tilde{v}}) - \mathcal{L}_P^{(n+1)}(\vec{\tilde{v}}\,') = \sum_{i=1}^{n} \delta\mathcal{L}_{\Sigma_n}^{(i)}.$$

By (XV.43), the hypotheses of Theorem XIV.12 with $\vec{F} = \tilde{\vec{v}}$, $\rho = \text{const}\,\lambda_0^{1-11\upsilon/7}$ and $\varepsilon = \frac{\aleph}{n_0}$ are satisfied. Hence, by (D.10),

$$\left| V_{\rm ph}\big(\delta\mathcal{L}^{(i)}\big)\right|^{\tilde{}}_{3,\Sigma_i} \le \text{const}\,\lambda_0^{3-33\upsilon/7}\mathfrak{l}_i\mathfrak{c}_i .$$

By Lemma XII.12 of [FKTo3],

$$\big\|\!\big| V_{\rm ph}\big(\delta\mathcal{L}^{(i)}\big)\big\|\!\big|^{\tilde{}} \le \text{const}\,\lambda_0^{3-33\upsilon/7}\mathfrak{l}_i \quad \text{and} \quad \big\|\!\big| \mathrm{D}V_{\rm ph}\big(\delta\mathcal{L}^{(i)}\big)\big\|\!\big|^{\tilde{}} \le \text{const}\,\lambda_0^{3-33\upsilon/7}\mathfrak{l}_i M^i$$

for all dd–operators, D, with $|\delta(\mathrm{D})| = 1$. By Lemma C.1, with $\alpha = \aleph$ and $\beta = 1-\aleph$, $\sum\limits_{i=2}^{\infty} P_\phi V_{\rm ph}\big(\delta\mathcal{L}^{(i)}\big)$ is $C^{\aleph}$. Thus $\sum\limits_{i=2}^{\infty} P_\phi\Big[\tilde{v}^{(i)} + \frac{1}{8}\mathrm{Ant}\big(V_{\rm ph}\big(\delta\mathcal{L}^{(i)}\big)\big)\Big]$ is $C^{\aleph/n_0}$. We choose $N_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}$ to be $\mathrm{sct}\big(\,\cdot\,,\ A_2(k)\big)$ applied to this sum and $L_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}(q_1,q_2,t)$ to be $\mathrm{sct}\big(\,\cdot\,,\ A_2(k)\big)$ applied to $\lim\limits_{n\to\infty} P_\phi V_{\rm ph}\big(\mathcal{L}_P^{(n+1)}(\tilde{\vec{v}}\,')\big)$ and evaluated at $k_1 = q_1 + \frac{t}{2}$, $k_2 = q_1 - \frac{t}{2}$, $k_3 = q_2 + \frac{t}{2}$ and $k_4 = q_2 + \frac{t}{2}$. By [FKTl, Theorem I.22] the limit exists pointwise for all $t \ne 0$ and is continuous there. The same theorem provides the existence of continuous extensions of $L_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}(q_1,q_2,(0,\mathbf{t}))$ and $L_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}(q_1,q_2,(t_0,\mathbf{0}))$ to $t = 0$. □

Appendix C. Hölder Continuity of Limits

Lemma C.1. *Let $\alpha,\ \beta,\ C_0,\ C_1 > 0$ and $M > 1$. There is a constant C' such that if*

$$f(t) = \sum_{j=0}^{\infty} f_j(t)$$

with

$$\sup_t \big|f_j(t)\big| \le \tfrac{C_0}{M^{\alpha j}} \qquad \sup_t \big|f_j'(t))\big| \le C_1 M^{\beta j},$$

then

$$\big|f(t) - f(t')\big| \le C'\, C_0^{\frac{\beta}{\alpha+\beta}}\, C_1^{\frac{\alpha}{\alpha+\beta}}\, |t-t'|^{\alpha/(\alpha+\beta)}.$$

Proof. Since

$$\big|f_j(t) - f_j(t')\big| \le 2\min\Big\{C_1 M^{\beta j}|t-t'|,\ \tfrac{C_0}{M^{\alpha j}}\Big\}$$

we have

$$\begin{aligned}
\big|f(t) - f(t')\big| &\le \sum_{j=0}^{\infty}\big|f_j(t) - f_j(t')\big| \\
&\le 2\sum_{\substack{0\le j<\infty \\ \frac{1}{M^{(\alpha+\beta)j}} \ge \frac{C_1}{C_0}|t-t'|}} C_1 M^{\beta j}|t-t'| + 2\sum_{\substack{0\le j<\infty \\ \frac{1}{M^{(\alpha+\beta)j}} \le \frac{C_1}{C_0}|t-t'|}} \frac{C_0}{M^{\alpha j}} \\
&\le 2C_1\tfrac{M^\beta}{M^\beta-1}\big(\tfrac{C_1}{C_0}|t-t'|\big)^{-\beta/(\alpha+\beta)}|t-t'|
\end{aligned}$$

$$+2C_0 \tfrac{1}{1-M^{-\alpha}} \big(\tfrac{C_1}{C_0}|t-t'|\big)^{\alpha/(\alpha+\beta)}$$
$$= C'\, C_0^{\frac{\beta}{\alpha+\beta}}\, C_1^{\frac{\alpha}{\alpha+\beta}}\, |t-t'|^{\alpha/(\alpha+\beta)}$$

with

$$C' = 2\big\{ \tfrac{M^\beta}{M^\beta-1} + \tfrac{M^\alpha}{M^\alpha-1}\big\}. \quad \square$$

Appendix D. Another Description of Particle–Hole Ladders

The estimates on iterated particle hole ladders in [FKTl] use cancellations between scales. For this reason, we want to replace the j–dependent functions u_j in the recursive Definition XIV.11 of iterated particle hole ladders by one j–independent function.

Definition D.1. *Let $\vec F = \big(F^{(2)}, F^{(3)}, \cdots\big)$ be a sequence of antisymmetric, spin independent, particle number conserving functions $F^{(i)} \in \check{\mathcal F}_{4,\Sigma_i}$ and $v(k)$ a function on $\mathbb R \times \mathbb R^2$ such that $|v(k)| \le \frac12 |\imath k_0 - e(\mathbf k)|$. We define, recursively on $0 \le j < \infty$, the compound particle hole (or wrong way) ladders up to scale j, denoted by $\mathcal L^{(j)} = \mathcal L^{(j)}_v(\vec F)$, as*

$$\mathcal L^{(0)} = 0,$$
$$\mathcal L^{(j+1)} = \mathcal L^{(j)}_{\Sigma_j} + 2 \sum_{\ell=1}^{\infty} (-1)^\ell (12)^{\ell+1} L_\ell\big(w;\, C_v^{(j)},\, C_v^{(\ge j+1)}\big)^{\rm ph},$$

where $w = \sum_{i=2}^j F^{(i)}_{\Sigma_j} + \frac18 \mathrm{Ant}\Big(V_{\rm ph}\big(\mathcal L^{(j)}\big)\Big)_{\Sigma_j}$.

In [FKTl, Remark I.21] we prove

Theorem D.2. *For every $\varepsilon > 0$ there are constants $\tilde\rho_0, \mathrm{const}$ such that the following holds. Let $\vec F = \big(F^{(2)}, F^{(3)}, \cdots\big)$ be a sequence of antisymmetric, spin independent, particle number conserving functions $F^{(i)} \in \check{\mathcal F}_{4,\Sigma_i}$ and $\vec p = \big(p^{(2)}, p^{(3)}, \cdots\big)$ be a sequence of antisymmetric, spin independent, particle number conserving functions $p^{(i)} \in \mathcal F_0(2, \Sigma_i)$. Assume that there is $\rho \le \tilde\rho_0$ such that for $i \ge 2$,*

$$|F^{(i)}|^{\sim}_{3,\Sigma_i} \le \tfrac{\rho}{M^{\varepsilon i}}\, \mathfrak c_i \qquad |p^{(i)}|_{1,\Sigma_i} \le \tfrac{\rho\, \mathfrak l_i}{M^i}\, \mathfrak c_i \qquad \check p^{(i)}(0,\mathbf k) = 0.$$

Set $v(k) = \sum_{i=2}^\infty \check p^{(i)}(k)$. Then for all $j \ge 1$,

$$\big|V_{\rm ph}\big(\mathcal L_v^{(j+1)}(\vec F)\big)\big|^{\sim}_{3,\Sigma_j} \le \mathrm{const}\, \rho^2\, \mathfrak c_j.$$

Proof that Theorem XIV.12 follows from Theorem D.2. Let $\vec F = \big(F^{(2)}, F^{(3)}, \cdots\big)$ and $\vec p = \big(p^{(2)}, p^{(3)}, \cdots\big)$ be as in the hypotheses of Theorem XIV.12. Because $\check p^{(i)}$ vanishes at $k_0 = 0$, Lemma XII.12 of [FKTo3] implies that

$$\big|\check p^{(i)}(k)\big| \le 2|k_0| \tfrac{\partial}{\partial t_{(1,0,0)}} |p^{(i)}|_{1,\Sigma_i}\Big|_{t=0} \le 2|\imath k_0 - e(\mathbf k)| \tfrac{\rho\,\mathfrak l_i}{M^i} M^i = 2\rho\, \mathfrak l_i |\imath k_0 - e(\mathbf k)|. \tag{D.1}$$

Set $w_j = \sum_{i=2}^{j} F_{\Sigma_j}^{(i)} + \frac{1}{8}\mathrm{Ant}\Big(V_{\rm ph}\big(\mathcal{L}^{(j)}(\vec p,\vec F)\big)\Big)_{\Sigma_j}$ and $v_j(k) = \sum_{i=2}^{j-1} \check p^{(i)}(k)$. By definition

$$\mathcal{L}^{(j+1)}(\vec p,\vec F) = \mathcal{L}^{(j)}(\vec p,\vec F)_{\Sigma_j} + 2\sum_{\ell=1}^{\infty} (-1)^\ell (12)^{\ell+1}\Big(L_\ell\big(w_j; C_{v_j}^{(j)}, C_{v_j}^{(\ge j+1)}\big)\Big)^{\rm ph}. \quad \text{(D.2)}$$

For $j \ge 1$ set

$$\begin{aligned} F'^{(j+1)} = F^{(j+1)} - \tfrac14 \sum_{\ell=1}^{\infty} (-1)^\ell (12)^{\ell+1}\mathrm{Ant} \\ \times V_{\rm ph}\Big(L_\ell(w_j; C_v^{(j)}, C_v^{(\ge j+1)})^{\rm ph} - L_\ell(w_j; C_{v_j}^{(j)}, C_{v_j}^{(\ge j+1)})^{\rm ph}\Big)_{\Sigma_{j+1}}. \end{aligned}$$

By induction ones sees that for $j \ge 2$,

$$w_j = \sum_{i=2}^{j} F'^{(i)}_{\Sigma_j} + \tfrac18 \mathrm{Ant}\Big(V_{\rm ph}\big(\mathcal{L}_v^{(j)}(\vec F')\big)\Big)_{\Sigma_j}. \quad \text{(D.3)}$$

Since $\check p^{(i)}(k)$ is supported on the $i^{\rm th}$ extended neighbourhood (defined in Definition I.2) and $\nu^{(j)}$ is supported on the $j^{\rm th}$ shell, $C_v^{(j)} = C_{v_j + \check p^{(j)} + \check p^{(j+1)}}^{(j)}$. By Proposition XIX.4.iii of [FKTo4],

$$\big|p^{(j)} + p_{\Sigma_j}^{(j+1)}\big|_{1,\Sigma_j} \le \text{const}\ \Big(|p^{(j)}|_{1,\Sigma_j} + \tfrac{\mathfrak{l}_j}{\mathfrak{l}_{j+1}}\mathfrak{c}_j \big|p^{(j+1)}\big|_{1,\Sigma_{j+1}}\Big) \le \text{const}\ \tfrac{\rho\,\mathfrak{l}_j}{M^j}\mathfrak{c}_{j+1}.$$

It follows from (D.1) that

$$|v(k) - v_j(k)| \le \mathit{const}\ \rho\,\mathfrak{l}_j\,|\imath k_0 - e(\mathbf{k})|.$$

By Corollary XIX.13 of [FKTo4],

$$\Big|\sum_{i=2}^{j-1} p_{\Sigma_j}^{(i)}\Big|_{1,\Sigma_j},\ \Big|\sum_{i=2}^{j+1} p_{\Sigma_j}^{(i)}\Big|_{1,\Sigma_j} \le \text{const}\ \tfrac{\rho}{M^j}\mathfrak{c}_{j+1}.$$

We apply Proposition D.7.ii of [FKTo3] with $u = \sum_{i=2}^{j+1} p_{\Sigma_j}^{(i)}$, $v = \sum_{i=2}^{\infty} p_{\Sigma_j}^{(i)}$, $u' = v' = \sum_{i=2}^{j-1} p_{\Sigma_j}^{(i)}$, $\varepsilon = \text{const}\ \rho\,\mathfrak{l}_j$, $X = \tau_2\,\mathfrak{c}_{j+1}$ (where τ_2 was defined in Lemma XIII.6 of [FKTo3]), $f = w_j$ and get, that for all $\ell \ge 1$,

$$\begin{aligned} &\Big|V_{\rm ph}\Big(L_\ell\big(w_j; C_v^{(j)}, C_v^{(\ge j+1)}\big)^{\rm ph} - L_\ell\big(w_j; C_{v_j}^{(j)}, C_{v_j}^{(\ge j+1)}\big)^{\rm ph}\Big)\Big|_{3,\Sigma_j}^{\sim} \\ &\quad \le \rho\,\mathfrak{l}_j\,\big(\text{const}\ \mathfrak{c}_{j+1}\big)^\ell\,|w_j|_{3,\Sigma_j}^{\sim\,\ell+1}. \end{aligned} \quad \text{(D.4)}$$

Proposition XIX.4.ii of [FKTo4] implies that also

$$\begin{aligned} &\Big|\mathrm{Ant}\ V_{\rm ph}\Big(L_\ell\big(w_j; C_v^{(j)}, C_v^{(\ge j+1)}\big)^{\rm ph} - L_\ell\big(w_j; C_{v_j}^{(j)}, C_{v_j}^{(\ge j+1)}\big)^{\rm ph}\Big)_{\Sigma_{j+1}}\Big|_{3,\Sigma_{j+1}}^{\sim} \\ &\quad \le \rho\,\mathfrak{l}_j\,\big(\text{const}\ \mathfrak{c}_{j+1}\big)^\ell\,|w_j|_{3,\Sigma_j}^{\sim\,\ell+1}. \end{aligned} \quad \text{(D.5)}$$

We now prove by induction that, if ρ_0 is small enough, then for all $j \ge 2$,

$$\begin{aligned} \big| F'^{(j)} \big|^{\sim}_{3,\Sigma_j} &\le 2\,\rho\ \max\big\{ \mathfrak{l}_j,\ \tfrac{1}{M^{\varepsilon j}} \big\} \mathfrak{c}_j , \\ \big| F'^{(j)} - F^{(j)} \big|^{\sim}_{3,\Sigma_j} &\le \text{const}\ \rho^3\ \mathfrak{l}_{j-1} \mathfrak{c}_j . \end{aligned} \tag{D.6}$$

The induction beginning is trivial since $F'^{(2)} = F^{(2)}$. For the induction step, assume that (D.6) holds for $j' \le j$. Then, using Proposition XIX.4.ii of [FKTo4],

$$\Big| \sum_{i=2}^{j} F'^{(i)}_{\Sigma_j} \Big|^{\sim}_{3,\Sigma_j} \ \le\ \mathit{const}\ \rho\ \sum_{i=2}^{j} \max\big\{ \mathfrak{l}_i,\ \tfrac{1}{M^{\varepsilon i}} \big\} \mathfrak{c}_j \ \le\ \mathit{const}\ \rho\ \mathfrak{c}_j .$$

Since $\mathcal{L}_v^{(j)}(\vec{F}')$ only depends on $F'^{(2)}, \cdots, F'^{(j-1)}$, Theorem D.2 applies whenever $\rho \le \frac{1}{2}\tilde{\rho}_0$, and

$$\big| V_{\rm ph}\big(\mathcal{L}_v^{(j)}(\vec{F}') \big) \big|^{\sim}_{3,\Sigma_{j-1}} \le \text{const}_0\ \rho^2 \mathfrak{c}_{j-1} \tag{D.7}$$

with a j–independent constant const_0 . Therefore, by (D.3)

$$| w_j |^{\sim}_{3,\Sigma_j} \le \mathit{const} (1 + \text{const}_0\ \rho)\ \rho\ \mathfrak{c}_j . \tag{D.8}$$

Hence, by the Definition of $F'^{(j+1)}$ and (D.5),

$$\begin{aligned} \big| F'^{(j+1)} - F^{(j+1)} \big|^{\sim}_{3,\Sigma_{j+1}} &\le \text{const}\ \rho^2\ \mathfrak{l}_j\ \mathfrak{c}_j\ \sum_{\ell=1}^{\infty} \big(\text{const}\ \rho\ \mathfrak{c}_j\ \mathfrak{c}_{j+1} \big)^{\ell} \\ &\le \text{const}'\ \rho^3\ \mathfrak{l}_j\ \mathfrak{c}_{j+1} . \end{aligned}$$

Therefore

$$\big| F'^{(j+1)} \big|^{\sim}_{3,\Sigma_{j+1}} \le \big| F^{(j+1)} \big|^{\sim}_{3,\Sigma_{j+1}} + \text{const}''\ \rho^3\ \mathfrak{l}_{j+1}\ \mathfrak{c}_{j+1} \le 2\,\rho\ \max\big\{ \mathfrak{l}_{j+1},\ \tfrac{1}{M^{\varepsilon(j+1)}} \big\} \mathfrak{c}_{j+1}$$

and (D.6) is proven for all j. Set

$$\delta\mathcal{L}^{(j)} = 2 \sum_{\ell=1}^{\infty} (-1)^{\ell} (12)^{\ell+1} \Big(L_\ell\big(w_j;\, C_{v_j}^{(j)}, C_{v_j}^{(\ge j+1)} \big) - L_\ell\big(w_j;\, C_v^{(j)}, C_v^{(\ge j+1)} \big) \Big)^{\rm ph} .$$

Subtracting Definition D.1 from (D.2),

$$\mathcal{L}^{(j+1)}(\vec{p}, \vec{F}) - \mathcal{L}_v^{(j+1)}(\vec{F}') = \mathcal{L}^{(j)}(\vec{p}, \vec{F})_{\Sigma_j} - \mathcal{L}_v^{(j)}(\vec{F}')_{\Sigma_j} + \delta\mathcal{L}^{(j)} .$$

Since $\mathcal{L}^{(1)}(\vec{p}, \vec{F}) = \mathcal{L}_v^{(1)}(\vec{F}') = 0$,

$$\mathcal{L}^{(j+1)}(\vec{p}, \vec{F}) - \mathcal{L}_v^{(j+1)}(\vec{F}') = \sum_{i=1}^{j} \delta\mathcal{L}^{(i)}_{\Sigma_j} . \tag{D.9}$$

By (D.4) and (D.8),

$$\begin{aligned}\left|V_{\rm ph}\big(\delta\mathcal{L}^{(j)}\big)\right|^{\sim}_{3,\Sigma_j} &\le \sum_{\ell=1}^{\infty} \rho\, \mathfrak{l}_j\, \big(\text{const }\ \mathfrak{c}_{j+1}\big)^{\ell}\, |w_j|^{\sim\,\ell+1}_{3,\Sigma_j} \\ &\le \sum_{\ell=1}^{\infty} const\ \rho^2 \mathfrak{c}_j\, \mathfrak{l}_j\, \big(\text{const } \rho\ \mathfrak{c}_j \mathfrak{c}_{j+1}\big)^{\ell} \\ &\le \text{const } \rho^3 \mathfrak{l}_j \mathfrak{c}_j. \end{aligned} \tag{D.10}$$

Hence by Proposition XIX.4.ii of [FKTo4].

$$\begin{aligned}\left|V_{\rm ph}\big(\mathcal{L}^{(j)}(\vec p,\vec F) - \mathcal{L}^{(j)}_v(\vec F')\big)\right|^{\sim}_{3,\Sigma_{j-1}} &\le \sum_{i=1}^{j-1} \left|V_{\rm ph}\big(\delta\mathcal{L}^{(i)}\big)_{\Sigma_{j-1}}\right|^{\sim}_{3,\Sigma_{j-1}} \\ &\le \sum_{i=1}^{j-1} const\ \mathfrak{c}_{j-2} \left|V_{\rm ph}\big(\delta\mathcal{L}^{(i)}\big)\right|^{\sim}_{3,\Sigma_i} \\ &\le \sum_{i=1}^{j-1} \text{const } \rho^3 \mathfrak{l}_i \mathfrak{c}_i \mathfrak{c}_{j-2} \\ &\le \text{const } \rho^3 \mathfrak{c}_{j-1}. \end{aligned}$$

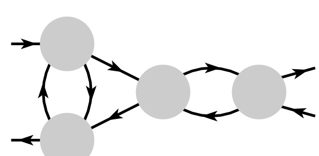
As pointed out in (D.7) , (D.6) implies that

$$\left|V_{\rm ph}\big(\mathcal{L}^{(j)}_v(\vec F')\big)\right|^{\sim}_{3,\Sigma_{j-1}} \le \text{const}_0\, \rho^2 \mathfrak{c}_{j-1}$$

for all $j \ge 2$ and Theorem XIV.12 follows by yet another application of Proposition XIX.4.ii of [FKTo4]. □

The inductive Definition XIV.11 of iterated particle–hole ladders is suited to direct application in the renormalization group analysis of §IX and §XV. Its relation to Definition D.1 of compound particle hole ladders has been exhibited above. In [FKTl] we use a more conceptual description of particle–hole ladders than that of Definition D.1. Corollary D.7 below and Remark I.21 of [FKTl] show that the description of particle–hole ladders in Definition D.1 agrees with that of Definition I.19 in [FKTl]. Therefore, Theorem D.2 follows from Theorem I.20 in [FKTl].

A compound particle–hole ladder of scale j may have a rung which is a particle–hole ladder of a scale $i < j$, and this rung may be "perpendicular" to the direction of the big ladder.



To make this concept precise, we define

Definition D.3. *Let $\mathfrak{X}$ be a measure space and F a four legged kernel on $\mathfrak{X}$. The associated flipped kernel is*

$$F^f(x_1,x_2,x_3,x_4) = -F(x_1,x_3,x_2,x_4).$$

The kernel F is called inversion symmetric if

$$F(x_4,x_3,x_2,x_1) = F(x_1,x_2,x_3,x_4).$$

Lemma D.4. *Let L be an inversion symmetric four legged kernel over $\mathfrak{Y}^{\updownarrow}_{\Sigma}$. Then*

$$\big(\mathrm{Ant}\, V_{\mathrm{ph}}(L)\big)^{\mathrm{ph}} = \tfrac{1}{3}(L + L^f).$$

The proof is left to the reader.

Example D.5.

i) The propagators $\mathcal{C}\big(C_u^{(j)}, C_u^{(\geq j+1)}\big)$ over $\mathcal{B}$ and $\mathcal{C}\big(C_u^{(j)}, C_u^{(\geq j+1)}\big)^{\mathrm{ph}}$ over $\mathcal{B}^{\updownarrow}$ are inversion symmetric.
ii) If f is an antisymmetric kernel over $\mathfrak{X}_\Sigma$ then its particle hole reduction f^{ph} is inversion symmetric.
iii) If $f_1',\ f_2'$ are inversion symmetric four-legged kernels over $\mathfrak{Y}^{\updownarrow}_{\Sigma}$ and P is an inversion symmetric bubble propagator over $\mathcal{B}^{\updownarrow}$ then

$$(f_1' \bullet P \bullet f_2')(z_4',z_3',z_2',z_1') = (f_2' \bullet P \bullet f_1')(z_1',z_2',z_3',z_4').$$

iv) If f' is an inversion symmetric four-legged kernel over $\mathfrak{Y}^{\updownarrow}_{\Sigma}$, P an inversion symmetric bubble propagator over $\mathcal{B}^{\updownarrow}$ and $\ell \geq 1$, then $(f' \bullet P)^\ell \bullet f'$ is also inversion symmetric.
v) One proves by induction on j that the compound particle hole ladders $\mathcal{L}^{(j)}(\vec{p}, \vec{F})$ are inversion symmetric.

Proposition D.6. *Let Σ be a sectorization. Let f be a particle number conserving, antisymmetric four legged kernel over $\mathfrak{X}_\Sigma$, L an inversion symmetric four legged kernel over $\mathfrak{Y}^{\updownarrow}_{\Sigma}$ and P a particle number conserving, inversion symmetric bubble propagator over $\mathcal{B}$ that obeys $P(\xi_1,\xi_2,\xi_3,\xi_4) = P(\xi_2,\xi_1,\xi_4,\xi_3)$. Set $w = f + \frac{1}{8}\mathrm{Ant}\big(V_{\mathrm{ph}}(L)\big)$. Then for $\ell \geq 1$,*

$$(12)^{\ell+1}\big([w \bullet P]^\ell \bullet w\big)^{\mathrm{ph}} = \tfrac{1}{2}\Big[\big(24\, f^{\mathrm{ph}} + L + L^f\big) \bullet {}^{\mathrm{ph}}P\Big]^\ell \bullet \big(24\, f^{\mathrm{ph}} + L + L^f\big).$$

Proof. By Lemma VII.5.i and Lemma D.4,

$$\begin{aligned}
&(12)^{\ell+1}\big([w \bullet P]^\ell \bullet w\big)^{\mathrm{ph}} \\
&= \tfrac{1}{2}(24)^{\ell+1}\big(f + \tfrac{1}{8}\mathrm{Ant}\, V_{\mathrm{ph}}L\big)^{\mathrm{ph}} \bullet {}^{\mathrm{ph}}P \bullet \cdots {}^{\mathrm{ph}}P \bullet \big(f + \tfrac{1}{8}\mathrm{Ant}\, V_{\mathrm{ph}}L\big)^{\mathrm{ph}} \\
&= \tfrac{1}{2}\big(24 f^{\mathrm{ph}} + 3(\mathrm{Ant}\, V_{\mathrm{ph}}L)^{\mathrm{ph}}\big) \bullet {}^{\mathrm{ph}}P \bullet \cdots {}^{\mathrm{ph}}P \bullet \big(24 f^{\mathrm{ph}} + 3(\mathrm{Ant}\, V_{\mathrm{ph}}L)^{\mathrm{ph}}\big) \\
&= \tfrac{1}{2}\big(24 f^{\mathrm{ph}} + L + L^f\big) \bullet {}^{\mathrm{ph}}P \bullet \cdots {}^{\mathrm{ph}}P \bullet \big(24 f^{\mathrm{ph}} + L + L^f\big). \quad \square
\end{aligned}$$

Corollary D.7. *Let $\vec{F} = \big(F^{(2)}, F^{(3)}, \cdots\big)$ be a sequence of antisymmetric, spin independent, particle number conserving functions $F^{(i)} \in \check{\mathcal{F}}_{4,\Sigma_i}$ and $v(k)$ a function on $\mathbb{R} \times \mathbb{R}^2$ such that $|v(k)| \le \frac{1}{2}|\imath k_0 - e(\mathbf{k})|$. As in Definition D.1, let $\mathcal{L}^{(j)} = \mathcal{L}_v^{(j)}(\vec{F})$ be the compound particle hole ladder up to scale j. Then for $j \ge 0$,*

$$\mathcal{L}^{(j+1)} = \mathcal{L}^{(j)}_{\Sigma_j} + \sum_{\ell=1}^{\infty} (-1)^\ell \big(24F + \mathcal{L}^{(j)}_{\Sigma_j} + \mathcal{L}^{(j)\,f}_{\Sigma_j}\big) \bullet \mathcal{C} \bullet \cdots \mathcal{C} \bullet \big(24F + \mathcal{L}^{(j)}_{\Sigma_j} + \mathcal{L}^{(j)\,f}_{\Sigma_j}\big),$$

where $F = \sum_{i=2}^{j} F^{(i)\,\mathrm{ph}}_{\Sigma_j}$ and $\mathcal{C} = {}^{\mathrm{ph}}\mathcal{C}\big(C_v^{(j)}, C_v^{(\ge j+1)}\big)$.

Proof. Since $\mathcal{L}^{(j)}$ and $\mathcal{C}$ are inversion symmetric by Example D.5 and $P = \mathcal{C}\big(C_v^{(j)}, C_v^{(\ge j+1)}\big)$ obeys $P(\xi_1, \xi_2, \xi_3, \xi_4) = P(\xi_2, \xi_1, \xi_4, \xi_3)$, the corollary follows from Definition D.1 and Proposition D.6. □

Notation

Norms

Norm	Characteristics	Reference
$\vert\vert\vert \cdot \vert\vert\vert_{1,\infty}$	no derivatives, external positions, acts on functions	Definition V.3
$\Vert \cdot \Vert_{1,\infty}$	derivatives, external positions, acts on functions	Definition V.3
$\Vert \cdot \Vert^{\tilde{}}$	derivatives, external momenta, acts on functions	Definition XIII.7
$\vert\vert\vert \cdot \vert\vert\vert_{\infty}$	no derivatives, external positions, acts on functions	Definition VI.7
$\vert \cdot \vert_{p,\Sigma}$	derivatives, external positions, all but p sectors summed	Definition VI.6
$\vert\vert\vert \cdot \vert\vert\vert_{1,\Sigma}$	no derivatives, all but 1 sector summed	(II.6)
$\vert\vert\vert \cdot \vert\vert\vert_{3,\Sigma}$	no derivatives, all but 3 sectors summed	(II.14)
$\vert \cdot \vert^{\tilde{}}_{p,\Sigma}$	derivatives, external momenta, all but ($p-$# external momenta) sectors summed	Definition XIII.12
$\Vert \cdot \Vert_{1,\Sigma}$	like $\vert \cdot \vert_{1,\Sigma}$, but for functions on $\big(\mathbb{R}^2 \times \Sigma\big)^2$	[Def'n E.3, [FKTo4]]
$\vert\varphi\vert_j$	$\rho_{m;n} \begin{cases} \Vert\varphi\Vert_{1,\Sigma_j} + \frac{1}{\mathfrak{l}_j} \vert\varphi\vert_{3,\Sigma_j} + \frac{1}{\mathfrak{l}_j^2} \vert\varphi\vert_{5,\Sigma_j} & \text{if } m = 0 \\ \frac{\mathfrak{l}_j}{M^{2j}} \vert\varphi\vert_{1,\Sigma_j} & \text{if } m \ne 0 \end{cases}$	Definition VI.6
$N_j(w, \alpha, X)$	$\frac{M^{2j}}{\mathfrak{l}_j}\, \mathfrak{e}_j(X) \sum_{m,n\ge 0} \alpha^n \big(\frac{\mathfrak{l}_j \mathrm{B}}{M^j}\big)^{n/2} \vert w_{m,n}\vert_j$	Definition VI.7
$N(\mathcal{G})$	$\sum_{m>0} \frac{1}{\lambda_0^{(1-\upsilon)\max\{m-2,2\}/2}} \vert\vert\vert G_m \vert\vert\vert_\infty$	Definition VI.7
$\vert f\vert^{\tilde{}}_j$	$\tilde{\rho}_{m;n} \begin{cases} \sum_{p=1}^{6} \frac{1}{\mathfrak{l}_j^{[(p-1)/2]}} \vert f\vert^{\tilde{}}_{p,\Sigma_j} & \text{if} m \ne 0 \\ \vert f\vert^{\tilde{}}_{1,\Sigma_j} + \frac{1}{\mathfrak{l}_j} \vert f\vert^{\tilde{}}_{3,\Sigma_j} + \frac{1}{\mathfrak{l}_j^2} \vert f\vert^{\tilde{}}_{5,\Sigma_j} & \text{if } m = 0 \end{cases}$	Definition XIII.15
$\vert f\vert^{\tilde{}}_{p,\Sigma_j,\tilde{\rho}}$	$\tilde{\rho}_{m;n} \vert f\vert^{\tilde{}}_{p,\Sigma_j}$	Definition XIII.15
		Definition XIV.6
$N_j^{\sim}(w, \alpha, X)$	$\frac{M^{2j}}{\mathfrak{l}_j}\, \mathfrak{e}_j(X) \sum_{m,n\ge 0} \alpha^{m+n} \big(\frac{\mathfrak{l}_j\, \mathrm{B}}{M^j}\big)^{(m+n)/2} \vert w^{\sim}_{m,n}\vert^{\tilde{}}_j$	Definition XIII.15

Spaces

Not'n	Description	Reference
$\mathcal{E}$	counterterm space	Definition I.1
$\mathfrak{K}_j$	space of future counterterms for scale j	Definition VI.9
$\mathcal{B}$	$\mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\} \times \{0, 1\}$ viewed as position space	before Def VII.1
$\check{\mathcal{B}}$	$\mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\} \times \{0, 1\}$ viewed as momentum space	beginning of §VI
$\check{\mathcal{B}}_m$	$\left\{ (\check{\eta}_1, \cdots, \check{\eta}_m) \in \check{\mathcal{B}}^m \;\middle|\; \check{\eta}_1 + \cdots + \check{\eta}_m = 0 \right\}$	Definition XIII.5
$\mathfrak{X}_\Sigma$	$= \mathfrak{X}_0 \cup \mathfrak{X}_1 = \check{\mathcal{B}} \cup (\mathcal{B} \times \Sigma)$	Definition XIV.1
$\mathfrak{X}'_\Sigma$	$= \mathfrak{X}_{-1} \cup \mathfrak{X}_0 \cup \mathfrak{X}_1 = \check{\mathcal{B}} \cup \check{\mathcal{B}} \cup (\mathcal{B} \times \Sigma)$	Definition XIV.16
$\mathcal{B}^\updownarrow$	$\mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\}$ viewed as position space	Definition VII.3
$\check{\mathcal{B}}^\updownarrow$	$\mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\}$ viewed as momentum space	Definition XIV.1
$\mathfrak{Y}^\updownarrow_\Sigma$	$\check{\mathcal{B}}^\updownarrow \cup (\mathcal{B}^\updownarrow \times \Sigma)$	Definition XIV.1
$\mathcal{F}_m(n; \Sigma)$	functions on $\mathcal{B}^m \times \left(\mathcal{B} \times \Sigma\right)^n$, internal momenta in sectors	Definition VI.3.ii
$\check{\mathcal{F}}_m(n; \Sigma)$	functions on $\check{\mathcal{B}}^m \times \left(\mathcal{B} \times \Sigma\right)^n$, internal momenta in sectors	Definition XIII.14
$\check{\mathcal{F}}_{n;\Sigma}$	functions on $\mathfrak{X}^n_\Sigma$ that reorder to $\check{\mathcal{F}}_m(n-m; \Sigma)$'s	Definition XIV.6
$\check{\mathcal{F}}_{m',m}(n; \Sigma)$	fns on $\mathfrak{X}^{m'}_{-1} \times \mathfrak{X}^m_0 \times \left(\mathcal{B} \times \Sigma\right)^n$, internal momenta in sectors	Definition XIV.17
$\check{\mathcal{F}}'_{n;\Sigma}$	functions on $\mathfrak{X}'^n_\Sigma$ that reorder to $\check{\mathcal{F}}_{m,m'}(n-m-m'; \Sigma)$'s	Definition XIV.17
$\mathcal{D}^{(j,\mathfrak{form})}_{\rm in}$	formal input data for scale j	Definition III.8
$\mathcal{D}^{(j,\mathfrak{form})}_{\rm out}$	formal output data for scale j	Definition III.9
$\mathcal{D}^{(j)}_{\rm in}$	input data for scale j	Definition IX.1
$\mathcal{D}^{(j)}_{\rm out}$	output data for scale j	Definition IX.2
$\tilde{\mathcal{D}}^{(j)}_{\rm in}$	more input data for scale j	Definition XV.1
$\tilde{\mathcal{D}}^{(j)}_{\rm out}$	more output data for scale j	Definition XV.3

Other Notation

Not'n	Description	Reference
r_0	number of k_0 derivatives tracked	following (I.3)
r	number of $\mathbf{k}$ derivatives tracked	following (I.3)
M	scale parameter, $M > 1$	before Definition I.2
const	generic constant, independent of scale	
const	generic constant, independent of scale and M	
$\nu^{(j)}(k)$	$j^{\rm th}$ scale function	Definition I.2
$\nu^{(\ge j)}(k)$	$\sum_{i \ge j} \nu^{(j)}(k)$	Definition I.2
n_0	degree of asymmetry	Definition I.10
J	particle/hole swap operator	(III.3)
$\Omega_S(\mathcal{W})(\phi, \psi)$	$\log \frac{1}{Z} \int e^{\mathcal{W}(\phi,\psi+\zeta)}\, d\mu_S(\zeta)$	Definition III.1
$\tilde{\Omega}_C(\mathcal{W})(\phi, \psi)$	$\log \frac{1}{Z} \int e^{\phi J \zeta}\, e^{\mathcal{W}(\phi,\psi+\zeta)} d\mu_C(\zeta)$	Definition III.1
$\aleph$	$\frac{1}{2} < \aleph < \frac{2}{3}$	following Definition VI.3
$\aleph'$	$\aleph < \aleph' < \frac{2}{3}$	Theorem XII.1
λ_0	maximum allowed "coupling constant"	Theorem VIII.5
υ	$0 < \upsilon < \frac{1}{4}$, power of λ_0 eaten by bounds	Definition V.6
$\rho_{m;n}(\lambda)$	$\lambda^{-(1-\upsilon)\max\{m+n-2,2\}/2}$	Definition V.6
$\rho_{m;n}$	$\rho_{m;n}(\lambda_0)\big\{1 \text{ if } m = 0\,;\ \sqrt[4]{\mathfrak{l}_j M^j} \text{ if } m > 0$	Definition VI.6.ii
$\tilde{\rho}_{m;n}$	$\rho_{m;n}(\lambda_0)\lambda_0^{m\upsilon/7}$	Theorem XIII.11
$\mathfrak{l}_j$	$= \frac{1}{M^{\aleph j}}$ = length of sectors of scale j	following Definition VI.3
Σ_j	the sectorization at scale j of length $\mathfrak{l}_j$	following Definition VI.3
B	j–independent constant	Definitions VI.7, XIII.15

Not'n	Description	Reference
$\mathfrak{c}_j$	$= \sum_{\substack{\lvert\boldsymbol{\delta}\rvert\le r \\ \lvert\delta_0\rvert\le r_0}} M^{j\lvert\delta\rvert} t^\delta + \sum_{\substack{\lvert\boldsymbol{\delta}\rvert>r \\ \text{or } \lvert\delta_0\rvert>r_0}} \infty\, t^\delta \in \mathfrak{N}_{d+1}$	Definition V.2
$\mathfrak{c}_{i,j}$	$= \sum_{\substack{\lvert\boldsymbol{\delta}\rvert\le r \\ \lvert\delta_0\rvert\le r_0}} M^{i\delta_0} M^{j\lvert\boldsymbol{\delta}\rvert} t^\delta + \sum_{\substack{\lvert\boldsymbol{\delta}\rvert>r \\ \text{or } \lvert\delta_0\rvert>r_0}} \infty\, t^\delta \in \mathfrak{N}_{d+1}$	(XV.1)
$\mathfrak{e}_j(X)$	$= \frac{\mathfrak{c}_j}{1-M^j X}$	Definition V.2
$\mathfrak{e}_{i,j}(X)$	$= \frac{\mathfrak{c}_{i,j}}{1-M^j X}$	(XV.1)
$f_{\rm ext}$	extends $f(\mathbf{x},\mathbf{x}')$ to $f_{\rm ext}\big((x_0,\mathbf{x},\sigma,a),(x_0',\mathbf{x}',\sigma',a')\big)$	[Definition E.1, [FKTo4]]
$*$	convolution	Definition VIII.6
$\bullet$	ladder convolution	Definition VII.2, Definition XIV.7
$\text{shear}(\,\cdot\,,B)$	maps kernel of $\mathcal{W}(\phi,\psi)$ to kernel of $\mathcal{W}(\phi,\psi+B\phi)$	Definition XIV.14
$\text{shear}'(\,\cdot\,,B)$	maps kernel of $\mathcal{W}(\phi,\psi)$ to kernel of $\mathcal{W}(\phi,\psi+B\phi')$	Definition XIV.19
$\text{sct}'(\,\cdot\,,B)$	maps kernel of $\mathcal{W}(\phi',\phi,\psi)$ to kernel of $\mathcal{W}(B\phi',\phi,\psi)$	Definition XIV.19
Π	maps kernel of $\mathcal{W}(\phi',\phi,\psi)$ to kernel of $\mathcal{W}(\phi,\phi,\psi)$	Definition XIV.17
$\hat{\mu}$	Fourier transform	Notation V.4

References

[FKTf1] Feldman, J., Knörrer, H., Trubowitz, E.: A Two Dimensional Fermi Liquid, Part 1: Overview. Commun. Math. Phys. **247**, 1–47 (2004)

[FKTf2] Feldman, J., Knörrer, H., Trubowitz, E.: A Two Dimensional Fermi Liquid, Part 2: Convergence. Commun. Math. Phys. **247**, 49–111 (2004)

[FKTo1] Feldman, J., Knörrer, H., Trubowitz, E.: Single Scale Analysis of Many Fermion Systems, Part 1: Insulators. Rev. Math. Phys. **15**, 949–993 (2003)

[FKTo2] Feldman, J., Knörrer, H., Trubowitz, E.: Single Scale Analysis of Many Fermion Systems, Part 2: The First Scale. Rev. Math. Phys. **15**, 995–1037 (2003)

[FKTo3] Feldman, J., Knörrer, H., Trubowitz, E.: Single Scale Analysis of Many Fermion Systems, Part 3: Sectorized Norms. Rev. Math. Phys. **15**, 1039–1120 (2003)

[FKTo4] Feldman, J., Knörrer, H., Trubowitz, E.: Single Scale Analysis of Many Fermion Systems, Part 4: Sector Counting. Rev. Math. Phys. **15**, 1121–1169 (2003)

[FKTr1] Feldman, J., Knörrer, H., Trubowitz, E.: Convergence of Perturbation Expansions in Fermionic Models, Part 1: Nonperturbative Bounds. Commun. Math. Phys. **247**, 195–242 (2004)

[FST3] Feldman, J., Salmhofer, M., Trubowitz, E.: Regularity of Interacting Nonspherical Fermi Surfaces: The Full Self–Energy. Commun. Pure Appl. Math. **LII**, 273–324 (1999)

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