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Particle–Hole Ladders

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Abstract: A self contained analysis demonstrates that the sum of all particle-hole ladder contributions for a two dimensional, weakly coupled fermion gas with a strictly convex Fermi curve at temperature zero is bounded. This is used in our construction of two dimensional Fermi liquids. This summary article contains the statements of the main results. The proofs are contained in the full, electronic, article.

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Contents

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Contents of the Electronic Supplement

I. Introduction

This article is one of a series, starting with [FKTf1], that provides a construction of a class of two dimensional Fermi liquids. The concept of a Fermi liquid was introduced by L. D. Landau in [L1, L2, L3] and has become the generally accepted explanation for the success of the independent electron approximation. The phenomenological implications of Fermi liquid theory are derived from the structure of the single particle density $n_{\mathbf{k}}$ and Landau's quasiparticle interaction and forward scattering amplitude. The single particle density is constructed as a relatively straightforward limit of the one particle Green's function. The quasiparticle interaction and forward scattering amplitude, by contrast, are defined through two different limits of the transfer momentum flowing through the particle/hole channel of the two particle Green's function. This subtlety arises because the two particle Green's function is bounded but not continuous at transfer momentum zero.

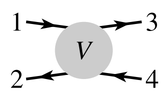
In [FKTr2] we showed that the leading contributions to the two particle Green's function are the, so-called, ladders. In this paper we extract sufficiently detailed information about particle/hole ladders to demonstrate the existence of the limits defining the quasiparticle interaction and forward scattering amplitude. In fact, in our construction of the full models, we are forced to this level of detail to formulate hypotheses on the sequence of effective interactions (see [FKTf2, Def. IX.1 and IX.2]) that enable us to make an inductive construction. In other words, we would not be able to construct any of the Green's functions without the present fine analysis of the particle/hole channel.

The control of the particle hole channel is in some ways the most subtle part of the argument in the construction of a Fermi liquid in that it results from a cancellation involving essentially all scales. Roughly speaking, it is like picking up the singularity of a Fourier series from its partial trigonometric sums.

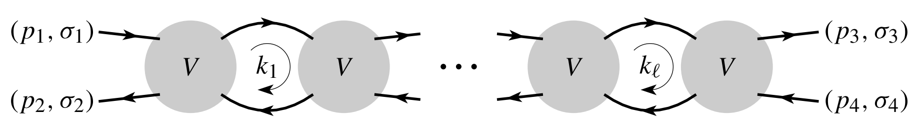
Philip Anderson [A1, A2] suggested that, because of an instability arising from the particle/hole channel, two dimensional Fermi gases should exhibit behavior similar to a one dimensional Luttinger liquid, with the single particle density $n_{\mathbf{k}}$ having a vertical tangent at the Fermi surface rather than a jump discontinuity. In this series of papers, we show that this is not the case for the class of models considered here.

This summary article contains the definitions and statements of main results of the full article, which is available at http://dx.doi.org/10.1007/s00220-004-1038-2. We use the same numbering here as in the full article.

Formally, the amputated four–point Green's function, $G_4\big((p_1,\sigma_1),(p_2,\sigma_2),(p_3,\sigma_3),(p_4,\sigma_4)\big)$ with incoming particles of momenta $p_1,\ p_4 \in \mathbb{R}\times\mathbb{R}^d$ and spins $\sigma_1, \sigma_4 \in \{\uparrow, \downarrow\}$ and outgoing particles of momenta $p_2,\ p_3$ and spins σ_2, σ_3, can be written as a sum of values of Feynman diagrams with four external legs. The propagator of these diagrams is $C(k) = \frac{1}{\imath k_0 - e(\mathbf{k})}$, where $k = (k_0, \mathbf{k}) \in \mathbb{R}\times\mathbb{R}^d$ and the dispersion relation $e(\mathbf{k})$ (into which the chemical potential has been absorbed) characterizes the independent fermion approximation. The interaction of the model determines the diagram vertices, $V\big((k_1,\sigma_1),(k_2,\sigma_2),(k_3,\sigma_3),(k_4,\sigma_4)\big)$, $k_1+k_4 = k_2+k_3$. Here, the incoming momenta are $k_1,\ k_4$ and the outgoing momenta are $k_2,\ k_3$.



1. Ladders in Momentum Space. The most important contributions to this four–point function are ladders. The contribution of the particle–hole ladder with $\ell + 1$ rungs

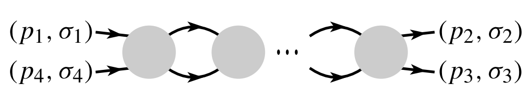


is

$$\sum_{\substack{\tau_{i,1},\tau_{i,2}\in\{\uparrow,\downarrow\}\\ i=1,\cdots,\ell}} \int \frac{d^{d+1}k_1}{(2\pi)^{d+1}}\cdots\frac{d^{d+1}k_\ell}{(2\pi)^{d+1}} \quad V\big((p_1,\sigma_1),(p_2,\sigma_2),(p_1+k_1,\tau_{1,1}),(p_2+k_1,\tau_{1,2})\big)C(p_1+k_1)C(p_2+k_1)$$

$$V\big((p_1+k_1,\tau_{1,1}),(p_2+k_1,\tau_{1,2}),\cdots\big)\cdots V\big(\cdots,(p_1+k_\ell,\tau_{\ell,1}),(p_2+k_\ell,\tau_{\ell,2})\big)$$
$$C(p_1+k_\ell)C(p_2+k_\ell)V\big((p_1+k_\ell,\tau_{\ell,1}),(p_2+k_\ell,\tau_{\ell,2}),(p_3,\sigma_3),(p_4,\sigma_4)\big).$$

The contribution of the particle–particle ladder with $\ell + 1$ rungs



is

$$\sum_{\substack{\tau_{i,1},\tau_{i,2}\in\{\uparrow,\downarrow\}\\ i=1,\cdots,\ell}} \int \frac{d^{d+1}k_i}{(2\pi)^{d+1}} \cdots \frac{d^{d+1}k_\ell}{(2\pi)^{d+1}} \; V\big((p_1,\sigma_1),(p_1+k_1,\tau_{1,1}),(p_4-k_1,\tau_{1,2}),(p_4,\sigma_4)\big)C(p_1+k_1)C(p_4-k_1)$$
$$V\big((p_1+k_1,\tau_{1,1}),\cdots,(p_4-k_1,\tau_{1,2})\big)\cdots V\big(\cdots,(p_1+k_\ell,\tau_{\ell,1}),(p_4-k_\ell,\tau_{\ell,2}),\cdots\big)$$
$$C(p_1+k_\ell)C(p_4-k_\ell)V\big((p_1+k_\ell,\tau_{\ell,1}),(p_2,\sigma_2),(p_3,\sigma_3),(p_4-k_\ell,\tau_{\ell,2})\big).$$

Ladders with two rungs are called bubbles. The values of the bubbles with dispersion relation $e(\mathbf{k}) = \frac{|\mathbf{k}|^2}{2m} - \mu$ and interaction $V((p_1,\sigma_1),(p_2,\sigma_2),(p_3,\sigma_3),(p_4,\sigma_4)) = \lambda\big(\delta_{\sigma_1,\sigma_2}\delta_{\sigma_3,\sigma_4} - \delta_{\sigma_1,\sigma_3}\delta_{\sigma_2,\sigma_4}\big)$ are well–known for $d = 2, 3$ [FHN]. The particle–particle bubble has a logarithmic singularity [FKST, Prop. II.1b] at transfer momentum $p_1 + p_4 = 0$ which is responsible for the formation of Cooper pairs and the onset of superconductivity. This singularity persists in models having dispersion relations that are symmetric about the origin, i.e. $e(\mathbf{k}) = e(-\mathbf{k})$. On the other hand, if $e(\mathbf{k})$ is strongly asymmetric in the sense of Definition **I**.10 of [FKTf1] then the particle–particle bubble remains continuous and, in particular, bounded [FKLT1, p. 297].

For the particle–hole bubble with $d = 2$ and $e(\mathbf{k}) = \frac{|\mathbf{k}|^2}{2m} - \mu$,

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3}\, C(k+p_1)\, C(k+p_2)\\
&\quad = \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} \frac{1}{i(k_0+t_0/2)-e(\mathbf{k}+\mathbf{t}/2)} \frac{1}{i(k_0-t_0/2)-e(\mathbf{k}-\mathbf{t}/2)}\\
&\quad = \begin{cases} -\frac{m}{2\pi} + \frac{m}{2\pi|\mathbf{t}|^2}\,\mathrm{Re}\sqrt{|\mathbf{t}|^2(|\mathbf{t}|^2-4k_F^2)-4m^2t_0^2-4\imath m t_0|\mathbf{t}|^2} & \text{if } t_0,\ |\mathbf{t}|\neq 0 \text{ or } |\mathbf{t}|\geq 2k_F\\ -\frac{m}{2\pi} & \text{if } t_0 = 0 \text{ and } 0<|\mathbf{t}|\leq 2k_F\ ,\\ 0 & \text{if } t_0\neq 0 \text{ and } \mathbf{t}=0 \end{cases}
\end{aligned}$$

where $t = p_1 - p_2$ is the transfer momentum, $k_F = \sqrt{2m\mu}$ is the radius of the Fermi surface and $\sqrt{\ }$ is the square root with nonnegative real part and cut along the negative real axis. See, for example, [FHN (2.22) or FKST, Prop. II.1a]. This is C^∞ on $\big\{\, t \in \mathbb{R}\times\mathbb{R}^2 \,\big|\, t_0 \neq 0 \text{ or } |\mathbf{t}| > 2k_F \,\big\}$, is Hölder continuous of degree 1 in a neighbourhood of any t with $t_0 = 0,\ 0 < |\mathbf{t}| < 2k_F$, and is Hölder continuous of degree $\frac{1}{2}$ in a neighbourhood of any t with $t_0 = 0,\ |\mathbf{t}| = 2k_F$, but cannot be continuously extended to $t = 0$. However its restriction to $t_0 = 0$ does have a C^∞ extension at the point $\mathbf{t} = 0$. The discontinuity at $t = 0$ persists for general, even strongly asymmetric, $e(\mathbf{k})$. For this reason, bounds on particle–hole ladders in position space are not straightforward.

That the restriction of the particle–hole bubble to $t_0 = 0$ does have a C^∞ extension for a large class of smooth dispersion relations may be seen by the following argument, which was shown to us by Manfred Salmhofer [S]. A generalization of this argument is used in Prop. III.27 of the full article.

Lemma I.1. *Choose a "scale parameter" $M > 1$ and a function $\nu \in C_0^\infty\big(\big[\frac{1}{M},\, 2M\big]\big)$ that takes values in* $[0, 1]$*, is identically 1 on $\big[\frac{2}{M}, M\big]$, is monotone on $\big[\frac{1}{M}, \frac{2}{M}\big]$ and* $[M, 2M]$*, and obeys*

$$\sum_{j=0}^{\infty} \nu\big(M^{2j}x\big) = 1 \tag{I.1}$$

for $0 < x < 1$. *Set* $\nu_0^{[0,j]}(k_0) = \sum_{\ell=0}^{j} \nu(M^{2\ell} k_0^2)$ *and let* $u(k, \mathbf{t})$ *be a bounded* C^∞ *function with compact support in* $\mathbf{k}$ *and bounded derivatives. Let* $e(\mathbf{k})$ *be a* C^∞ *function that obeys* $\lim_{|\mathbf{k}|\to\infty} e(\mathbf{k}) = +\infty$. *Assume that the gradient of* $e(\mathbf{k})$ *does not vanish on the Fermi surface* $F = \left\{ \mathbf{k} \in \mathbb{R}^d \;\middle|\; e(\mathbf{k}) = 0 \right\}$. *Then*

$$B(\mathbf{t}) = \lim_{j\to\infty} \int dk \; \frac{\nu_0^{[0,j]}(k_0) u(k, \mathbf{t})}{[ik_0 - e(\mathbf{k})][ik_0 - e(\mathbf{k} + \mathbf{t})]}$$

is C^∞ *for* $\mathbf{t}$ *in a neighbourhood of* 0.

Proof. Write

$$\begin{aligned}
B_j(\mathbf{t}) &= \int dk \; \tfrac{\nu_0^{[0,j]}(k_0) u(k,\mathbf{t})}{[ik_0 - e(\mathbf{k})][ik_0 - e(\mathbf{k}+\mathbf{t})]} \\
&= \int dk \; \tfrac{\nu_0^{[0,j]}(k_0) u(k,\mathbf{t})}{e(\mathbf{k}) - e(\mathbf{k}+\mathbf{t})} \Big[\tfrac{1}{ik_0 - e(\mathbf{k})} - \tfrac{1}{ik_0 - e(\mathbf{k}+\mathbf{t})} \Big] \\
&= \int dk \; \tfrac{\nu_0^{[0,j]}(k_0) u(k,\mathbf{t})}{e(\mathbf{k}) - e(\mathbf{k}+\mathbf{t})} \int_0^1 ds \; \tfrac{d}{ds} \tfrac{1}{ik_0 - E(\mathbf{k},\mathbf{t},s)} \\
&= \int dk \int_0^1 ds \; \tfrac{\nu_0^{[0,j]}(k_0) u(k,\mathbf{t})}{[ik_0 - E(\mathbf{k},\mathbf{t},s)]^2},
\end{aligned}$$

where

$$E(\mathbf{k}, \mathbf{t}, s) = s e(\mathbf{k}) + (1 - s) e(\mathbf{k} + \mathbf{t}).$$

Make, for each fixed s and k_0, the change of variables from $\mathbf{k}$ to E and $d - 1$ variables θ on F. Denote by $J(E, \mathbf{t}, \theta, s)$ the Jacobian of this change of variables and set

$$f(k_0, E, \theta, \mathbf{t}, s) = u\big((k_0, \mathbf{k}(E, \theta, \mathbf{t}, s)), \mathbf{t}\big) J(E, \theta, \mathbf{t}, s).$$

Because u has compact support in $\mathbf{k}$, f vanishes unless $|E| \le \mathcal{E}$, for some finite $\mathcal{E}$. Thus

$$B_j(\mathbf{t}) = \int_0^1 ds \int d\theta \int dk_0 \int_{-\mathcal{E}}^{\mathcal{E}} dE \; \tfrac{\nu_0^{[0,j]}(k_0) f(k_0, E, \theta, \mathbf{t}, s)}{[ik_0 - E]^2}.$$

Set

$$B_j'(\mathbf{t}) = \int_0^1 ds \int d\theta \int dk_0 \int_{-\mathcal{E}}^{\mathcal{E}} dE \; \tfrac{\nu_0^{[0,j]}(k_0) f(k_0, 0, \theta, \mathbf{t}, s)}{[ik_0 - E]^2}.$$

Since

$$\Big| \partial_{\mathbf{t}}^\alpha \Big[\tfrac{\nu_0^{[0,j]}(k_0) f(k_0, E, \theta, \mathbf{t}, s)}{[ik_0 - E]^2} - \tfrac{\nu_0^{[0,j]}(k_0) f(k_0, 0, \theta, \mathbf{t}, s)}{[ik_0 - E]^2} \Big] \Big| \le \mathrm{const}_\alpha \tfrac{|E|}{k_0^2 + E^2}$$

is integrable on $\mathbb{R} \times [-\mathcal{E}, \mathcal{E}]$, $\lim_{j\to\infty} B_j(\mathbf{t}) - B_j'(\mathbf{t})$ exists and is C^∞ by the Lebesgue dominated convergence theorem. So it suffices to consider

$$B_j'(\mathbf{t}) = -2\mathcal{E} \int_0^1 ds \int d\theta \int dk_0 \; \tfrac{\nu_0^{[0,j]}(k_0) f(k_0, 0, \theta, \mathbf{t}, s)}{k_0^2 + \mathcal{E}^2}.$$

Since

$$\left|\partial_{\mathbf{t}}^{\alpha} \frac{\nu_0^{[0,j]}(k_0) f(k_0,0,\theta,\mathbf{t},s)}{k_0^2+\mathcal{E}^2}\right| \le \mathrm{const}_\alpha \frac{1}{k_0^2+\mathcal{E}^2}$$

is integrable on $\mathbb{R}$, $\lim_{j\to\infty} B'_j(\mathbf{t})$ exists and is C^∞ by the Lebesgue dominated convergence theorem. □

2. Scales and Sectors. In this paper, we derive position space bounds for generalized particle–hole ladders in two space dimensions as they arise in a multiscale analysis. The main result is Theorem I.20, which is used in [FKTf2], under the name Theorem D.2, to help construct a Fermi liquid. We assume that the dispersion relation $e(\mathbf{k})$ is C^{r_e+3} for some $r_e \ge 6$, that its gradient does not vanish on the Fermi curve $F = \{ \mathbf{k} \in \mathbb{R}^2 \mid e(\mathbf{k}) = 0 \}$ and that the Fermi curve is nonempty, connected, compact and strictly convex (meaning that its curvature does not vanish anywhere). We also fix the number $r_0 \ge 6$ of derivatives in k_0 that we wish to control.

We introduce scales as in [FKTf1, Def. I.2] and [FKTo2, §VIII]:

Definition I.2. *i) For $j \ge 1$, the jth scale function on $\mathbb{R} \times \mathbb{R}^2$ is defined as*

$$\nu^{(j)}(k) = \nu\Big(M^{2j}(k_0^2 + e(\mathbf{k})^2)\Big),$$

where ν is the function of (I.1). It may be constructed by choosing a function $\varphi \in C_0^\infty\big((-2,2)\big)$ that is identically one on $[-1,1]$ and setting $\nu(x) = \varphi(x/M) - \varphi(Mx)$ for $x > 0$ and zero otherwise. By construction, $\nu^{(j)}$ is identically one on

$$\Big\{ k = (k_0,\mathbf{k}) \in \mathbb{R}\times\mathbb{R}^2 \ \Big|\ \sqrt{\tfrac{2}{M}}\,\tfrac{1}{M^j} \le |ik_0 - e(\mathbf{k})| \le \sqrt{M}\,\tfrac{1}{M^j} \Big\}.$$

The support of $\nu^{(j)}$ is called the jth shell. By construction, it is contained in

$$\Big\{ k \in \mathbb{R}\times\mathbb{R}^2 \ \Big|\ \tfrac{1}{\sqrt{M}}\,\tfrac{1}{M^j} \le |ik_0 - e(\mathbf{k})| \le \sqrt{2M}\,\tfrac{1}{M^j} \Big\}$$

The momentum k is said to be of scale j if k lies in the j^{th} shell.

ii) For $j \ge 1$, set

$$\nu^{(\ge j)}(k) = \sum_{i \ge j} \nu^{(i)}(k)$$

for $|ik_0 - e(\mathbf{k})| > 0$ and $\nu^{(\ge j)}(k) = 1$ for $|ik_0 - e(\mathbf{k})| = 0$. Equivalently, $\nu^{(\ge j)}(k) = \varphi\big(M^{2j-1}(k_0^2 + e(\mathbf{k})^2)\big)$. By construction, $\nu^{(\ge j)}$ is identically 1 on

$$\Big\{ k \in \mathbb{R}\times\mathbb{R}^2 \ \Big|\ |ik_0 - e(\mathbf{k})| \le \sqrt{M}\,\tfrac{1}{M^j} \Big\}.$$

The support of $\nu^{(\ge j)}$ is called the jth neighbourhood of the Fermi surface. By construction, it is contained in

$$\Big\{ k \in \mathbb{R}\times\mathbb{R}^2 \ \Big|\ |ik_0 - e(\mathbf{k})| \le \sqrt{2M}\,\tfrac{1}{M^j} \Big\}.$$

The support of $\varphi\big(M^{2j-2}(k_0^2 + e(\mathbf{k})^2)\big)$ is called the jth extended neighbourhood. It is contained in

$$\Big\{ k \in \mathbb{R}\times\mathbb{R}^d \ \Big|\ |ik_0 - e(\mathbf{k})| \le \sqrt{2}M\,\tfrac{1}{M^j} \Big\}.$$

To estimate functions in position space and still make use of conservation of momentum, we use sectorization. See [FKTf1, Example A.1]. The following definition is also made in [FKTf2, §VI] and [FKTo3, §XII].

Definition I.3 (Sectors and sectorizations).

i) Let I be an interval on the Fermi surface F and $j \geq 1$. Then

$$s = \left\{ k \text{ in the } j\text{th neighbourhood} \,\middle|\, \pi_F(k) \in I \right\}$$

is called a sector of length $|I|$ at scale j. Here $k \mapsto \pi_F(k)$ is a projection on the Fermi surface. Two different sectors s and s' are called neighbours if $s' \cap s \neq \emptyset$.

ii) A sectorization of length $\mathfrak{l}$ at scale j is a set Σ of sectors of length $\mathfrak{l}$ at scale j that obeys

- *the set Σ of sectors covers the Fermi surface*
- *each sector in Σ has precisely two neighbours in Σ, one to its left and one to its right*
- *if $s,\ s' \in \Sigma$ are neighbours then $\frac{1}{16}\mathfrak{l} \leq |s \cap s' \cap F| \leq \frac{1}{8}\mathfrak{l}$.*

Observe that there are at most $2\,\mathrm{length}(F)/\mathfrak{l}$ sectors in Σ.

In the renormalization group map of [FKTf1] and [FKTo3], we integrate over fields whose arguments (x, σ, s) lie in $\mathcal{B}^{\updownarrow} \times \Sigma$, where $\mathcal{B}^{\updownarrow} = (\mathbb{R} \times \mathbb{R}^2) \times \{\uparrow, \downarrow\}$ is the set of all "(positions, spins)". On the other hand, we are interested in the dependence of the two and four–point functions on external momenta. To distinguish between the set of all positions and the set of all momenta, we denote by $\mathbb{M} = \mathbb{R} \times \mathbb{R}^2$, the set of all possible momenta. The set of all possible positions shall still be denoted $\mathbb{R} \times \mathbb{R}^2$. Thus the external variables (k, σ) lie in $\check{\mathcal{B}}^{\updownarrow} = \mathbb{M} \times \{\uparrow, \downarrow\}$. In total, legs of four–legged kernels may lie in the disjoint union $\mathfrak{Y}^{\updownarrow}_{\Sigma} = \check{\mathcal{B}}^{\updownarrow} \,\dot\cup\, (\mathcal{B}^{\updownarrow} \times \Sigma)$ for some sectorization Σ. The four–legged kernels over $\mathfrak{Y}^{\updownarrow}_{\Sigma}$ that we consider here arise in [FKTf2, §VII] as particle–hole reductions (as in Definition VII.4 of [FKTf2]) of four–legged kernels on $\mathfrak{X}_{\Sigma} = \check{\mathcal{B}} \,\dot\cup\, (\mathcal{B} \times \Sigma)$, where $\check{\mathcal{B}} = \check{\mathcal{B}}^{\updownarrow} \times \{0, 1\}$ and $\mathcal{B} = \mathcal{B}^{\updownarrow} \times \{0, 1\}$ and $\{0, 1\}$ is the set of creation/annihilation indices. Particle–hole reduction sets the creation/annihilation index to zero for legs number one and four and to one for legs number two and three. To simplify the notation in this paper, we shall eliminate the spin variables so that the legs lie in

$$\mathfrak{Y}_{\Sigma} = \mathbb{M} \,\dot\cup \big((\mathbb{R} \times \mathbb{R}^2) \times \Sigma\big).$$

Sometimes a four–legged kernel will have different sectorizations Σ, Σ' on its two left hand legs and on its two right-hand legs. Therefore, we introduce the space

$$\mathfrak{Y}^{(4)}_{\Sigma,\Sigma'} = \mathfrak{Y}^2_{\Sigma} \times \mathfrak{Y}^2_{\Sigma'}.$$

Since $\mathfrak{Y}_{\Sigma}$ is the disjoint union of $\mathbb{M}$ and $(\mathbb{R} \times \mathbb{R}^2) \times \Sigma$, the space $\mathfrak{Y}^{(4)}_{\Sigma,\Sigma'}$ is the disjoint union

$$\mathfrak{Y}^{(4)}_{\Sigma,\Sigma'} = \dot{\bigcup_{i_1,i_2,i_3,i_4 \in \{0,1\}}} \mathfrak{Y}_{i_1,\Sigma} \times \mathfrak{Y}_{i_2,\Sigma} \times \mathfrak{Y}_{i_3,\Sigma'} \times \mathfrak{Y}_{i_4,\Sigma'}, \tag{I.2}$$

where $\mathfrak{Y}_{0,\Sigma} = \mathbb{M}$ and $\mathfrak{Y}_{1,\Sigma} = (\mathbb{R} \times \mathbb{R}^2) \times \Sigma$. If f is a function on $\mathfrak{Y}^{(4)}_{\Sigma,\Sigma'}$, we denote by $f\big|_{(i_1,\cdots,i_4)}$ its restriction to $\mathfrak{Y}_{i_1,\Sigma} \times \mathfrak{Y}_{i_2,\Sigma} \times \mathfrak{Y}_{i_3,\Sigma'} \times \mathfrak{Y}_{i_4,\Sigma'}$ under the identification (I.2).

Definition I.4 (Translation invariance). *Let Σ and Σ' be sectorizations.*

i) Let $y \in \mathfrak{Y}_\Sigma$ and $t \in \mathbb{R} \times \mathbb{R}^2$. We set

$$T_t y = \begin{cases} k & \text{if } y = k \in \mathbb{M} \\ (x+t, s) & \text{if } y = (x,s) \in \left(\mathbb{R} \times \mathbb{R}^2\right) \times \Sigma \end{cases}.$$

ii) Let $i_1, \cdots, i_4 \in \{0,1\}$. A function f on $\mathfrak{Y}_{i_1,\Sigma} \times \mathfrak{Y}_{i_2,\Sigma} \times \mathfrak{Y}_{i_3,\Sigma'} \times \mathfrak{Y}_{i_4,\Sigma'}$ is called translation invariant, if for all $t \in \mathbb{R} \times \mathbb{R}^2$,

$$f(T_t y_1, \cdots, T_t y_4) = \Bigg(\prod_{\substack{1 \le \mu \le 4 \\ i_\mu = 0}} e^{\imath (-1)^{b_\mu} < y_\mu, t >_-} \Bigg) f(y_1, \cdots, y_4),$$

where

$$b_\mu = \begin{cases} 0 & \text{if } \mu = 1,4 \\ 1 & \text{if } \mu = 2,3 \end{cases} \tag{I.3}$$

and $< k, x >_- = -k_0 x_0 + \mathbf{k}_1 \mathbf{x}_1 + \mathbf{k}_2 \mathbf{x}_2$. This choice of b_μ reflects our image of f as a particle–hole kernel, with first and fourth, resp. second and third, arguments being creation, resp. annihilation, arguments.

iii) A function f on $\mathfrak{Y}^{(4)}_{\Sigma,\Sigma'}$ is translation invariant if $f\big|_{(i_1,\cdots,i_4)}$ is translation invariant for all $i_1, \cdots, i_4 \in \{0,1\}$.

A function f on $\left(\mathfrak{Y}^{\updownarrow}_\Sigma\right)^4$ is translation invariant if $f\,((\,\cdot\,,\sigma_1),(\,\cdot\,,\sigma_2),(\,\cdot\,,\sigma_3),(\,\cdot\,,\sigma_4))$ is translation invariant for all $\sigma_1, \cdots, \sigma_4 \in \{\uparrow, \downarrow\}$.

Definition I.5 (Fourier transform). *Let Σ, Σ' be sectorizations. Set $\mathfrak{Y}_{2,\Sigma} = \mathbb{M} \times \Sigma$.*

i) Let $i_1, \cdots, i_4 \in \{0,1,2\}$ and $1 \le \mu \le 4$ such that $i_\mu = 1$. The Fourier transform of a function f on $\mathfrak{Y}_{i_1,\Sigma} \times \mathfrak{Y}_{i_2,\Sigma} \times \mathfrak{Y}_{i_3,\Sigma'} \times \mathfrak{Y}_{i_4,\Sigma'}$ with respect to the μth variable is the function on $\mathfrak{Y}_{i'_1,\Sigma} \times \mathfrak{Y}_{i'_2,\Sigma} \times \mathfrak{Y}_{i'_3,\Sigma'} \times \mathfrak{Y}_{i'_4,\Sigma'}$ with

$$i'_\nu = \begin{cases} i_\nu & \text{if } \nu \ne \mu \\ 2 & \text{if } \nu = \mu \end{cases}$$

defined by

$$(\Phi_\mu f)(y_1,\cdots,y_{\mu-1},(k,s),y_{\mu+1},\cdots,y_4) = \int e^{\imath(-1)^{b_\mu} <k,x>_-} f(y_1,\cdots,y_{\mu-1},(x,s),y_{\mu+1},\cdots,y_4)\, d^3x.$$

ii) Let $i_1, \cdots, i_4 \in \{0,1\}$ with $i_\mu = 1$ for at least one $1 \le \mu \le 4$. The total Fourier transform $\check{f}$ of a translation invariant function f on $\mathfrak{Y}_{i_1,\Sigma} \times \mathfrak{Y}_{i_2,\Sigma} \times \mathfrak{Y}_{i_3,\Sigma'} \times \mathfrak{Y}_{i_4,\Sigma'}$ is defined by

$$\check{f}(y_1, y_2, y_3, y_4)\, (2\pi)^3 \delta(k_1 - k_2 - k_3 + k_4) = \Bigg(\prod_{\substack{1 \le \mu \le 4 \\ i_\mu = 1}} \Phi_\mu\, f \Bigg)(y_1, y_2, y_3, y_4),$$

where $y_\mu = k_\mu$ when $i_\mu = 0$ and $y_\mu = (k_\mu, s_\mu)$ when $i_\mu = 1$. $\check{f}$ is defined on the set of all $(y_1, y_2, y_3, y_4) \in \mathfrak{Y}_{2i_1,\Sigma} \times \mathfrak{Y}_{2i_2,\Sigma} \times \mathfrak{Y}_{2i_3,\Sigma'} \times \mathfrak{Y}_{2i_4,\Sigma'}$ for which $k_1 - k_2 = k_3 - k_4$.

Definition I.6 (Sectorized functions). *Let Σ and Σ' be sectorizations.*

i) Let $i_1, \cdots, i_4 \in \{0,1\}$. A translation invariant function f on $\mathfrak{Y}_{i_1,\Sigma} \times \mathfrak{Y}_{i_2,\Sigma} \times \mathfrak{Y}_{i_3,\Sigma'} \times \mathfrak{Y}_{i_4,\Sigma'}$ is sectorized if, for each $1 \le \mu \le 4$ with $i_\mu = 1$, the total Fourier transform $\check{f}(y_1,\cdots,y_{\mu-1},(k,s),y_{\mu+1},\cdots,y_4)$ vanishes unless k is in the jth extended neighbourhood and $\pi_F(k) \in s$.

ii) A translation invariant function f on $\mathfrak{Y}^{(4)}_{\Sigma,\Sigma'}$ is sectorized if $f\big|_{(i_1,\cdots,i_4)}$ is sectorized for all $i_1, \cdots, i_4 \in \{0,1\}$.

A translation invariant function f on $\big(\mathfrak{Y}^{\updownarrow}_{\Sigma}\big)^4$ is sectorized if $f((\,\cdot\,,\sigma_1),(\,\cdot\,,\sigma_2),(\,\cdot\,,\sigma_3),(\,\cdot\,,\sigma_4))$ is sectorized for all $\sigma_1, \cdots, \sigma_4 \in \{\uparrow, \downarrow\}$.

Remark I.7. If f is a function in the space $\check{\mathcal{F}}_{4,\Sigma}$ of Definition XIV.6 of [FKTf2] (or Definition XVI.7.iii of [FKTo3]), then its particle–hole reduction is a sectorized function on $\big(\mathfrak{Y}^{\updownarrow}_{\Sigma}\big)^4$.

3. Particle–Hole Ladders.

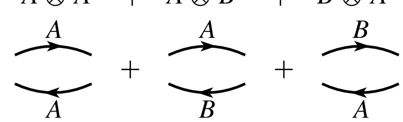
Definition I.8. *i) A (spin independent) propagator is a translation invariant function on $\big(\mathbb{R}\times\mathbb{R}^2\big)^2$. If $A(x,x')$ is a propagator, then its transpose is $A^t(x,x') = A(x',x)$.*

ii) A (spin independent) bubble propagator is a translation invariant function on $\big(\mathbb{R}\times\mathbb{R}^2\big)^4$. If A and B are propagators, we define the bubble propagator

$$A \otimes B(x_1,x_2,x_3,x_4) = A(x_1,x_3)B(x_2,x_4).$$

We set

$$\begin{aligned} \mathcal{C}(A,B) &= (A+B)\otimes(A+B)^t - B\otimes B^t \\ &= A\otimes A^t + A\otimes B^t + B\otimes A^t \end{aligned}$$

$=$ 

iii) Let $\Sigma, \Sigma', \Sigma''$ be sectorizations, P be a bubble propagator and F be a function on $\mathfrak{Y}_{i_1,\Sigma''} \times \mathfrak{Y}_{i_2,\Sigma''} \times (\mathbb{R}\times\mathbb{R}^2)^2$. If K is a function on $\mathfrak{Y}_\Sigma \times \mathfrak{Y}_\Sigma \times \mathfrak{Y}_{1,\Sigma'} \times \mathfrak{Y}_{1,\Sigma'}$, we set

$$(K \bullet P)(y_1,y_2;x_3,x_4) = \sum_{s'_1,s'_2\in\Sigma'} \int dx'_1 dx'_2\ K(y_1,y_2,(x'_1,s'_1),(x'_2,s'_2))\ P(x'_1,x'_2;x_3,x_4).$$

If K is a function on $\mathfrak{Y}_{1,\Sigma} \times \mathfrak{Y}_{1,\Sigma} \times \mathfrak{Y}_{i_3,\Sigma'} \times \mathfrak{Y}_{i_4,\Sigma'}$, we set, when i_1, i_2, i_3, i_4 are not all 0,

$$(F \bullet K)(y_1,y_2,y_3,y_4) = \sum_{s_1,s_2\in\Sigma} \int dx_1 dx_2\ F(y_1,y_2;x_1,x_2)K((x_1,s_1),(x_2,s_2),y_3,y_4)$$

and when $i_1, i_2, i_3, i_4 = 0$,

$$\begin{aligned} &(F \bullet K)(k_1,k_2,k_3,k_4)(2\pi)^3\delta(k_1-k_2-k_3+k_4) \\ &\quad = \sum_{s_1,s_2\in\Sigma} \int dx_1 dx_2\ F(k_1,k_2;x_1,x_2)K((x_1,s_1),(x_2,s_2),k_3,k_4). \end{aligned}$$

Observe that $K \bullet P$ is a function on $\mathfrak{Y}_\Sigma^2 \times (\mathbb{R}\times\mathbb{R}^2)^2$ and $F \bullet K$ is a function on $\mathfrak{Y}^{(4)}_{\Sigma,\Sigma'}$. If K' is a function on $(\mathfrak{Y}^\updownarrow_\Sigma)^4$ and F' is a function on $(\mathfrak{Y}^\updownarrow_\Sigma)^2 \times (\mathcal{B}^\updownarrow)^2$ we set

$$(K' \bullet P)\big((\cdot,\sigma_1),(\cdot,\sigma_2),(\cdot,\sigma_3),(\cdot,\sigma_4)\big) = K'\big((\cdot,\sigma_1),(\cdot,\sigma_2),(\cdot,\sigma_3),(\cdot,\sigma_4)\big) \bullet P$$

and

$$\begin{aligned}(F' \bullet K')&\big((\cdot,\sigma_1),(\cdot,\sigma_2),(\cdot,\sigma_3),(\cdot,\sigma_4)\big)\\ &= \sum_{\tau_1,\tau_2\in\{\uparrow,\downarrow\}} F'\big((\cdot,\sigma_1),(\cdot,\sigma_2),(\cdot,\tau_1),(\cdot,\tau_2)\big) \bullet K'\big((\cdot,\tau_1),(\cdot,\tau_2),(\cdot,\sigma_3),(\cdot,\sigma_4)\big).\end{aligned}$$

iv) Let $\ell \ge 1$. Let, for $1 \le i \le \ell+1$, $\Sigma^{(i)}$, $\Sigma'^{(i)}$ be sectorizations and K_i a function on $\mathfrak{Y}^{(4)}_{\Sigma^{(i)},\,\Sigma'^{(i)}}$. Furthermore, let $P_1,\cdots,P_\ell$ be bubble propagators. The ladder with rungs $K_1,\cdots,K_{\ell+1}$ and bubble propagators $P_1,\cdots,P_\ell$ is defined to be

$$K_1 \bullet P_1 \bullet K_2 \bullet P_2 \bullet \cdots \bullet K_\ell \bullet P_\ell \bullet K_{\ell+1}.$$

If Σ is a sectorization and $K'_1,\ \cdots,\ K'_{\ell+1}$ are functions on $\big(\mathfrak{Y}^\updownarrow_\Sigma\big)^4$, the ladder with rungs $K'_1,\cdots,K'_{\ell+1}$ and bubble propagators $P_1,\cdots,P_\ell$ is defined to be

$$K'_1 \bullet P_1 \bullet K'_2 \bullet P_2 \bullet \cdots \bullet K'_\ell \bullet P_\ell \bullet K'_{\ell+1}.$$

Remark I.9. We typically use $\mathcal{C}(A,B)$ with A being the part, $\nu^{(j)}(k)C(k)$, of the propagator, $C(k)$, having momentum in the jth shell and B being the part, $\nu^{(\ge j+1)}(k)C(k)$, of the propagator having momentum in the $(j+1)$st neighbourhood. The bubble propagator $\mathcal{C}(A,B)$ always contains at least one "hard line" A and may or may not contain one "soft line" B. The latter are created by Wick ordering. See [FKTf1, §II, Subsect. 9].

Remark I.10. If $F_1,\ F_2$ are functions on $\big(\mathfrak{X}_\Sigma\big)^4$ and $A,\ B$ are propagators over $\mathcal{B}$ in the sense of Definition VII.1.i of [FKTf2], then the particle–hole reduction of $F_1 \bullet \mathcal{C}(A,B) \bullet F_2$ (with the $\mathcal{C}(A,B)$ of Definition VII.1.i of [FKTf2]) is equal to

$$-F_1^{\rm ph} \bullet \mathcal{C}\big(A((\cdot\,1),(\cdot\,0)),\, B((\cdot\,1),(\cdot\,0))\big) \bullet F_2^{\rm ph}$$

(with the $\mathcal{C}$ of Definition I.8) since $B((x,\sigma,0),(x',\sigma',1)) = -B((\cdot\,1),(\cdot\,0))^t((x,\sigma),(x',\sigma'))$.

4. Norms. In the momentum space variables, we take suprema of the function and its derivatives. In the position space variables, we will apply the L^1–L^∞ norm of Definition I.11, below, to the function and to the function multiplied by various coordinate differences.

Definition I.11. *Let f be a function on $\big(\mathbb{R}\times\mathbb{R}^2\big)^n$. Its L^1–L^∞ norm is*

$$|||f|||_{1,\infty} = \max_{1\le j_0\le n}\ \sup_{x_{j_0}\in\mathbb{R}\times\mathbb{R}^2} \int \prod_{\substack{j=1,\cdots,n\\ j\ne j_0}} dx_j\ |f(x_1,\cdots,x_n)|.$$

Multiple derivatives are labeled by a multiindex $\delta = (\delta_0,\delta_1,\delta_2) \in \mathbb{N}_0\times\mathbb{N}_0^2$. For such a multiindex, we set $|\delta| = \delta_0+\delta_1+\delta_2$, $\delta! = \delta_0!\,\delta_1!\,\delta_2!$ and $x^\delta = x_0^{\delta_0}x_1^{\delta_1}x_2^{\delta_2}$ for $x\in\mathbb{R}\times\mathbb{R}^2$.

Definition I.12. *Let Σ be a sectorization and A a function on $\big((\mathbb{R}\times\mathbb{R}^2)\times\Sigma\big)^2$. For a multiindex $\delta\in\mathbb{N}_0\times\mathbb{N}_0^2$, we define*

$$|A|_{1,\Sigma}^{\delta}=\max_{i=1,2}\ \max_{s_i\in\Sigma}\sum_{s_{3-i}\in\Sigma}\Big|\Big|\Big|(x-y)^{\delta}\,A\big((x,s_1),(y,s_2)\big)\Big|\Big|\Big|_{1,\infty}.$$

Variables for four–point functions may be momenta or position/sector pairs. Therefore we introduce differential–decay operators that differentiate momentum space variables and multiply position space variables by coordinate differences. We again use the identification

$$\mathfrak{Y}_{\Sigma,\Sigma'}^{(4)}=\dot{\bigcup_{i_1,i_2,i_3,i_4\in\{0,1\}}}\mathfrak{Y}_{i_1,\Sigma}\times\mathfrak{Y}_{i_2,\Sigma}\times\mathfrak{Y}_{i_3,\Sigma'}\times\mathfrak{Y}_{i_4,\Sigma'}$$

of (I.2).

Definition I.13 (Differential–decay operators). *Let Σ and Σ' be sectorizations, $\delta=(\delta_0,\delta_1,\delta_2)\in\mathbb{N}_0\times\mathbb{N}_0^2$ a multiindex and $\mu,\mu'\in\{1,2,3,4\}$ with $\mu\neq\mu'$.*

i) Let $i_1,\cdots,i_4\in\{0,1\}$ and f be a function on $\mathfrak{Y}_{i_1,\Sigma}\times\mathfrak{Y}_{i_2,\Sigma}\times\mathfrak{Y}_{i_3,\Sigma'}\times\mathfrak{Y}_{i_4,\Sigma'}$. If $i_\mu=0$, multiplication by the δth power of the position variable dual to k_μ (see Definition I.5) is implemented by

$$\mathrm{D}_{\mu}^{\delta}f(\cdots,k_\mu,\cdots)=(-1)^{\delta_1+\delta_2}(-1)^{b_\mu|\delta|}\imath^{|\delta|}\ \frac{\partial^{\delta_0}}{\partial k_{\mu,0}^{\delta_0}}\frac{\partial^{\delta_1}}{\partial\mathbf{k}_{\mu,1}^{\delta_1}}\frac{\partial^{\delta_2}}{\partial\mathbf{k}_{\mu,2}^{\delta_2}}f(\cdots,k_\mu,\cdots).$$

In general, set

$$\mathrm{D}_{\mu;\mu'}^{\delta}f=\begin{cases}\big(\mathrm{D}_{\mu}^{\delta}-\mathrm{D}_{\mu'}^{\delta}\big)f & \text{if } i_\mu=i_{\mu'}=0\\ \big(\mathrm{D}_{\mu}^{\delta}-x_{\mu'}^{\delta}\big)f & \text{if } i_\mu=0,\,i_{\mu'}=1\\ \big(x_{\mu}^{\delta}-\mathrm{D}_{\mu'}^{\delta}\big)f & \text{if } i_\mu=1,\ i_{\mu'}=0\\ \big(x_{\mu}^{\delta}-x_{\mu'}^{\delta}\big)f & \text{if } i_\mu=i_{\mu'}=1\end{cases}.$$

Here, when $i_\mu=1$, the μ^{th} argument of f is (x_μ,s_μ).

ii) If f is a function on $\mathfrak{Y}_{\Sigma,\Sigma'}^{(4)}$, then $\big(\mathrm{D}_{\mu;\mu'}^{\delta}f\big)\big|_{(i_1,\cdots,i_4)}=\mathrm{D}_{\mu;\mu'}^{\delta}\big(f\big|_{(i_1,\cdots,i_4)}\big)$ for all $i_1,\cdots,i_4\in\{0,1\}$.

Definition I.14. *Let Σ, Σ' be sectorizations.*

i) Let $i_1,\cdots,i_4\in\{0,1\}$ and f be a function on $\mathfrak{Y}_{i_1,\Sigma}\times\mathfrak{Y}_{i_2,\Sigma}\times\mathfrak{Y}_{i_3,\Sigma'}\times\mathfrak{Y}_{i_4,\Sigma'}$. For multiindices $\delta_{\mathrm{l}},\ \delta_{\mathrm{c}},\ \delta_{\mathrm{r}}\in\mathbb{N}_0\times\mathbb{N}_0^2$, we define

$$|f|_{\Sigma,\Sigma'}^{(\delta_{\mathrm{l}},\delta_{\mathrm{c}},\delta_{\mathrm{r}})}=\max_{\substack{s_\nu\in\Sigma\\ \nu=1,2\\ \text{with } i_\nu=1}}\ \max_{\substack{s_\nu\in\Sigma'\\ \nu=3,4\\ \text{with } i_\nu=1}}\ \sup_{\substack{k_\nu\in\mathbb{M}\\ \nu=1,2,3,4\\ \text{with } i_\nu=0}}\ \max_{\substack{\mu=1,2\\ \mu'=3,4}}\Big|\Big|\Big|\mathrm{D}_{1;2}^{\delta_{\mathrm{l}}}\mathrm{D}_{\mu;\mu'}^{\delta_{\mathrm{c}}}\mathrm{D}_{3;4}^{\delta_{\mathrm{r}}}f\Big|\Big|\Big|_{1,\infty}.$$

Here, the ν^{th} argument of f is k_ν when $i_\nu=0$ and (x_ν,s_ν) when $i_\nu=1$. The $|||\cdot|||_{1,\infty}$ of Definition I.11 is applied to all spatial arguments of $\mathrm{D}_{1;2}^{\delta_{\mathrm{l}}}\mathrm{D}_{\mu;\mu'}^{\delta_{\mathrm{c}}}\mathrm{D}_{3;4}^{\delta_{\mathrm{r}}}f$.

ii) If f is a function on $\mathfrak{Y}_{\Sigma,\Sigma'}^{(4)}$, we define

$$|f|_{\Sigma,\Sigma'}^{(\delta_{\mathrm{l}},\delta_{\mathrm{c}},\delta_{\mathrm{r}})}=\sum_{i_1,i_2,i_3,i_4\in\{0,1\}}\big|f\big|_{(i_1,\cdots,i_4)}\big|_{\Sigma,\Sigma'}^{(\delta_{\mathrm{l}},\delta_{\mathrm{c}},\delta_{\mathrm{r}})}.$$

In this definition, the system $(\delta_{\rm l}, \delta_{\rm c}, \delta_{\rm r})$ of multiindices indicates, roughly speaking, that one takes $\delta_{\rm l}$ derivatives with respect to the momentum flowing between the two left legs, $\delta_{\rm r}$ derivatives with respect to the momentum flowing between the two right legs and $\delta_{\rm c}$ derivatives with respect to momenta flowing from the left-hand side to the right-hand side.

In [FKTf1, FKTf2, FKTf3] and [FKTo1, FKTo2, FKTo3, FKTo4], we combine the norms of all derivatives of a function in a formal power series. We denote by $\mathfrak{N}_3$ the set of all formal power series $X = \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} X_\delta t^\delta$ in the variables $t = (t_0, t_1, t_2)$ with coefficients $X_\delta \in \mathbb{R}_+\cup\{\infty\}$. See Definition V.2 of [FKTf2] or Definition II.4 of [FKTo1].

A quantity in $\mathfrak{N}_3$ characteristic of the power counting for derivatives in scale j is

$$\mathfrak{c}_j = \sum_{\substack{\delta_1+\delta_2\le r_e \\ |\delta_0|\le r_0}} M^{j|\delta|}\, t^\delta + \sum_{\substack{\delta_1+\delta_2> r_e \\ \text{or } |\delta_0|> r_0}} \infty\, t^\delta. \tag{I.4}$$

Definition I.15. *Let Σ be a sectorization.*

i) For a function A on $\big((\mathbb{R}\times\mathbb{R}^2)\times\Sigma\big)^2$, we define

$$|A|_{1,\Sigma} = \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \tfrac{1}{\delta!}|A|^{\delta}_{1,\Sigma}\, t^\delta.$$

ii) For a function f on $\mathfrak{Y}^4_\Sigma = \mathfrak{Y}^{(4)}_{\Sigma,\Sigma}$, we define

$$|f|_\Sigma = \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \tfrac{1}{\delta!}\Big(\max_{\delta_{\rm l}+\delta_{\rm c}+\delta_{\rm r}=\delta} |f|^{(\delta_{\rm l},\delta_{\rm c},\delta_{\rm r})}_{\Sigma,\Sigma}\Big) t^\delta.$$

iii) For a function f on $\big(\mathfrak{Y}^{\updownarrow}_\Sigma\big)^4$, we define

$$|f|_\Sigma = \sum_{\sigma_1,\cdots,\sigma_4\in\{\uparrow,\downarrow\}} |f((\,\cdot\,,\sigma_1),(\,\cdot\,,\sigma_2),(\,\cdot\,,\sigma_3),(\,\cdot\,,\sigma_4))|_\Sigma.$$

The following lemma, whose proof follows immediately from the various definitions and Lemma D.2.ii of [FKTo3], compares these norms with the norms of Definition VI.6 of [FKTf2].

Lemma I.16. *Let Σ be a sectorization.*

(i) Let f be a sectorized, translation invariant function on $\big(\mathfrak{Y}^{\updownarrow}_\Sigma\big)^4$ and $V_{\rm ph}(f)$ its particle–hole value as in Definition VII.4 of [FKTf2]. Let $|\cdot|^{\sim}_{3,\Sigma}$ be the norm of Definition XIII.12 of [FKTf3] (or Definition XVI.4 of [FKTo3]). Then there is a constant const, that depends only on r_0 and r, such that

$$|V_{\rm ph}(f)|^{\sim}_{3,\Sigma} \le \mathit{const}\, |f|_\Sigma + \sum_{\substack{\delta_1+\delta_2>r \\ \text{or } \delta_0>r_0}} \infty\, t^\delta.$$

(ii) Let g be a function in the space $\check{\mathcal{F}}_{4,\Sigma}$ of Definition XIV.6 of [FKTf2] (or Definition XVI.7.iii of [FKTo3]) and $g^{\rm ph}$ its particle–hole reduction as in Definition VII.4 of [FKTf2]. Then there is a universal const such that

$$|g^{\rm ph}|_\Sigma \le \mathit{const}\, |g|^{\sim}_{3,\Sigma}.$$

5. The Propagators. The propagators we use in the multiscale analysis of [FKTf1, FKTf2, FKTf3] are of the form

$$C_v^{(j)}(k) = \frac{\nu^{(j)}(k)}{ik_0 - e(\mathbf{k}) - v(k)} \qquad C_v^{(\geq j)}(k) = \frac{\nu^{(\geq j)}(k)}{ik_0 - e(\mathbf{k}) - v(k)}$$

with functions $v(k)$ satisfying $|v(k)| \leq \frac{1}{2}|\imath k_0 - e(\mathbf{k})|$. Their Fourier transforms are

$$\begin{aligned}
C_v^{(j)}(x,y) &= \int \tfrac{d^3k}{(2\pi)^3}\, e^{\imath <k,x-y>_-} C_v^{(j)}(k) \\
C_v^{(\geq j)}(x,y) &= \int \tfrac{d^3k}{(2\pi)^3}\, e^{\imath <k,x-y>_-} C_v^{(\geq j)}(k), \\
C_v^{(j)}(y) &= \int \tfrac{d^3k}{(2\pi)^3}\, e^{-\imath <k,y>_-} C_v^{(j)}(k) \\
C_v^{(\geq j)}(y) &= \int \tfrac{d^3k}{(2\pi)^3}\, e^{-\imath <k,y>_-} C_v^{(\geq j)}(k).
\end{aligned}$$

The function $v(k)$ will be the sum of Fourier transforms of sectorized, translation invariant functions $p\big((x,s),(x,s')\big)$ on $\Big(\big(\mathbb{R}\times\mathbb{R}^2\big)\times\Sigma\Big)^2$ for various sectorizations Σ. The Fourier transform of such a function is defined as

$$\check{p}(k) = \sum_{s,s'\in\Sigma} \int d^3x\; e^{\imath <k,x>_-}\; p\big((0,s),(x,s')\big).$$

6. Resectorization. We now fix $\frac{1}{2} < \aleph < \frac{2}{3}$ and set $\mathfrak{l}_j = \frac{1}{M^{\aleph j}}$. Furthermore, we select, for each $j \geq 1$, a sectorization Σ_j of length $\mathfrak{l}_j$ at scale j and a partition of unity $\big\{\, \chi_s \,\big|\, s\in\Sigma_j \,\big\}$ of the $j^{\rm th}$ neighbourhood which fulfills Lemma XII.3 of [FKTo3] with $\Sigma = \Sigma_j$. The Fourier transform of χ_s is

$$\hat{\chi}_s(x) = \int e^{-\imath <k,x>_-}\, \chi_s(k)\, \tfrac{d^3k}{(2\pi)^3}.$$

Definition I.17 (Resectorization). *Let $j, j', j_{\rm l}, j'_{\rm l}, j_{\rm r}, j'_{\rm r} \geq 1$.*

i) Let p be a sectorized, translation invariant function on $\Big(\big(\mathbb{R}\times\mathbb{R}^2\big)\times\Sigma_j\Big)^2$. Then, for $j'\neq j$, the j'–resectorization of p is

$$p_{\Sigma_{j'}}\big((x_1,s_1),(x_2,s_2)\big) = \sum_{s'_1,s'_2\in\Sigma_j} \int dx'_1\, dx'_2\; \hat{\chi}_{s_1}(x_1-x'_1)\; p\big((x'_1,s'_1),(x'_2,s'_2)\big)\; \hat{\chi}_{s_2}(x'_2-x_2).$$

It is a sectorized, translation invariant function on $\Big(\big(\mathbb{R}\times\mathbb{R}^2\big)\times\Sigma_{j'}\Big)^2$. If $j=j'$, we set $p_{\Sigma_{j'}} = p$.

ii) Let $i_1,\cdots,i_4\in\{0,1\}$ and f be a function on $\mathfrak{Y}_{i_1,\Sigma_{j_{\rm l}}}\times\mathfrak{Y}_{i_2,\Sigma_{j_{\rm l}}}\times\mathfrak{Y}_{i_3,\Sigma_{j_{\rm r}}}\times\mathfrak{Y}_{i_4,\Sigma_{j_{\rm r}}}$ that is sectorized and translation invariant. Then the $(j'_{\rm l}, j'_{\rm r})$–resectorization of f is the sectorized, translation invariant function on $\mathfrak{Y}_{i_1,\Sigma_{j'_{\rm l}}}\times\mathfrak{Y}_{i_2,\Sigma_{j'_{\rm l}}}\times\mathfrak{Y}_{i_3,\Sigma_{j'_{\rm r}}}\times\mathfrak{Y}_{i_4,\Sigma_{j'_{\rm r}}}$ defined by

$$f_{\Sigma_{j'_{\rm l}},\Sigma_{j'_{\rm r}}}(y_1,y_2,y_3,y_4) = \sum_{\substack{s'_\mu\in\Sigma_{j_{\rm l}}\\ \mu\in\{1,2\}\cap S}}\ \sum_{\substack{s'_\mu\in\Sigma_{j_{\rm r}}\\ \mu\in\{3,4\}\cap S}} \int \prod_{\mu\in S}\Big(dx'_\mu\; \hat{\chi}_{s_\mu}\big((-1)^{b_\mu}(x_\mu-x'_\mu)\big)\Big)\, f\big(y'_1,y'_2,y'_3,y'_4\big),$$

where

$$S = \{\ \mu \mid i_\mu = 1\ \} \cap \begin{cases} \{1,2,3,4\} & \text{if } j'_{\rm l} \neq j_{\rm l},\ j'_{\rm r} \neq j_{\rm r} \\ \{1,2\} & \text{if } j'_{\rm l} \neq j_{\rm l},\ j'_{\rm r} = j_{\rm r} \\ \{3,4\} & \text{if } j'_{\rm l} = j_{\rm l},\ j'_{\rm r} \neq j_{\rm r} \\ \emptyset & \text{if } j'_{\rm l} = j_{\rm l},\ j'_{\rm r} = j_{\rm r} \end{cases}$$

and $y'_\mu = y_\mu$ *for* $\mu \notin S$ *and, for* $\mu \in S$,

$$y_\mu = (x_\mu, s_\mu) \qquad y'_\mu = (x'_\mu, s'_\mu).$$

iii) If f *is a sectorized, translation invariant function on* $\mathfrak{Y}^{(4)}_{\Sigma_{j_{\rm l}},\Sigma_{j_{\rm r}}}$, *then* $\big(f_{\Sigma_{j'_{\rm l}},\Sigma_{j'_{\rm r}}}\big)\big|_{(i_1,\cdots,i_4)} = \big(f\big|_{(i_1,\cdots,i_4)}\big)_{\Sigma_{j'_{\rm l}},\Sigma_{j'_{\rm r}}}$ *for all* $i_1,\cdots,i_4 \in \{0,1\}$. *If* $j'_{\rm l} = j'_{\rm r} = j'$, *we set* $f_{\Sigma_{j'}} = f_{\Sigma_{j'},\Sigma_{j'}}$.

iv) If f *is a sectorized, translation invariant function on* $\big(\mathfrak{Y}^{\updownarrow}_{\Sigma_j}\big)^4$, *then*

$$f_{\Sigma_{j'}}((\cdot,\sigma_1),(\cdot,\sigma_2),(\cdot,\sigma_3),(\cdot,\sigma_4)) = \Big(f((\cdot,\sigma_1),(\cdot,\sigma_2),(\cdot,\sigma_3),(\cdot,\sigma_4))\Big)_{\Sigma_{j'}}$$

for all $\sigma_1,\cdots,\sigma_4 \in \{\uparrow,\downarrow\}$.

Remark I.18. Let K and H be sectorized translation invariant functions on $\mathfrak{Y}^{(4)}_{\Sigma_{i_{\rm l}},\Sigma_{j_{\rm l}}}$ and $\mathfrak{Y}^{(4)}_{\Sigma_{i_{\rm r}},\Sigma_{j_{\rm r}}}$ respectively. Let P be a bubble propagator. If the Fourier transform

$$\int \prod_{\mu=1}^{4} dx_\mu \ \prod_{\mu=1}^{4} e^{-\imath(-1)^{b_\mu}<k_\mu,x_\mu>_-}\ P(x_1,x_2,x_3,x_4)$$

of P is supported on the $\max\{j'_{\rm l}, i'_{\rm r}\}^{\rm th}$ neighbourhood, then

$$\Big[K \bullet P \bullet H\Big]_{\Sigma_{i'_{\rm l}},\Sigma_{j'_{\rm r}}} = K_{\Sigma_{i'_{\rm l}},\Sigma_{j'_{\rm l}}} \bullet P \bullet H_{\Sigma_{i'_{\rm r}},\Sigma_{j'_{\rm r}}}.$$

7. Compound Particle–Hole Ladders. Define, for any set $\mathcal{Z}$ and any function K on $\mathcal{Z}^4$, the flipped function

$$K^f(z_1,z_2,z_3,z_4) = -K(z_1,z_3,z_2,z_4). \tag{I.5}$$

Definition I.19. *Let* $\vec{F} = \big(F^{(2)}, F^{(3)}, \cdots\big)$ *be a sequence of sectorized, translation invariant functions* $F^{(i)}$ *on* $\big(\mathfrak{Y}^{\updownarrow}_{\Sigma_i}\big)^4$ *and* $v(k)$ *a function on* $\mathbb{M}$ *such that* $|v(k)| \le \frac{1}{2}|\imath k_0 - e(\mathbf{k})|$. *We define, recursively on* $0 \le j < \infty$, *the compound particle–hole (or wrong way) ladders up to scale* j, *denoted by* $\mathcal{L}^{(j)} = \mathcal{L}^{(j)}_v(\vec{F})$, *as*

$$\mathcal{L}^{(0)} = 0,$$
$$\mathcal{L}^{(j+1)} = \mathcal{L}^{(j)}_{\Sigma_j} + \sum_{\ell=1}^{\infty} \big(F + \mathcal{L}^{(j)}_{\Sigma_j} + \mathcal{L}^{(j)\,f}_{\Sigma_j}\big) \bullet \mathcal{C}^{(j)} \bullet \cdots \mathcal{C}^{(j)} \bullet \big(F + \mathcal{L}^{(j)}_{\Sigma_j} + \mathcal{L}^{(j)\,f}_{\Sigma_j}\big),$$

where $F = \sum_{i=2}^{j} F^{(i)}_{\Sigma_j}$ *and the* ℓ*th term has* ℓ *bubble propagators* $\mathcal{C}^{(j)} = \mathcal{C}\big(C^{(j)}_v, C^{(\ge j+1)}_v\big)$. *Observe that* $\mathcal{L}^{(1)} = \mathcal{L}^{(2)} = 0$.

Theorem I.20. *For every $\varepsilon > 0$ there are constants ρ_0, const*[1] *such that the following holds. Let $\vec{F} = \big(F^{(2)}, F^{(3)}, \cdots\big)$ be a sequence of sectorized, translation invariant spin independent*[2] *functions $F^{(i)}$ on $\big(\mathfrak{Y}^{\updownarrow}_{\Sigma_i}\big)^4$ and $\vec{p} = \big(p^{(2)}, p^{(3)}, \cdots\big)$ be a sequence of sectorized, translation invariant functions $p^{(i)}$ on $\Big(\big(\mathbb{R}\times\mathbb{R}^2\big)\times\Sigma_i\Big)^2$. Assume that there is $\rho \le \rho_0$ such that for $i \ge 2$,*

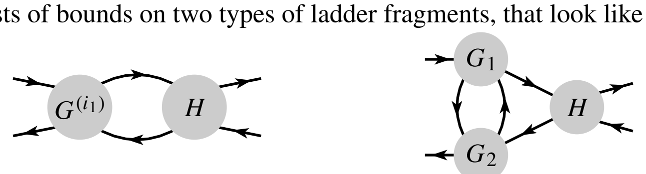
$$|F^{(i)}|_{\Sigma_i} \le \tfrac{\rho}{M^{\varepsilon i}}\,\mathfrak{c}_i \qquad |p^{(i)}|_{1,\Sigma_i} \le \tfrac{\rho\,\mathfrak{l}_i}{M^i}\,\mathfrak{c}_i \qquad \check{p}^{(i)}(0,\mathbf{k}) = 0.$$

Set $v(k) = \sum_{i=2}^{\infty} \check{p}^{(i)}(k)$. Then for all $j \ge 1$,

$$|\mathcal{L}_v^{(j+1)}(\vec{F})|_{\Sigma_j} \le \text{const}\,\rho^2\,\mathfrak{c}_j.$$

Remark I.21. Theorem I.20 and Theorem D.2 of [FKTf3] are equivalent. If one replaces the functions $F^{(i)}$ of Theorem D.2 of [FKTf3] by 24 times their particle–hole reductions, then, by Corollary D.7 of [FKTf3] and Remark I.10, the concepts of compound ladders of Definition I.19 and Definition D.1 of [FKTf3] coincide. Hence Theorem I.20 and Theorem D.2 of [FKTf3] are equivalent by Lemma I.16.

Theorem I.20 is proven in the full article following Corollary II.24. The core of the proof consists of bounds on two types of ladder fragments, that look like



and are called particle–hole bubbles and double bubbles, and a combinatorial result, Corollary II.12, that enables one to express general ladders in terms of these fragments. The most subtle part of the bound, Theorem II.19, on particle–hole bubbles is a generalization of Lemma I.1. The bound, Theorem II.20, on double bubbles also exploits "volume improvement due to overlapping loops". A simple introduction to this phenomenon is provided at the beginning of §IV of the full article.

Ladders with external momenta have an infrared limit that behaves much like the model bubble of Lemma I.1.

Theorem I.22. *Under the hypotheses of Theorem I.20, the limit*

$$\mathfrak{L}(q,q',t,\sigma_1,\cdots\sigma_4) = \lim_{j\to\infty} \mathcal{L}_v^{(j)}(\vec{F})\big|_{i_1,i_2,i_3,i_4=0}\big((q+\tfrac{t}{2},\sigma_1),(q-\tfrac{t}{2},\sigma_2),(q'+\tfrac{t}{2},\sigma_3),(q'-\tfrac{t}{2},\sigma_4)\big)$$

exists for transfer momentum $t \ne 0$ and is continuous in (q,q',t) for $t \ne 0$. The restrictions to $\mathbf{t} = 0$ and to $t_0 = 0$, namely, $\mathfrak{L}(q,q',(t_0,\mathbf{0}),\sigma_1,\cdots\sigma_4)$ and $\mathfrak{L}(q,q',(0,\mathbf{t}),\sigma_1,\cdots\sigma_4)$, have continuous extensions to $t = 0$.

This theorem is proven in the full article following Lemma II.29.

[1] Throughout this paper we use "const" to denote unimportant constants that depend only on the dispersion relation $e(\mathbf{k})$ and the scale parameter M. In particular, they do not depend on the scale j.

[2] "Spin independence" is formally defined in Definition II.6 of the full article.

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