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# SINGLE SCALE ANALYSIS OF MANY FERMION SYSTEMS PART 2: THE FIRST SCALE

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The first renormalization group map arising from the momentum space decomposition of a weakly coupled system of fermions at temperature zero differs from all subsequent maps. Namely, the component of momentum dual to temperature may be arbitrarily large — there is no ultraviolet cutoff. The methods of Part 1 are supplemented to control this special case.

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## Contents



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## VI. Introduction to Part 2

We continue our analysis of models for weakly interacting fermions in $d$-dimensions given in terms of

- a single particle dispersion relation $e(\mathbf{k})$ on $\mathbb{R}^d$,
- an ultraviolet cutoff $U(\mathbf{k})$ on $\mathbb{R}^d$,
- an interaction.

From now on, we fix $r \geq 2$ and assume that the dispersion relation is at least $r+d+1$ times differentiable. As discussed in Part 1, formally, the generating functional for the connected amputated Green's functions is

$$\mathcal{G}_{\mathrm{amp}}(\phi) = \log \frac{1}{Z} \int e^{\mathcal{V}(\psi+\phi)} d\mu_C(\psi)$$

where $Z = \int e^{\mathcal{V}(\psi)} d\mu_C(\psi)$. In this Grassmann integral, there are anticommuting fields $\psi(\xi)$, where $\xi = (x_0, \mathbf{x}, \sigma, a) \in \mathcal{B} = \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\} \times \{0,1\}$. See the beginning of Sec. II. The covariance of the Grassmann Gaussian measure $d\mu_C$ is the Fourier transform $C(\xi, \xi')$ of

$$C(k_0, \mathbf{k}) = \frac{U(\mathbf{k})}{\imath k_0 - e(\mathbf{k})}$$

as in Proposition IV.8. The interaction is

$$\mathcal{V}(\psi) = \int_{(\mathbb{R}\times\mathbb{R}^2\times\{\uparrow,\downarrow\})^4} V_0(x_1, x_2, x_3, x_4)$$
$$\times \psi((x_1,1))\psi((x_2,0))\psi((x_3,1))\psi((x_4,0)) dx_1 dx_2 dx_3 dx_4 \,.$$

We shall, at various places, assume that $V_0$ has a number of symmetries, that we abbreviate by single letters — translation invariance (T), spin independence (S), conservation of particle number (N), "$k_0$-reversal reality" (R) and "bar/unbar exchange invariance" (B). Precise definitions and a discussion of the properties of these symmetries are given in Appendix B.

Formally, the Green's functions of the many fermion system are

$$S_{2n}(x_1, y_1, \ldots, x_n, y_n) = \frac{1}{Z} \int \prod_{i=1}^{n} \psi((x_i, 0))\psi((y_i, 1)) e^{\mathcal{V}(\psi)} d\mu_C(\psi)$$

where $Z = \int e^{\mathcal{V}(\psi)} d\mu_C(\psi)$. The generating functional for these Green's functions is

$$\mathcal{S}(\phi) = \frac{1}{Z} \int e^{\phi J \psi} e^{\mathcal{V}(\psi)} d\mu_C(\psi)$$

where the operator $J$ has kernel

$$J((x_0, \mathbf{x}, \sigma, a), (x_0', \mathbf{x}', \sigma', a'))$$
$$= \delta(x_0 - x_0')\delta(\mathbf{x} - \mathbf{x}')\delta_{\sigma,\sigma'} \begin{cases} 1 & \text{if } a = 1,\ a' = 0 \\ -1 & \text{if } a = 0,\ a' = 1 \\ 0 & \text{otherwise} \end{cases} \tag{VI.1}$$

so that the source term has the form

$$\phi J\psi = \int d\xi d\xi' \phi(\xi) J(\xi,\xi')\psi(\xi') = \int dx \bar{\phi}(x)\psi(x) + \bar{\psi}(x)\phi(x) = \psi J\phi\,. \qquad \text{(VI.2)}$$

The generating functional for the connected Green's functions is

$$\mathcal{G}(\phi) = \log \frac{1}{Z}\int e^{\phi J\psi} e^{\mathcal{V}(\psi)} d\mu_C(\psi)$$

and the connected Green's functions themselves are determined by

$$\mathcal{G}(\phi) = \sum_{n=1}^{\infty} \frac{1}{(n!)^2} \int \prod_{i=1}^{n} dx_i dy_i G_{2n}(x_1,y_1,\ldots,x_n,y_n) \prod_{i=1}^{n} \bar{\phi}(x_i)\phi(y_i)\,.$$

The relation between the connected Green's functions and the amputated connected Green's functions is

$$G_{2n}(x_1,y_1,\ldots,x_n,y_n)$$
$$= \int \prod_{i=1}^{n} dx_i' dy_i' \left(\prod_{i=1}^{n} C(x_i,x_i')C(y_i',y_i)\right) G_{2n}^{\mathrm{amp}}(x_1',y_1',\ldots,x_n',y_n')$$

for $n \geq 2$, and

$$G_2(x,y) - C(x,y) = \int dx' dy' C(x,x')C(y',y) G_2^{\mathrm{amp}}(x',y')\,.$$

In a multiscale analysis we shall estimate the position space supremum norm of connected Green's functions and the momentum space supremum norm of connected amputated Green's functions. We fix $r_0 \geq 2$ and control the Green's functions, including up to $r_0$ derivatives in the $k_0$ direction. In Sec. VII, we introduce a variant, $\tilde{\Omega}$, of the renormalization group map $\Omega$ for use with the connected Green's functions. In Sec. VIII, we introduce the scale decomposition that will be used for the multiscale analysis. Using the results of Part 1, we discuss the map $\tilde{\Omega}$, for the first few scales. The discussion will be sufficiently general to allow the absorption of a (renormalization) counterterm in the dispersion relation. In Sec. X, we introduce norms for use with the amputated Green's functions and discuss the map $\Omega$, for the first few scales. Notation tables are provided at the end of the paper.

## VII. Amputated and Nonamputated Green's Functions

**Definition VII.1.** The (unamputated) renormalization group map $\tilde{\Omega}_C$ with respect to the covariance $C$ associates the Grassmann function

$$\tilde{\Omega}_C(\mathcal{W})(\phi,\psi) = \log \frac{1}{Z}\int e^{\phi J\zeta} e^{\mathcal{W}(\phi,\psi+\zeta)} d\mu_C(\zeta) \quad \text{where } Z = \int e^{W(0,\zeta)} d\mu_C(\zeta) \neq 0$$

to the Grassmann function $\mathcal{W}(\phi,\psi)$. As was the case with $\Omega_C(\mathcal{W})$, [1, Theorem II.28] implies that, under hypotheses that we will make explicit later, the formal Taylor expansion of $\tilde{\Omega}_C(\mathcal{W})$ converges to an analytic function of $\mathcal{W}$.

**Remark VII.2.** (i) In the situation described in the introduction, the generating functional for the connected Green's functions is

$$\mathcal{G}(\phi) = \tilde{\Omega}_C(\mathcal{V})(\phi, 0)\,.$$

(ii) $\tilde{\Omega}$ obeys the semigroup property

$$\tilde{\Omega}_{C_1+C_2} = \tilde{\Omega}_{C_1} \circ \tilde{\Omega}_{C_2}\,.$$

In order to use the results of Part 1 and [1], we note the following relationship between $\tilde{\Omega}_C$ and the renormalization group map

$$\Omega_C(\mathcal{W})(\phi,\psi) = \log \frac{1}{Z}\int e^{\mathcal{W}(\phi,\psi+\zeta)} d\mu_C(\zeta) \quad \text{where } Z = \int e^{W(0,\zeta)} d\mu_C(\zeta)$$

of Part 1.

**Lemma VII.3.**

$$\begin{aligned}\tilde{\Omega}_C(\mathcal{W})(\phi,\psi) &= \frac{1}{2}\phi JCJ\phi + \Omega_C(\mathcal{W})(\phi,\psi + CJ\phi)\\ &= \frac{1}{2}\phi JCJ\phi + \Omega_C(\mathcal{W})(\phi,\psi) + \int :\Omega_C(\mathcal{W})(\phi,\psi+\zeta)\\ &\qquad - \Omega_C(\mathcal{W})(\phi,\psi):_\zeta :e^{\phi J\zeta}:_\zeta\, d\mu_C(\zeta)\end{aligned}$$

*where for any kernel* $B(\eta,\eta')$, $B\phi = \int d\eta B(\eta,\eta')\phi(\eta')$ *and* $\phi B\phi = \int d\eta d\eta' \phi(\eta)$ $B(\eta,\eta')\phi(\eta')$.

**Proof.** By Lemma C.1, with $\phi$ replaced by $J\phi$, and (VI.2),

$$\begin{aligned}\tilde{\Omega}_C(\mathcal{W})(\phi,\psi) &= \log e^{-\frac{1}{2}(J\phi)C(J\phi)} \frac{1}{Z}\int e^{\mathcal{W}(\phi,\zeta+\psi+CJ\phi)} d\mu_C(\zeta)\\ &= \log e^{\frac{1}{2}\phi JCJ\phi} e^{\Omega_C(\mathcal{W})(\phi,\psi+CJ\phi)}\\ &= \frac{1}{2}\phi JCJ\phi + \Omega_C(\mathcal{W})(\phi,\psi+CJ\phi)\,.\end{aligned}$$

Also by Lemma C.1

$$\begin{aligned}\Omega_C(\mathcal{W})(\phi,\psi+CJ\phi) &= \int :\Omega_C(\mathcal{W})(\phi,\psi+\zeta+CJ\phi):_\zeta\, d\mu_C(\zeta)\\ &= e^{\frac{1}{2}(J\phi)C(J\phi)} \int :\Omega_C(\mathcal{W})(\phi,\psi+\zeta):_\zeta\, e^{\zeta J\phi} d\mu_C(\zeta)\\ &= \int :\Omega_C(\mathcal{W})(\phi,\psi+\zeta):_\zeta :e^{\zeta J\phi}:_\zeta\, d\mu_C(\zeta)\\ &= \Omega_C(\mathcal{W})(\phi,\psi) + \int :\Omega_C(\mathcal{W})(\phi,\psi+\zeta)\\ &\qquad - \Omega_C(\mathcal{W})(\phi,\psi):_\zeta :e^{\phi J\zeta}:_\zeta\, d\mu_C(\zeta)\,. \qquad \text{(VII.1)}\end{aligned}$$

□

The aim of the next section is to estimate the (unamputated) renormalization group map, $\tilde{\Omega}$, with respect to the norms of Definition III.9. The difference between the maps $\Omega_C$ and $\tilde{\Omega}_C$ lies in the source terms as is described in Lemma VII.3. The estimates for this difference are similar to, but easier than, the estimates for the map $\Omega_C$ itself.

**Definition VII.4 (External Improving).** Let $\|\cdot\|$ be a family of symmetric seminorms on the spaces $\mathcal{F}_m(n)$. We say that the covariance $C$ is $\Gamma$-external improving with respect to this family of seminorms if, for each $m \geq 0$, $n \geq 1$, there is an $i$ with $1 \leq i \leq n$ such that

$$\left\|\mathrm{Ant}_{\mathrm{ext}} \int d\zeta d\zeta' J(\eta_{m+1}, \zeta) C(\zeta, \zeta') f(\eta_1, \ldots, \eta_m; \xi_1, \ldots, \xi_{i-1}, \zeta', \xi_i, \ldots, \xi_{n-1})\right\| \leq \Gamma \|f\|$$

for all $f \in \mathcal{F}_m(n)$. Recall that $\mathrm{Ant}_{\mathrm{ext}}$ was introduced in [2, Definition II.9]. Observe that the function on the left-hand side is in $\mathcal{F}_{m+1}(n-1)$.

**Lemma VII.5.** *Let $\|\cdot\|$ be a family of symmetric seminorms and let the covariance $C$ be $\Gamma$-external improving with respect to this family of seminorms. Let $f(\phi, \psi, \zeta)$ be of degree $p'$ in $\zeta$. The integral $\int : f(\phi, \psi, \zeta) :_{\zeta, C} : [\phi J \zeta]^p :_{\zeta, C} d\mu_C(\zeta)$ vanishes unless $p = p'$ and then*

$$\left\| \int : f(\phi, \psi, \zeta) :_{\zeta, C} : [\phi J \zeta]^p :_{\zeta, C} d\mu_C(\zeta) \right\| \leq p! \Gamma^p \|f\| .$$

**Proof.** Observe that $\phi J \psi = \int d\xi d\xi' \phi(\xi) J(\xi, \xi') \psi(\xi') \in A_1 \otimes V$. By Definition VII.4 and [2, Definition III.1],

$$\left\| \underset{i \to n+1}{\mathcal{C}\mathrm{on}_C} (\mathrm{Ant}_{\mathrm{ext}}(h \otimes \phi J \psi)) \right\| \leq \Gamma \|h\| \tag{VII.2}$$

for all $h \in A_m \otimes V^{\otimes n}$, $m \geq 0$, $n \geq 1$ and some $1 \leq i \leq n$. Observe that $h \otimes \phi J \psi \in (A_m \otimes V^{\otimes n}) \otimes (A_1 \otimes V) \cong A_m \otimes A_1 \otimes V^{\otimes n+1}$, so that $\mathrm{Ant}_{\mathrm{ext}}(h \otimes \phi J \psi) \in A_{m+1} \otimes V^{\otimes n+1}$ and $\underset{i \to n+1}{\mathcal{C}\mathrm{on}_C}(\mathrm{Ant}_{\mathrm{ext}}(h \otimes \phi J \psi)) \in A_{m+1} \otimes V^{\otimes n-1}$.

Set

$$g(\phi, \psi, \zeta, \zeta') = f(\phi, \psi, \zeta) [\phi J \zeta']^p .$$

By Lemma II.13 and [1, Remark II.12], $p$ times, starting with $f(\xi, \xi', \xi'') = : g(\phi, \psi, \xi', \xi'') :_{\xi''}$

$$\begin{aligned} \int : f(\phi, \psi, \zeta) :_{\zeta, C} : [\phi J \zeta]^p :_{\zeta, C} d\mu_C(\zeta) &= \int [: g(\phi, \psi, \zeta, \zeta') :_{\zeta, \zeta'}]_{\zeta' = \zeta} d\mu_C(\zeta) \\ &= \int \left[ : \underset{\zeta \to \zeta'}{\mathrm{Con}_C} \, g(\phi, \psi, \zeta, \zeta') :_{\zeta, \zeta'} \right]_{\zeta' = \zeta} d\mu_C(\zeta) \end{aligned}$$

$$= \int \left[ : \operatorname*{Con}_{\zeta \to \zeta'} {}_C{}^{p'} \, g(\phi, \psi, \zeta, \zeta') :_{\zeta, \zeta'} \right]_{\zeta' = \zeta} d\mu_C(\zeta)$$

$$= \operatorname*{Con}_{\zeta \to \zeta'} {}_C{}^{p'} \, g(\phi, \psi, \zeta, \zeta')$$

if $p' \geq p$, since then

$$\operatorname*{Con}_{\zeta \to \zeta'} {}_C{}^{p'} \, g(\phi, \psi, \zeta, \zeta')$$

is independent of $\zeta$ and $\zeta'$. If

$$p' > p \,, \quad \operatorname*{Con}_{\zeta \to \zeta'} {}_C{}^{p'} \, g(\phi, \psi, \zeta, \zeta') = 0 \,.$$

If

$$p' < p \,, \quad \operatorname*{Con}_{\zeta \to \zeta'} {}_C{}^{p'} \, g(\phi, \psi, \zeta, \zeta')$$

is of degree 0 in $\zeta$ and of degree $p - p' > 0$ in $\zeta'$ and the integral

$$= \int \left[ : \operatorname*{Con}_{\zeta \to \zeta'} {}_C{}^{p'} \, g(\phi, \psi, \zeta, \zeta') :_{\zeta, \zeta'} \right]_{\zeta' = \zeta} d\mu_C(\zeta)$$

again vanishes. It now suffices to apply [1, Definition II.9] and (VII.2), $p$ times. □

**Proposition VII.6.** *Let* $\gamma > 0$ *and* $\alpha \geq 1$ *obey* $\frac{\gamma}{\alpha} \leq \frac{1}{3}$. *Let*

$$\mathcal{W}'(\phi, \psi) = \mathcal{W}(\phi, \psi + CJ\phi) \,.$$

*If* $C$ *is* $\gamma\mathrm{b}$*-external improving*

$$N(\mathcal{W}' - \mathcal{W}; \mathfrak{c}, \mathrm{b}, \alpha) \leq \frac{\gamma}{\alpha} N(\mathcal{W}; \mathfrak{c}, \mathrm{b}, 2\alpha) \,.$$

**Proof.** Write $\mathcal{W}(\phi, \psi) = \sum_{m,n} \mathcal{W}_{m,n}(\phi, \psi)$ with $\mathcal{W}_{m,n} \in A_m[n]$ and

$$\mathcal{W}(\phi, \psi + \zeta) = \sum_{m,n} \mathcal{W}_{m,n}(\phi, \psi + \zeta) = \sum_{m,n} \sum_{p=0}^{n} \mathcal{W}_{m,n-p,p}(\phi, \psi, \zeta)$$

with $\mathcal{W}_{m,n-p,p} \in A_m[n-p, p]$. By [1, Lemma II.22(iii)]

$$\|\mathcal{W}_{m,n-p,p}\| \leq \binom{n}{p} \|\mathcal{W}_{m,n}\| \,.$$

Then, by (VII.1),

$$\mathcal{W}'(\phi, \psi) - \mathcal{W}(\phi, \psi) = \mathcal{W}(\phi, \psi + CJ\phi) - \mathcal{W}(\phi, \psi)$$

$$= \int : \mathcal{W}(\phi, \psi + \zeta) - \mathcal{W}(\phi, \psi) :_{\zeta, C} : e^{\phi J \zeta} :_{\zeta, C} d\mu_C(\zeta)$$

$$= \sum_{\substack{m,p\geq 0\\ n\geq 1}} \sum_{p'=1}^{n} \frac{1}{p!} \int :\mathcal{W}_{m,n-p',p'}(\phi,\psi,\zeta):_{\zeta,C} :(\phi J\zeta)^p:_{\zeta,C} d\mu_C(\zeta)$$

$$= \sum_{\substack{m\geq 0\\ n\geq 1}} \sum_{p=1}^{n} \frac{1}{p!} \int :\mathcal{W}_{m,n-p,p}(\phi,\psi,\zeta):_{\zeta,C} :(\phi J\zeta)^p:_{\zeta,C} d\mu_C(\zeta)\,.$$

By Lemma VII.5

$$\left\| \int :\mathcal{W}_{m,n-p,p}(\phi,\psi,\zeta):_{\zeta,C} :(\phi J\zeta)^p:_{\zeta,C} d\mu_C(\zeta) \right\| \leq p!(\gamma \mathrm{b})^p \|\mathcal{W}_{m,n-p,p}\|$$

$$= \leq p! \binom{n}{p} (\gamma \mathrm{b})^p \|\mathcal{W}_{m,n}\|$$

so that

$$N(\mathcal{W}'-\mathcal{W};\mathfrak{c},\mathrm{b},\alpha) \leq \frac{\mathfrak{c}}{\mathrm{b}^2} \sum_{\substack{m\geq 0\\ n\geq 1}} \sum_{p=1}^{n} \frac{1}{p!} \alpha^{n-p}\mathrm{b}^{n-p}$$

$$\times \left\| \int :\mathcal{W}_{m,n-p,p}:_{\zeta,C} :(\phi J\zeta)^p:_{\zeta,C} d\mu_C(\zeta) \right\|$$

$$\leq \frac{\mathfrak{c}}{\mathrm{b}^2} \sum_{\substack{m\geq 0\\ n\geq 1}} \sum_{p=1}^{n} \binom{n}{p} \left(\frac{\gamma}{\alpha}\right)^p \alpha^n \mathrm{b}^n \|W_{m,n}\|$$

$$= \frac{\mathfrak{c}}{\mathrm{b}^2} \sum_{\substack{m\geq 0\\ n\geq 1}} \left[\left(1+\frac{\gamma}{\alpha}\right)^n - 1\right] \alpha^n \mathrm{b}^n \|W_{m,n}\|\,.$$

Applying

$$\left(1+\frac{\gamma}{\alpha}\right)^n - 1 \leq \frac{\gamma}{\alpha} n \left(1+\frac{\gamma}{\alpha}\right)^{n-1} \leq \frac{\gamma}{\alpha}\left(\frac{3}{2}\right)^n \left(1+\frac{\gamma}{\alpha}\right)^{n-1} \leq \frac{\gamma}{\alpha} 2^n$$

we have

$$N(\mathcal{W}'-\mathcal{W};\mathfrak{c},\mathrm{b},\alpha) \leq \frac{\mathfrak{c}}{\mathrm{b}^2} \sum_{\substack{m\geq 0\\ n\geq 1}} \frac{\gamma}{\alpha} 2^n \alpha^n \mathrm{b}^n \|W_{m,n}\|$$

$$\leq \frac{\gamma}{\alpha} N(\mathcal{W};\mathfrak{c},\mathrm{b},2\alpha)\,. \qquad \square$$

**Corollary VII.7.** *Let* $\gamma,\gamma' > 0$ *and* $\alpha > 1$ *with* $\frac{\gamma}{\alpha} \leq \frac{1}{6}$. *Let*

$$\mathcal{W}'_\kappa(\phi,\psi) = \mathcal{W}(\phi,\psi + C_\kappa J\phi)\,.$$

*If $C_0$ is $\gamma$b-external improving and $\frac{d}{d\kappa}C_\kappa|_{\kappa=0}$ is $\gamma'$b-external improving*

$$N\left(\frac{d}{d\kappa}\mathcal{W}'_\kappa\bigg|_{\kappa=0};\mathfrak{c},\mathrm{b},\alpha\right)\le 2\frac{\gamma'}{\alpha}N(\mathcal{W};\mathfrak{c},\mathrm{b},2\alpha)\,.$$

**Proof.** Define $D_z = C_0 + z\frac{d}{d\kappa}C_\kappa|_{\kappa=0}$ and $\mathcal{W}''_z(\phi,\psi)=\mathcal{W}(\phi,\psi+D_zJ\phi)-\mathcal{W}(\phi,\psi)$. Then $\frac{d}{d\kappa}\mathcal{W}'_\kappa|_{\kappa=0} = \frac{d}{dz}\mathcal{W}''_z|_{z=0}$. Furthermore, applying the triangle inequality directly to the Definition VII.4 of "external improving", we see that $D_z$ is $(\gamma+|z|\gamma')$b-external improving. As $\frac{\gamma+|z|\gamma'}{\alpha}\le\frac{1}{3}$ for all $|z|\le\frac{\alpha}{6\gamma'}$, Proposition VII.6 implies that

$$N(\mathcal{W}''_z;\mathfrak{c},\mathrm{b},\alpha)\le\frac{1}{3}N(\mathcal{W};\mathfrak{c},\mathrm{b},2\alpha)$$

for all $|z|\le\frac{\alpha}{6\gamma'}$. The corollary now follows by the Cauchy integral theorem to express $N(\frac{d}{d\kappa}\mathcal{W}'_\kappa|_{\kappa=0};\mathfrak{c},\mathrm{b},\alpha)$ as an integral of $N(\mathcal{W}''_z;\mathfrak{c},\mathrm{b},\alpha)$ over the circle of radius $\frac{\alpha}{6\gamma'}$ centered on the origin. □

Similarly to Lemma V.1, we have:

**Lemma VII.8.** *Let $\rho_{m;n}$ be a sequence of nonnegative real numbers such that $\rho_{m;n'}\le\rho_{m;n}$ for $n'\le n$. Define for $f\in\mathcal{F}_m(n)$*

$$\|f\|=\rho_{m;n}\|f\|_{1,\infty}$$

*where $\|f\|_{1,\infty}$ is the $L_1$–$L_\infty$-norm introduced in Example* II.6. *Let $C$ be a covariance and $\Gamma$ obey*

$$\Gamma\ge\frac{\rho_{1;n-1}}{\rho_{0;n}}|||C|||_{1,\infty}\qquad\text{for all } n\ge1$$

$$\Gamma\ge\frac{\rho_{m+1;n-1}}{\rho_{m;n}}|||C|||_{\infty}\qquad\text{for all } m,n\ge1\,.$$

*Then $C$ is $\Gamma$-external improving with respect to the family of seminorms $\|\cdot\|$, in the sense of Definition* VII.4.

**Proof.** Let $f\in\mathcal{F}_m(n)$ and set

$$g(\eta_1,\ldots,\eta_{m+1};\xi_1,\ldots,\xi_{n-1})$$
$$=\mathrm{Ant}_{\mathrm{ext}}\int d\zeta d\zeta' J(\eta_{m+1},\zeta)C(\zeta,\zeta')f(\eta_1,\ldots,\eta_m;\xi_1\ldots,\xi_{i-1},\zeta',\xi_i,\ldots,\xi_{n-1})\,.$$

If $m=0$

$$|||g|||_{1,\infty}\le|||f|||_{1,\infty}|||JC|||_{1,\infty}=|||f|||_{1,\infty}|||C|||_{1,\infty}$$

by Lemma II.7. If $m\ne0$,

$$|||g|||_{1,\infty}\le|||f|||_{1,\infty}|||JC|||_{\infty}=|||f|||_{1,\infty}|||C|||_{\infty}\,.$$

Since $g\in\mathcal{F}_{m+1}$, $\|g\|_{1,\infty}=|||g|||_{1,\infty}$. The lemma now follows from the hypothesis on $\Gamma$. □

## VIII. Scales

From now on we discuss the situation that the dispersion relation $e(\mathbf{k})$ has zeroes on the support of the ultraviolet cutoff $U(\mathbf{k})$; in other words, that the Fermi surface $F$ is not empty. Then a single scale analysis as for insulators is not possible because there is an infrared problem due to the singularity of the propagator $\frac{U(\mathbf{k})}{\imath k_0 - e(\mathbf{k})}$ on the set

$$\{(k_0, \mathbf{k}) \in \mathbb{R} \times \mathbb{R}^d | k_0 = 0,\ e(\mathbf{k}) = 0\}$$

which can be canonically identified with the Fermi surface. This singularity causes the $L_1$–$L_\infty$ norm (in position space) of the propagator to be infinite.

To analyze the singularity at the Fermi surface, we introduce scales by slicing momentum space into shells around the Fermi surface. We choose a "scale parameter" $M > 1$ and a function $\nu \in C_0^\infty([\frac{1}{M}, 2M])$ that takes values in $[0, 1]$, is identically one on $[\frac{2}{M}, M]$ and obeys

$$\sum_{j=0}^{\infty} \nu(M^{2j} x) = 1$$

for $0 < x < 1$.

The scale parameter $M$ is chosen sufficiently big (depending on the dispersion relation $e(\mathbf{k})$ and the ultraviolet cutoff $U(\mathbf{k})$). The function $\nu$ may be constructed by choosing a function $\varphi \in C_0^\infty((-2, 2))$ that is identically one on $[-1, 1]$ and setting $\nu(x) = \varphi(x/M) - \varphi(Mx)$ for $x > 0$ and zero otherwise. Then $\nu(x)$ vanishes for $x \geq 2M$ and $x \leq \frac{1}{M}$ and is identically one for $\frac{2}{M} \leq x \leq M$ and $\sum_{j=0}^{\infty} \nu(M^{2j} x) = \varphi(x/M)$ for $x > 0$.

**Definition VIII.1.** (i) For $j \geq 1$, the $j$th scale function on $\mathbb{R} \times \mathbb{R}^d$ is defined as

$$\nu^{(j)}(k) = \nu(M^{2j}(k_0^2 + e(\mathbf{k})^2))\,.$$

By construction, $\nu^{(j)}$ is identically one on

$$\left\{ k = (k_0, \mathbf{k}) \in \mathbb{R} \times \mathbb{R}^d \middle| \sqrt{\frac{2}{M}} \frac{1}{M^j} \leq |ik_0 - e(\mathbf{k})| \leq \sqrt{M} \frac{1}{M^j} \right\}.$$

The support of $\nu^{(j)}$ is called the $j$th shell. By construction, it is contained in

$$\left\{ k \in \mathbb{R} \times \mathbb{R}^d \middle| \frac{1}{\sqrt{M}} \frac{1}{M^j} \leq |ik_0 - e(\mathbf{k})| \leq \sqrt{2M} \frac{1}{M^j} \right\}.$$

The momentum $k$ is said to be of scale $j$ if $k$ lies in the $j$th shell.

(ii) For real $j \geq 1$, set

$$\nu^{(\geq j)}(k) = \varphi(M^{2j-1}(k_0^2 + e(\mathbf{k})^2))$$

with the function $\varphi$ introduced just before this definition. By construction, $\nu^{(\geq j)}$ is identically one on

$$\left\{ k \in \mathbb{R} \times \mathbb{R}^d \middle| |ik_0 - e(\mathbf{k})| \leq \sqrt{M} \frac{1}{M^j} \right\}.$$

Observe that if $j$ is an integer, then for $|ik_0 - e(\mathbf{k})| > 0$

$$\nu^{(\geq j)}(k) = \sum_{i \geq j} \nu^{(i)}(k)\,.$$

The support of $\nu^{(\geq j)}$ is called the $j$th neighborhood of the Fermi surface. By construction, it is contained in

$$\left\{ k \in \mathbb{R} \times \mathbb{R}^d \,\middle|\, |ik_0 - e(\mathbf{k})| \leq \sqrt{2M}\frac{1}{M^j} \right\}.$$

**Remark VIII.2.** Since the scale parameter $M > 1$, the shells near the Fermi curve have $j$ near $+\infty$, and the neighborhoods shrink as $j \to \infty$.

**Conventions VIII.3.** (i) We choose $M$ so big that $\nu^{(\geq 1)}(k) \leq U(\mathbf{k})$ for all $k = (k_0, \mathbf{k}) \in \mathbb{R} \times \mathbb{R}^d$.

(ii) We also use the notations

$$\nu^{(\leq j)}(k) = \sum_{i=0}^{j} \nu^{(i)}(k)$$

$$\nu^{(<j)}(k) = \nu^{(\leq j-1)}(k)\,, \quad \nu^{(>j)}(k) = \nu^{(\geq j+1)}(k)\,.$$

(iii) Generic constants that depend only on the dispersion relation $e(\mathbf{k})$ and the ultraviolet cutoff $U(\mathbf{k})$ will be denoted by "$_{const}$". Generic constants that may also depend on the scale parameter $M$, but still not on the scale $j$, will be denoted "const".

For technical discussions we also need a set of functions that "envelope" the various shells. We set $\tilde{\nu} = \varphi(x/M^2) - \varphi(M^2 x)$ for $x > 0$ and zero otherwise. It is in $C_0^\infty((\frac{1}{M^2}, 2M^2))$, takes values in $[0, 1]$ and is identically one on $[\frac{2}{M^2}, M^2]$ and hence on the support of $\nu$, assuming that $M \geq 2$.

**Definition VIII.4.** (i) For $j \geq 1$, the $j$th extended scale function on $\mathbb{R} \times \mathbb{R}^d$ is defined as

$$\tilde{\nu}^{(j)}(k) = \tilde{\nu}(M^{2j}(k_0^2 + e(\mathbf{k})^2))\,.$$

The support of $\tilde{\nu}^{(j)}$ is called the $j$th extended shell. It is (for $j \geq 2$) contained in the union of the $(j-1)$st, $j$th and $(j+1)$st shells. In fact, if $M \geq 2$, $\tilde{\nu}^{(j)}$ is identically one on the $j$th shell and, if $j, M \geq 2$, $\nu^{(j-1)} + \nu^{(j)} + \nu^{(j+1)}$ is identically one on the $j$th extended shell.

(ii) By definition, the $j$th extended neighborhood is the union of the $i$th extended shells with $i \geq j$. It is (for $j \geq 2$) contained in the $(j-1)$st neighborhood of the Fermi surface. The function

$$\tilde{\nu}^{(\geq j)}(k) = \varphi(M^{2j-2}(k_0^2 + e(\mathbf{k})^2))$$

is supported on the $j$th extended neighborhood and identically one on the $j$th neighborhood.

(iii) Set $\bar{\nu}^{(\geq j)}(k) = \varphi(M^{2j-3}(k_0^2+e(\mathbf{k})^2)) = \nu^{(\geq j-1)}$. Then $\bar{\nu}^{(\geq j)}(k)$ is identically one on the $j$th extended neighborhood. The support of $\bar{\nu}^{(\geq j)}$ is called the $j$th doubly extended neighborhood and is contained in $\{k \in \mathbb{R}\times\mathbb{R}^d \,|\, |ik_0 - e(\mathbf{k})| \leq \sqrt{2}M^{3/2}\frac{1}{M^j}\}$.

Observe that the ultraviolet cutoff $U(\mathbf{k})$ does not depend on $k_0$, so that the propagator $\frac{U(\mathbf{k})}{\imath k_0 - e(\mathbf{k})}$ is not compactly supported. However, we can use the results of Part 1 to integrate out the ultraviolet part of the model in the $k_0$-direction and to pass to a model whose propagator is supported in the second neighborhood. We choose to pass to a model supported in the second, rather than first, neighborhood because the second doubly extended neighborhood can be made arbitrarily small by choosing $M$ sufficiently large.

In the renormalization group analysis we shall add a counterterm $\delta e(\mathbf{k})$ to the dispersion relation $e(\mathbf{k})$.

**Definition VIII.5.** Let $\mu > 0$. The space of counterterms, $\mathcal{E}_\mu$, consists of all functions $\delta e(\mathbf{k})$ on $\mathbb{R}^d$ that are supported in $\{\mathbf{k} \in \mathbb{R}^d | U(\mathbf{k}) \neq 0\}$ and obey

$$\|\delta\hat{e}\|_{1,\infty} < \mu + \sum_{\delta \neq \mathbf{0}} \infty t^\delta$$

where $\delta\hat{e}$ was defined just before Definition IV.10 and the norm $\|\cdot\|_{1,\infty}$ was defined in Example II.6.

Recall, from Definition IV.10 that $\mathfrak{c}_0 = \sum_{\substack{|\boldsymbol{\delta}|\leq r \\ |\delta_0|\leq r_0}} t^\delta + \sum_{\substack{|\boldsymbol{\delta}|>r \\ \text{or } |\delta_0|>r_0}} \infty t^\delta \in \mathfrak{N}_{d+1}$ and $\mathfrak{e}_0(X) = \frac{\mathfrak{c}_0}{1-X}$, for $X \in \mathfrak{N}_{d+1}$ with $X_\mathbf{0} < 1$.

**Theorem VIII.6.** *Fix $j_0 \geq 1$ and set, for $\delta e \in \mathcal{E}_\mu$,*

$$C_0(k;\delta e) = \frac{U(\mathbf{k}) - \nu^{(>j_0)}(k)}{\imath k_0 - e(\mathbf{k}) + \delta e(\mathbf{k})}\,.$$

*Define the covariance $C_0(\delta e)$ by*

$$C_0(\xi,\xi';\delta e) = \begin{cases} \delta_{\sigma,\sigma'} \displaystyle\int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{\imath\langle k, x-x'\rangle_-} C_0(k;\delta e) & \text{if } a=0,\ a'=1 \\ 0 & \text{if } a=a' \\ -C_0(\xi',\xi;\delta e) & \text{if } a=1,\ a'=0 \end{cases}$$

*for $\xi = (x,a) = (x_0,\mathbf{x},\sigma,a)$, $\xi' = (x',a') = (x_0',\mathbf{x}',\sigma',a')$. Then there are ($M$ and $j_0$-dependent) constants* b, $\beta_0$, $\varepsilon_0$, const *and $\mu > 0$ such that, for all $\beta \geq \beta_0$ and $\varepsilon \leq \varepsilon_0$, the following holds:*

> *Choose a system $\boldsymbol{\rho} = (\rho_{m;n})_{m,n\in\mathbb{N}_0}$, of positive real numbers obeying $\rho_{m;n-1} \leq \rho_{m;n}$, $\rho_{m+1;n-1} \leq \rho_{m;n}$ and $\rho_{m+m';n+n'-2} \leq \rho_{m;n}\rho_{m';n'}$. For*

*an even Grassmann function*

$$\mathcal{W}(\phi,\psi) = \sum_{\substack{m,n\ge 0\\ m+n \text{ even}}} \int_{\mathcal{B}^{m+n}} d\eta_1\cdots d\eta_m d\xi_1\cdots d\xi_n$$

$$\times W_{m,n}(\eta_1\cdots\eta_m,\xi_1,\ldots,\xi_n)\phi(\eta_1)\cdots\phi(\eta_m)\psi(\xi_1)\cdots\psi(\xi_n)$$

*with kernels $W_{m,n}$ that are separately antisymmetric under permutations of their $\eta$ and $\xi$ arguments and $X \in \mathfrak{N}_{d+1}$ with $X_{\mathbf{0}} < 1$, set*

$$N_0(\mathcal{W};\beta;X,\boldsymbol{\rho}) = \mathfrak{e}_0(X) \sum_{\substack{m+n\ge 2\\ m+n \text{ even}}} \beta^n \rho_{m;n}\|W_{m,n}\|_{1,\infty}\,.$$

*Let $X \in \mathfrak{N}_{d+1}$ with $X_{\mathbf{0}} < \frac{1}{4}$. The formal Taylor series $\tilde\Omega_{C_0(\delta e)}(\mathcal{V})$ converges to an analytic map on $\{(\mathcal{V}(\psi),\delta e)|\mathcal{V}$ even, $N_0(\mathcal{V};32\beta;X,\boldsymbol{\rho})_{\mathbf{0}} \le \varepsilon\mathfrak{e}_0(X)_{\mathbf{0}}$, $\delta e\in\mathcal{E}_\mu$, $|||\delta\hat e|||_{1,\infty} \le X_{\mathbf{0}}\}$. Furthermore, for all $\delta e \in \mathcal{E}_\mu$ with $\|\delta\hat e\|_{1,\infty} \le X$ and all even Grassmann functions $\mathcal{V}(\psi)$ with $N_0(\mathcal{V};32\beta;X,\boldsymbol{\rho}) \le \varepsilon\mathfrak{e}_0(X)$, one has*

$$N_0\left(\tilde\Omega_{C_0(\delta e)}(\mathcal{V})(\phi,\psi) - \mathcal{V}(\psi) - \frac{1}{2}\phi J C_0(\delta e) J\phi;\beta;X,\boldsymbol{\rho}\right) \le \frac{\varepsilon \mathrm{b}}{\beta}\mathfrak{e}_0(X)$$

*and*

$$N_0\left(\frac{d}{ds}\left[\tilde\Omega_{C_0(\delta e+s\delta e')}(\mathcal{V})(\phi,\psi) - \frac{1}{2}\phi J C_0(\delta e+s\delta e')J\phi\right]_{s=0};\beta;X,\boldsymbol{\rho}\right)$$

$$\le \frac{\varepsilon \mathrm{b}}{\beta}\mathfrak{e}_0(X)\|\delta\hat e'\|_{1,\infty}\,.$$

**Proof.** By Proposition IV.5 (with $\chi = \nu^{(>j_0)}$ and $e$ replaced by $e-\delta e$) and Proposition IV.11, there is a constant $\mathrm{const}_1$ such that $S(C_0(\delta e)) \le \mathrm{const}_1$, $|||C_0(\delta e)|||_\infty \le \mathrm{const}_1$ and

$$\|C_0(\delta e)\|_{1,\infty} \le \frac{1}{4}\,\mathrm{const}_1\,\mathfrak{e}_0(\|\delta\hat e\|_{1,\infty})\,.$$

Set $\|\cdot\| = \rho_{m;n}\|\cdot\|_{1,\infty}$ for functions on $\mathcal{B}^m\times\mathcal{B}^n$. Set $\mathrm{b} = 4\,\mathrm{const}_1$ and $\mathfrak{c} = \mathrm{const}_1\,\mathfrak{e}_0(X)$. By Lemma V.1, with respect to this family of seminorms, b is an integral bound for $C_0(\delta e)$ and $\mathfrak{c}$ is a contraction bound for $C_0(\delta e)$. Furthermore, by Lemma VII.8, $C_0(\delta e)$ is $\mathrm{const}_1$-external improving.

For any Grassmann function $\mathcal{W}(\phi,\psi)$ let $N(\mathcal{W};\mathfrak{c},\mathrm{b},\alpha)$ be the norm of Definition III.9 with respect to the family of seminorms $\|\cdot\|$. Set $\alpha = \frac{\beta}{\mathrm{b}}$. Then, as $\mathfrak{c} = \frac{\mathrm{b}}{4}\mathfrak{e}_0(X)$, $N(\mathcal{W};\mathfrak{c},\mathrm{b},\alpha) = \frac{1}{4\mathrm{b}}N_0(\mathcal{W};\beta;X,\boldsymbol{\rho})$ and, if $\mathcal{V} = :\mathcal{V}':_{C_0(\delta e)}$,

$$N(\mathcal{V}';\mathfrak{c},\mathrm{b},\alpha) \le \frac{1}{4\mathrm{b}}N_0(\mathcal{V};2\beta;X,\boldsymbol{\rho})$$

$$N(\mathcal{V}'-\mathcal{V};\mathfrak{c},\mathrm{b},\alpha) \le \frac{\mathrm{b}}{2\beta^2}N_0(\mathcal{V};2\beta;X,\boldsymbol{\rho})$$

by [1, Corollary II.32].

To prove the first part of the Theorem, set $\mathcal{V} = :\mathcal{V}':_{C_0(\delta e)}$. Then the hypotheses of Theorem III.10 with $C = C_0(\delta e)$ and $\mathcal{W} = \mathcal{V}'$ are fulfilled. Therefore,

$$\begin{aligned}
&\frac{1}{4\mathrm{b}} N_0(\Omega_{C_0(\delta e)}(\mathcal{V}) - \mathcal{V}; \beta; X, \boldsymbol{\rho}) \\
&\quad \le N(\Omega_{C_0(\delta e)}(:\mathcal{V}':_{C_0(\delta e)}) - \mathcal{V}'; \mathfrak{c}, \mathrm{b}, \alpha) + N(\mathcal{V}' - \mathcal{V}; \mathfrak{c}, \mathrm{b}, \alpha) \\
&\quad \le \frac{2}{\alpha^2} \frac{N(\mathcal{V}'; \mathfrak{c}, \mathrm{b}, 8\alpha)^2}{1 - \frac{4}{\alpha^2} N(\mathcal{V}'; \mathfrak{c}, \mathrm{b}, 8\alpha)} + N(\mathcal{V}' - \mathcal{V}; \mathfrak{c}, \mathrm{b}, \alpha) \\
&\quad \le \frac{2}{\alpha^2} \frac{\frac{1}{16\mathrm{b}^2} N_0(\mathcal{V}; 16\beta; X, \boldsymbol{\rho})^2}{1 - \frac{4}{\alpha^2} \frac{1}{4\mathrm{b}} N_0(\mathcal{V}; 16\beta; X, \boldsymbol{\rho})} + \frac{\mathrm{b}}{2\beta^2} N_0(\mathcal{V}; 2\beta; X, \boldsymbol{\rho}) \\
&\quad \le \frac{\varepsilon^2}{8\beta^2} \frac{\mathfrak{e}_0(X)^2}{1 - \frac{\varepsilon \mathrm{b}}{\beta^2} \mathfrak{e}_0(X)} + \frac{\varepsilon \mathrm{b}}{2\beta^2} \mathfrak{e}_0(X) \\
&\quad \le \frac{\varepsilon \mathrm{b}}{\beta^2} \mathfrak{e}_0(X)
\end{aligned}$$

since $\varepsilon_0$ is chosen so that $\frac{\mathrm{b}}{\beta^2}\varepsilon < \frac{1}{4}$ and, by Corollary A.5(ii), $\frac{\mathfrak{e}_0^2(X)}{1 - \frac{1}{4}\mathfrak{e}_0(X)} \le \mathit{const}\, \mathfrak{e}_0(X)$. By Lemma VII.3 and Proposition VII.6,

$$\begin{aligned}
&N_0(\tilde{\Omega}_{C_0(\delta e)}(\mathcal{V}) - \frac{1}{2}\phi J C_0(\delta e) J \phi - \mathcal{V}; \beta; X, \boldsymbol{\rho}) \\
&\quad = N_0(\Omega_{C_0(\delta e)}(\mathcal{V})(\phi, \psi + C_0(\delta e) J \phi) - \mathcal{V}(\psi); \beta; X, \boldsymbol{\rho}) \\
&\quad \le N_0(\Omega_{C_0(\delta e)}(\mathcal{V})(\phi, \psi + C_0(\delta e) J \phi) - \Omega_{C_0(\delta e)}(\mathcal{V})(\phi, \psi); \beta; X, \boldsymbol{\rho}) \\
&\qquad + N_0(\Omega_{C_0(\delta e)}(\mathcal{V})(\phi, \psi) - \mathcal{V}(\psi); \beta; X, \boldsymbol{\rho}) \\
&\quad \le \frac{1}{4\alpha} N_0(\Omega_{C_0(\delta e)}(\mathcal{V}); 2\beta; X, \boldsymbol{\rho}) + \frac{4\varepsilon \mathrm{b}^2}{\beta^2} \mathfrak{e}_0(X) \\
&\quad \le \frac{1}{4\alpha} [N_0(\Omega_{C_0(\delta e)}(\mathcal{V}) - \mathcal{V}; 2\beta; X, \boldsymbol{\rho}) + N_0(\mathcal{V}; 2\beta; X, \boldsymbol{\rho})] + \frac{4\varepsilon \mathrm{b}^2}{\beta^2} \mathfrak{e}_0(X) \\
&\quad \le \frac{\mathrm{b}}{4\beta} \left[ \frac{\varepsilon \mathrm{b}^2}{\beta^2} \mathfrak{e}_0(X) + \varepsilon \mathfrak{e}_0(X) \right] + \frac{4\varepsilon \mathrm{b}^2}{\beta^2} \mathfrak{e}_0(X) \\
&\quad \le \frac{\varepsilon \mathrm{b}}{\beta} \mathfrak{e}_0(X) \,.
\end{aligned}$$

The joint analyticity in $\mathcal{V}$ and $\delta e$ follows from Proposition IV.11 and [1, Remark III.11].

Finally, we prove the bound on $\frac{d}{ds}[\tilde{\Omega}_{C_0(\delta e+s\delta e')}(\mathcal{V})(\phi,\psi) - \frac{1}{2}\phi J C_0(\delta e + s\delta e') J\phi]_{s=0}$. As

$$\frac{d}{ds}C_0(k;\delta e + s\delta e')\bigg|_{s=0} = -\frac{U(\mathbf{k}) - \nu^{(>j_0)}(k)}{[\imath k_0 - e(\mathbf{k}) + \delta e(\mathbf{k})]^2}\delta e'(\mathbf{k})\,.$$

Propositions IV.3(i), IV.8(i) and IV.11(ii) give that

$$S\left(\frac{d}{ds}C_0(\delta e + s\delta e')\bigg|_{s=0}\right) \le \mathrm{const}_1\sqrt{|||\delta\hat{e}'|||_{1,\infty}}$$

$$\left|\left|\left|\frac{d}{ds}C_0(\delta e + s\delta e')\bigg|_{s=0}\right|\right|\right|_\infty \le \mathrm{const}_1|||\delta\hat{e}'|||_{1,\infty}$$

$$\left\|\frac{d}{ds}C_0(\delta e + s\delta e')\bigg|_{s=0}\right\|_{1,\infty} \le \frac{1}{2}\,\mathrm{const}_1\,\mathfrak{e}_0(\|\delta\hat{e}\|_{1,\infty})\|\delta\hat{e}'\|_{1,\infty}\,.$$

Set $\mathrm{b}' = 4\,\mathrm{const}_1\sqrt{|||\delta\hat{e}'|||_{1,\infty}}$ and $\mathfrak{c}' = \mathrm{const}_1\,\mathfrak{e}_0(X)\|\delta\hat{e}'\|_{1,\infty}$. By Lemma V.1, $\frac{1}{2}\mathrm{b}'$ is an integral bound for $\frac{d}{ds}C_0(\delta e+s\delta e')|_{s=0}$ and $\mathfrak{c}'$ is a contraction bound for $\frac{d}{ds}C_0(\delta e+ s\delta e')|_{s=0}$. Furthermore, by Lemma VII.8, $\frac{d}{ds}C_0(\delta e + s\delta e')|_{s=0}$ is $\mathrm{const}_1|||\delta\hat{e}'|||_{1,\infty}$-external improving.

Define $C_\kappa = C_0(\delta e + \kappa\delta e')$ and $\mathcal{W}_\kappa$ by $:\mathcal{W}_\kappa:_{C_\kappa} = \mathcal{V}$. Even when $\alpha$ is replaced by $2\alpha$, the hypotheses of [1, Lemmas IV.5(i) and IV.7(i)], with $\mu = 1$, are satisfied. By these two lemmas, followed by [1, Corollary II.32(iii)],

$$\begin{aligned}
&N\left(\frac{d}{ds}\Omega_{C_s}(\mathcal{V})\bigg|_{s=0};\mathfrak{c},\mathrm{b},\alpha\right)\\
&\quad= N\left(\frac{d}{ds}\Omega_{C_s}(:\mathcal{W}_s:_{C_s})\bigg|_{s=0};\mathfrak{c},\mathrm{b},\alpha\right)\\
&\quad\le N\left(\frac{d}{ds}\Omega_{C_s}(:\mathcal{V}':_{C_s})\bigg|_{s=0};\mathfrak{c},\mathrm{b},\alpha\right) + N\left(\frac{d}{ds}\Omega_{C_0}(:\mathcal{W}_s:_{C_0})\bigg|_{s=0};\mathfrak{c},\mathrm{b},\alpha\right)\\
&\quad\le \frac{1}{2\alpha^2}\frac{N(\mathcal{V}';\mathfrak{c},\mathrm{b},8\alpha)}{1-\frac{4}{\alpha^2}N(\mathcal{V}';\mathfrak{c},\mathrm{b},8\alpha)}\,\mathrm{const}_1\,\mathfrak{e}_0(X)\|\delta\hat{e}'\|_{1,\infty}\\
&\qquad+\left\{1+\frac{2}{\alpha^2}\frac{N(\mathcal{V}';\mathfrak{c},\mathrm{b},8\alpha)}{1-\frac{4}{\alpha^2}N(\mathcal{V}';\mathfrak{c},\mathrm{b},8\alpha)}\right\}N\left(\frac{d}{ds}\mathcal{W}_s\bigg|_{s=0};\mathfrak{c},\mathrm{b},2\alpha\right)\\
&\quad\le \frac{1}{2\alpha^2}\frac{N(\mathcal{V}';\mathfrak{c},\mathrm{b},8\alpha)}{1-\frac{4}{\alpha^2}N(\mathcal{V}';\mathfrak{c},\mathrm{b},8\alpha)}\,\mathrm{const}_1\,\mathfrak{e}_0(X)\|\delta\hat{e}'\|_{1,\infty}\\
&\qquad+\left\{1+\frac{2}{\alpha^2}\frac{N(\mathcal{V}';\mathfrak{c},\mathrm{b},8\alpha)}{1-\frac{4}{\alpha^2}N(\mathcal{V}';\mathfrak{c},\mathrm{b},8\alpha)}\right\}\frac{\|\delta\hat{e}'\|_{1,\infty}}{(2\alpha-1)^2}N(\mathcal{V};\mathfrak{c},\mathrm{b},4\alpha)\\
&\quad\le \mathrm{const}\,\frac{\varepsilon}{\alpha^2}\mathfrak{e}_0(X)\|\delta\hat{e}'\|_{1,\infty}
\end{aligned}$$

as above. By Lemma VII.3, Proposition VII.6 and Corollary VII.7

$$\begin{aligned}
&N_0\left(\frac{d}{ds}\left[\tilde\Omega_{C_0(\delta e+s\delta e')}(\mathcal{V})(\phi,\psi)-\frac{1}{2}\phi JC_0(\delta e+s\delta e')J\phi\right]_{s=0};\beta;X,\boldsymbol{\rho}\right)\\
&\quad=N_0\left(\frac{d}{ds}\Omega_{C_0(\delta e+s\delta e')}(\mathcal{V})(\phi,\psi+C_0(\delta e+s\delta e')J\phi)\Big|_{s=0};\beta;X,\boldsymbol{\rho}\right)\\
&\quad\le N_0\left(\frac{d}{ds}\Omega_{C_0(\delta e+s\delta e')}(\mathcal{V})(\phi,\psi+C_0(\delta e)J\phi)\Big|_{s=0};\beta;X,\boldsymbol{\rho}\right)\\
&\qquad+N_0\left(\frac{d}{ds}\Omega_{C_0(\delta e)}(\mathcal{V})(\phi,\psi+C_0(\delta e+s\delta e')J\phi)\Big|_{s=0};\beta;X,\boldsymbol{\rho}\right)\\
&\quad\le\frac{1}{4\alpha}N_0\left(\frac{d}{ds}\Omega_{C_0(\delta e+s\delta e')}(\mathcal{V});2\beta;X,\boldsymbol{\rho}\right)\\
&\qquad+\frac{1}{2\alpha}N_0(\Omega_{C_0(\delta e)}(\mathcal{V});2\beta;X,\boldsymbol{\rho})|||\delta\hat e'|||_{1,\infty}\\
&\quad\le\text{const}\,\frac{\varepsilon}{\alpha^3}\mathfrak{e}_0(X)\|\delta\hat e'\|_{1,\infty}+\frac{\varepsilon}{2\alpha}\mathfrak{e}_0(X)\|\delta\hat e'\|_{1,\infty}\\
&\quad\le\frac{\varepsilon\mathrm{b}}{\beta}\mathfrak{e}_0(X)\|\delta\hat e'\|_{1,\infty}
\end{aligned}$$

as above. □

**Remark VIII.7.** (i) Let

$$\mathcal{V}(\psi)=\sum_{n\ \text{even}}\int_{\mathcal{B}^n}d\xi_1\cdots d\xi_n V_n(\xi_1,\ldots,\xi_n)\psi(\xi_1)\cdots\psi(\xi_n)$$

as in Theorem VIII.6. Since $\mathfrak{c}_0^2\ge{\scriptstyle const}\,\mathfrak{c}_0$, if

$$\sum(32\beta)^n\rho_{0;n}\|V_n\|_{1,\infty}\le\frac{\varepsilon}{const}\mathfrak{c}_0$$

then the hypothesis $N_0(\mathcal{V};32\beta;X,\boldsymbol{\rho})\le\varepsilon\mathfrak{e}_0(X)$ is satisfied.

(ii) Observe that $\Omega_{C_0}(\mathcal{V})$ does not depend on $\phi$, while $\tilde\Omega_{C_0}(\mathcal{V})$ does.

(iii) In the applications we have in mind, there is a small constant $\lambda>0$ (the coupling constant) and a small number $\upsilon>0$ such that

$$\rho_{m;n}=\begin{cases}\dfrac{1}{\lambda^{(1-\upsilon)(m+n-2)/2}} & \text{if } m+n\ge 4\\[2ex] \dfrac{1}{\lambda^{(1-\upsilon)}} & \text{if } m+n=2\,.\end{cases}$$

Then the hypotheses on $\rho_{m;n}$ in the theorem are fulfilled.

**Remark VIII.8.** The norms of Theorem VIII.6 are too coarse for a multi scale analysis of many fermion systems. See [3, Sec. II, Subsec. 7]. In the notation of this paper, this may be seen as follows. For simplicity, set $r = r_0 = 0$. Let

$$C^{(j)}(k) = \frac{\nu^{(j)}(k)}{\imath k_0 - e(\mathbf{k})}\,.$$

The condition

$$\rho_{0;n+n'-2} \le \rho_{0;n}\rho_{0;n'}$$

with $n = n' = 2$ implies that $\rho_{0;2} \ge 1$ and hence $\rho_{0;n} \ge 1$ for all $n \ge 2$. From Proposition IV.3(i) one deduces that $\frac{\mathrm{const}}{M^{j/2}}$ is an integral bound for $C^{(j)}$. A direct application of Proposition IV.8(i) and Lemma V.1 gives the poor estimate $M^{dj}t^{\mathbf{0}} + \sum_{|\delta|>0} \infty t^{\delta}$ for a contraction bound for $C^{(j)}$. A more careful argument in which one

- decomposes $C^{(j)} = \sum_{s\in\Sigma} C_s^{(j)}$ into $M^{(d-1)j/2}$ terms each having the projection of $\mathbf{k}$ onto the Fermi surface restricted to a roughly rectangular region of side $M^{-j/2}$ (see [4, Definition XII.1] for the precise construction)
- applies Proposition IV.8(i) to obtain $|||C_s^{(j)}|||_{1,\infty} \le \mathrm{const}\, M^j$ for all $s \in \Sigma$

yields $\mathrm{const}\, M^{\frac{d+1}{2}j} t^{\mathbf{0}} + \sum_{|\delta|>0} \infty t^{\delta}$ as a realistic contraction bound. Thus, for an even Grassmann function

$$\mathcal{W}(\psi) = \sum_{n=0}^{\infty} \int d\xi_1 \cdots d\xi_{2n} W_{2n}(\xi_1, \ldots, \xi_{2n}) \psi(\xi_1) \cdots \psi(\xi_{2n})$$

the norm $N(\mathcal{W}; \mathfrak{c}, \mathrm{b}, \alpha)$ of Definition III.9 has

$$N(\mathcal{W}; \mathfrak{c}, \mathrm{b}, \alpha)_{\mathbf{0}} = \mathrm{const}\, M^{\frac{d+3}{2}j} \Bigg\{ \frac{\alpha^2}{M^j} \rho_{0;2} |||W_2|||_{1,\infty} + \frac{\alpha^4}{M^{2j}} \rho_{0;4} |||W_4|||_{1,\infty}$$

$$+ \sum_{n\ge 3} \left( \mathrm{const}\, \frac{\alpha^2}{M^j} \right)^n \rho_{0;2n} |||W_{2n}|||_{1,\infty} \Bigg\}\,.$$

In particular, for $N(\mathcal{W}; \mathfrak{c}, \mathrm{b}, \alpha)_{\mathbf{0}}$ to be order one, it is necessary that $|||W_4|||_{1,\infty}$ is of order $\frac{1}{M^{\frac{d-1}{2}j}}$. For $d \ge 2$, this is not even the case for the original interaction $\mathcal{V}$, with all momenta restricted to the $j$th shell.

## IX. The Fourier Transform

Theorem VIII.6 and its higher scale analog, Theorem XV.3, shall be used in a renormalization group flow to obtain estimates on the suprema of connected Green's functions in position space. We wish to also obtain estimates on the suprema of certain connected amputated Green's functions in momentum space. In Sec. X, we introduce norms tailored to that purpose and prove an analog of Theorem VIII.6. In this section, we set up notation and concepts for the passage between position space and momentum space that will be needed to do that.

In Remark VIII.8, we pointed out that estimates based on integral and contraction bounds of the propagator $\frac{\nu^{(\geq j)}}{\imath k_0 - e(\mathbf{k})}$ are worse than those one expects by naive power counting in momentum space. The reason is that conservation of momentum is not exploited effectively. In the case $d = 2$, this difficulty can be completely overcome by introducing sectors, see [5], [6, Sec. X], [3, Sec. II, Subsec. 8] and Sec. XII. They localize momenta into small pieces of the shells introduced in Definition VIII.1. The discussion of this section is also useful for that purpose.

To systematically deal with Fourier transforms, we call

$$\check{\mathcal{B}} = \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\} \times \{0, 1\}$$

"momentum space". For $\check{\xi} = (k, \sigma', a') = (k_0, \mathbf{k}, \sigma', a') \in \check{\mathcal{B}}$ and $\xi = (x, a) = (x_0, \mathbf{x}, \sigma, a) \in \mathcal{B}$ we define the inner product

$$\langle \check{\xi}, \xi \rangle = \delta_{\sigma',\sigma}\delta_{a',a}(-1)^a \langle k, x\rangle_- = \delta_{\sigma',\sigma}\delta_{a',a}(-1)^a(-k_0x_0 + \mathbf{k}_1\mathbf{x}_1 + \cdots + \mathbf{k}_d\mathbf{x}_d)$$

"characters"

$$E_+(\check{\xi}, \xi) = \delta_{\sigma',\sigma}\delta_{a',a}e^{\imath\langle\check{\xi},\xi\rangle} = \delta_{\sigma',\sigma}\delta_{a',a}e^{\imath(-1)^a(-k_0x_0+\mathbf{k}_1\mathbf{x}_1+\cdots+\mathbf{k}_d\mathbf{x}_d)}$$

$$E_-(\check{\xi}, \xi) = \delta_{\sigma',\sigma}\delta_{a',a}e^{-\imath\langle\check{\xi},\xi\rangle} = \delta_{\sigma',\sigma}\delta_{a',a}e^{-\imath(-1)^a(-k_0x_0+\mathbf{k}_1\mathbf{x}_1+\cdots+\mathbf{k}_d\mathbf{x}_d)}$$

and integrals

$$\int d\xi \bullet = \sum_{\substack{a\in\{0,1\}\\ \sigma\in\{\uparrow,\downarrow\}}} \int_{\mathbb{R}\times\mathbb{R}^d} dx_0 d^d\mathbf{x} \bullet \qquad \int d\check{\xi} \bullet = \sum_{\substack{a\in\{0,1\}\\ \sigma\in\{\uparrow,\downarrow\}}} \int_{\mathbb{R}\times\mathbb{R}^d} dk_0 d^d\mathbf{k} \bullet \,.$$

For $\check{\xi} = (k, \sigma, a)$, $\check{\xi}' = (k', \sigma', a') \in \check{\mathcal{B}}$ we set

$$\check{\xi} + \check{\xi}' = (-1)^a k + (-1)^{a'} k' \in \mathbb{R} \times \mathbb{R}^d \,.$$

**Definition IX.1 (Fourier transforms).** Let $f(\eta_1, \ldots, \eta_m; \xi_1, \ldots, \xi_n)$ be a translation invariant function on $\mathcal{B}^m \times \mathcal{B}^n$.

(i) The total Fourier transform $\check{f}$ of $f$ is defined by

$$\check{f}(\check{\eta}_1, \ldots, \check{\eta}_m; \check{\xi}_1, \ldots, \check{\xi}_n)(2\pi)^{d+1}\delta(\check{\eta}_1 + \cdots + \check{\eta}_m + \check{\xi}_1 + \cdots + \check{\xi}_n)$$

$$= \int \prod_{i=1}^m E_+(\check{\eta}_i, \eta_i)d\eta_i \prod_{j=1}^n E_+(\check{\xi}_j, \xi_j)d\xi_j f(\eta_1, \ldots, \eta_m; \xi_1, \ldots, \xi_n)$$

or, equivalently, by

$$f(\eta_1, \ldots, \eta_m; \xi_1, \ldots, \xi_n)$$

$$= \int \prod_{i=1}^m \frac{E_-(\check{\eta}_i, \eta_i)d\check{\eta}_i}{(2\pi)^{d+1}} \prod_{j=1}^n \frac{E_-(\check{\xi}_j, \xi_j)d\check{\xi}_j}{(2\pi)^{d+1}}$$

$$\times \check{f}(\check{\eta}_1, \ldots, \check{\eta}_m; \check{\xi}_1, \ldots, \check{\xi}_n)(2\pi)^{d+1}\delta(\check{\eta}_1 + \cdots + \check{\eta}_m + \check{\xi}_1 + \cdots + \check{\xi}_n)$$

$\check{f}$ is defined on the set $\{(\check{\eta}_1, \ldots, \check{\eta}_m; \check{\xi}_1, \ldots, \check{\xi}_n) \in \check{\mathcal{B}}^m \times \mathcal{B}^n | \check{\eta}_1 + \cdots + \check{\eta}_m + \check{\xi}_1 + \cdots + \check{\xi}_n = 0\}$.

If $m = 0$, $n = 2$ and $f(\xi_1, \xi_2)$ conserves particle number and is spin independent and antisymmetric, we define $\check{f}(k)$ by

$$\check{f}((k, \sigma, 1), (k, \sigma', 0)) = \delta_{\sigma,\sigma'} \check{f}(k)$$

or equivalently by

$$\check{f}(k) = \int dy e^{\imath\langle k,y\rangle_-} f((0, \sigma, 1), (y, \sigma, 0))\,.$$

(ii) If $n \geq 1$, the partial Fourier transform $f^\sim$ is defined by

$$f^\sim(\check{\eta}_1, \ldots, \check{\eta}_m; \xi_1, \ldots, \xi_n) = \int \left(\prod_{i=1}^m E_+(\check{\eta}_i, \eta_i) d\eta_i\right) f(\eta_1, \ldots, \eta_m; \xi_1, \ldots, \xi_n)$$

or, equivalently, by

$$f(\eta_1, \ldots, \eta_m; \xi_1, \ldots, \xi_n) = \int \left(\prod_{i=1}^m E_-(\check{\eta}_i, \eta_i) \frac{d\check{\eta}_i}{(2\pi)^{d+1}}\right) f^\sim(\check{\eta}_1, \ldots, \check{\eta}_m; \xi_1, \ldots, \xi_n)\,.$$

If $n = 0$, we set $f^\sim = \check{f}$.

**Remark IX.2.** Translation invariance of $f$ implies that for all $t \in \mathbb{R} \times \mathbb{R}^d$

$$f^\sim(\check{\eta}_1, \ldots, \check{\eta}_m; \xi_1 + t, \ldots, \xi_n + t) = e^{\imath\langle \check{\eta}_1 + \cdots + \check{\eta}_m, t\rangle_-} f^\sim(\check{\eta}_1, \ldots, \check{\eta}_m; \xi_1, \ldots, \xi_n)\,.$$

This is what we mean when we say that "$f^\sim(\check{\eta}_1, \ldots, \check{\eta}_m; \xi_1, \ldots, \xi_n)$ is translation invariant".

There will be two different situations in which we wish to associate a two-point function $B(\xi, \xi')$ in position/spin/particle–hole space, $\mathcal{B}$, to a function $B(k)$ in momentum space that has no spin/particle–hole dependence. In the first case, treated in Definition IX.3 below, $B(\xi, \xi')$ is a propagator and so is spin-independent and particle number conserving, so that $B(\xi, \xi')$ vanishes unless one of $\xi, \xi'$ is particle and the other is hole. In the second case, treated in Definition IX.4 below, convolution with $B(\xi, \xi')$ corresponds to pure multiplication by $B(k)$ in momentum space. This is used to, for example, introduce partitions of unity in momentum space. In this case $B(\xi, \xi')$ is diagonal in the particle/hole indices.

**Definition IX.3 (Fourier transforms of covariances).** If $C(k)$ is a function on $\mathbb{R} \times \mathbb{R}^d$, we say that the **covariance** $C(\xi, \xi')$ on $\mathcal{B} \times \mathcal{B}$, defined by

$$C((x_0, \mathbf{x}, \sigma, a), (x_0', \mathbf{x}', \sigma', a')) = \begin{cases} \delta_{\sigma,\sigma'} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{\imath\langle k, x-x'\rangle_-} C(k) & \text{if } a = 0,\ a' = 1 \\ 0 & \text{if } a = a' \\ -C((x', \sigma', a'), (x, \sigma, a)) & \text{if } a = 1,\ a' = 0 \end{cases}$$

is the Fourier transform of $C(k)$.

As in part (ii) of Proposition IV.3, we use the notation

**Definition IX.4.** If $\chi(k)$ is a function on $\mathbb{R} \times \mathbb{R}^d$, we define the Fourier transform $\hat{\chi}$ by

$$\hat{\chi}(\xi, \xi') = \delta_{\sigma,\sigma'}\delta_{a,a'} \int e^{(-1)^a \imath \langle k, x-x' \rangle_-} \chi(k) \frac{d^{d+1}k}{(2\pi)^{d+1}}$$

for $\xi = (x, a) = (x_0, \mathbf{x}, \sigma, a)$, $\xi' = (x', a') = (x'_0, \mathbf{x}', \sigma', a') \in \mathcal{B}$.

**Lemma IX.5.** (i) *Let* $C(k)$ *be a function on* $\mathbb{R} \times \mathbb{R}^d$ *and* $C(\xi, \xi')$ *the associated covariance in the sense of Definition* IX.3. *Then*

$$CJ = \hat{C} \qquad (JCJ)\check{}(k) = C(k)$$

*where* $J$ *was defined in* (VI.1), $\hat{C}$ *was defined in Definition* IX.4 *and* $\check{f}(k)$ *was defined in Definition* IX.1(i).

(ii) *Let* $\chi(k)$ *and* $\chi'(k)$ *be functions on* $\mathbb{R} \times \mathbb{R}^d$. *Then*

$$\int d\xi'' \hat{\chi}(\xi, \xi'') \hat{\chi}'(\xi'', \xi') = \widehat{\chi\chi'}(\xi, \xi') \,.$$

(iii) *Let* $\chi(k)$ *be a function on* $\mathbb{R} \times \mathbb{R}^d$. *Then*

$$(J\hat{\chi})\check{}(k) = \chi(k)$$

$$(J\hat{\chi}J)(\xi, \xi') = -\hat{\chi}(\xi', \xi) \,.$$

**Proof.** The proof of this lemma consists of a number of three or four line computations. □

In a renormalization group analysis we will adjust the counterterms in such a way that, at each scale, the Fourier transform of the two point function is small on $\{k = (k_0, \mathbf{k}) \in \mathbb{R} \times \mathbb{R}^d | k_0 = 0\}$. Then the absolute value of the Fourier transform of the two point function at a point $(k_0, \mathbf{k})$ can be estimated in terms of $|k_0|$ and the $k_0$ derivative of the Fourier transform of the two point function. The following lemma is used to make an analogous estimate in position space.

For a function $f(x)$ on $\mathbb{R} \times \mathbb{R}^d$, we define

$$\|f\|_{L^1} = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^d} \frac{1}{\delta!} \left[ \int |x^\delta f(x)| d^{d+1}x \right] t^\delta \in \mathfrak{N}_{d+1} \,.$$

**Lemma IX.6.** *Let* $u(\xi, \xi')$ *be a translation invariant function on* $\mathcal{B}^2$ *that satisfies* $\check{u}(((0, \mathbf{k}), \sigma, a), ((0, \mathbf{k}'), \sigma', a')) = 0$. *Furthermore let* $\chi(k)$ *be a function on* $\mathbb{R} \times \mathbb{R}^d$. *For* $x \in \mathbb{R} \times \mathbb{R}^d$ *set*

$$\chi'(x) = \int e^{\imath \langle k, x \rangle_-} \chi(k) \frac{d^{d+1}k}{(2\pi)^{d+1}}$$

*so that for* $\xi = (x, \sigma, a)$, $\xi' = (x', \sigma', a') \in \mathcal{B}$

$$\hat{\chi}(\xi, \xi') = \delta_{\sigma,\sigma'}\delta_{a,a'}\chi'((-1)^a(x - x'))\,.$$

*Then*

(i)

$$\left\|\int d\eta\hat{\chi}(\xi,\eta)u(\eta,\xi')\right\|_{1,\infty} \le \left\|\frac{\partial\chi'}{\partial x_0}\right\|_{L^1} \|\mathcal{D}_{1,2}^{(1,0,\dots,0)}u\|_{1,\infty} + \sum_{\substack{\delta\in\mathbb{N}_0\times\mathbb{N}_0^d \\ \delta_0\neq 0}} \infty t^\delta\,.$$

(ii)

$$\left\|\mathcal{D}_{1,2}^{(1,0,\dots,0)}\int d\eta\hat{\chi}(\xi,\eta)u(\eta,\xi')\right\|_{1,\infty}$$

$$\le const\left(\|\hat{\chi}\|_{1,\infty} + \left\|x_0\frac{\partial\chi'}{\partial x_0}\right\|_{L^1}\right)\|\mathcal{D}_{1,2}^{(1,0,\dots,0)}u\|_{1,\infty} + \sum_{\substack{\delta\in\mathbb{N}_0\times\mathbb{N}_0^d \\ \delta_0 > r_0}} \infty t^\delta\,.$$

**Proof.** (i) Fix $\xi = (x, \sigma, a)$, $\xi' = (x', \sigma', a') \in \mathcal{B}$. By translation invariance

$$\begin{aligned}\int d\eta\hat{\chi}(\xi,\eta)u(\eta,\xi') &= \int dy\chi'((-1)^a(x-y))u((y,\sigma,a),(x',\sigma',a')) \\ &= \int dy\chi'((-1)^a(x-y))u((y-x',\sigma,a),(0,\sigma',a')) \\ &= \int dy\chi'((-1)^a y)v(x-x'-y)\end{aligned}$$

where $v(y) = u((y,\sigma,a),(0,\sigma',a'))$. By hypothesis

$$\int dy_0 v(y_0,\mathbf{y}) = 0 \quad \text{for all } \mathbf{y}\in\mathbb{R}^d\,.$$

Therefore

$$\begin{aligned}&\int d\eta\hat{\chi}(\xi,\eta)u(\eta,\xi') \\ &\quad= \int dy(\chi'((-1)^a y) - \chi'((-1)^a(x_0 - x_0', \mathbf{y})))v(x-x'-y) \\ &\quad= \int dy\frac{\chi'((-1)^a y) - \chi'((-1)^a(x_0 - x_0',\mathbf{y}))}{(x_0 - x_0') - y_0}[(x_0 - x_0' - y_0)v(x-x'-y)] \\ &\quad= (-1)^{a+1}\int dy\int_0^1 ds\frac{\partial\chi'}{\partial x_0}((-1)^a(sy_0 + (1-s)(x_0 - x_0'),\mathbf{y})) \\ &\qquad\times\mathcal{D}_{1,2}^{(1,0,\dots,0)}u((x-x'-y,\sigma,a),(0,\sigma',a'))\end{aligned}$$

$$= (-1)^{a+1} \int dy \int_0^1 ds \frac{\partial \chi'}{\partial x_0}((-1)^a (sy_0 + (1-s)(x_0 - x_0'), \mathbf{y}))$$

$$\times \mathcal{D}_{1,2}^{(1,0,\ldots,0)} u((x-y,\sigma,a),\xi')\,.$$

Consequently, for fixed $\xi' \in \mathcal{B}$

$$\int d\xi \left| \int d\eta \hat{\chi}(\xi,\eta) u(\eta,\xi') \right|$$

$$\le \sum_{\sigma,a} \int_0^1 ds \int d\mathbf{x} d\mathbf{y} \int dx_0 dy_0 \left| \frac{\partial \chi'}{\partial x_0}((-1)^a (sy_0 + (1-s)(x_0 - x_0'), \mathbf{y})) \right|$$

$$\times |\mathcal{D}_{1,2}^{(1,0,\ldots,0)} u((x_0 - y_0, \mathbf{x} - \mathbf{y}, \sigma, a), \xi')|$$

$$= \sum_{\sigma,a} \int_0^1 ds \int d\mathbf{x} d\mathbf{y} \int d\alpha d\beta \left| \frac{\partial \chi'}{\partial x_0}((-1)^a (\beta, \mathbf{y})) \right|$$

$$\times |\mathcal{D}_{1,2}^{(1,0,\ldots,0)} u((\alpha, \mathbf{x} - \mathbf{y}, \sigma, a), \xi')|$$

$$\le \left\| \frac{\partial \chi'}{\partial x_0} \right\|_{L^1} \|\mathcal{D}_{1,2}^{(1,0,\ldots,0)} u\|_{1,\infty}\,. \tag{IX.1}$$

Here we used, for each fixed $s$, the change of variables $\alpha = x_0 - y_0$, $\beta = sy_0 + (1-s)x_0$. The integral $\int d\xi' |\int d\eta \hat{\chi}(\xi,\eta) u(\eta,\xi')|$, for fixed $\xi \in \mathcal{B}$, is treated similarly.

By Leibniz's rule (Lemma II.2) and (IX.1),

$$\sum_{\delta_0=0} \frac{t^\delta}{\delta!} \int d\xi \left| \mathcal{D}_{1,2}^\delta \int d\eta \hat{\chi}(\xi,\eta) u(\eta,\xi') \right|$$

$$\le \sum_{\delta_0=0} \sum_{\substack{\alpha,\beta \in \mathbb{N}_0 \times \mathbb{N}_0^d \\ \alpha+\beta=\delta}} \begin{pmatrix} \delta \\ \alpha,\beta \end{pmatrix} \frac{t^\delta}{\delta!} \int d\xi \left| \int d\eta (\mathcal{D}_{1,2}^\alpha \hat{\chi})(\xi,\eta)(\mathcal{D}_{1,2}^\beta u)(\eta,\xi') \right|$$

$$\le \sum_{\alpha_0=0} \sum_{\beta_0=0} \frac{t^\alpha}{\alpha!} \frac{t^\beta}{\beta!} \left( \left\| \frac{\partial}{\partial x_0} x^\alpha \chi' \right\|_{L^1} \|\mathcal{D}_{1,2}^{(1,0,\ldots,0)} \mathcal{D}_{1,2}^\beta u\|_{1,\infty} \Big|_{t=0} \right)$$

$$= \sum_{\alpha_0=0} \sum_{\beta_0=0} \frac{t^\alpha}{\alpha!} \frac{t^\beta}{\beta!} \left( \left\| x^\alpha \frac{\partial}{\partial x_0} \chi' \right\|_{L^1} \|\mathcal{D}_{1,2}^\beta \mathcal{D}_{1,2}^{(1,0,\ldots,0)} u\|_{1,\infty} \Big|_{t=0} \right)$$

$$\le \left\| \frac{\partial \chi'}{\partial x_0} \right\|_{L^1} \|\mathcal{D}_{1,2}^{(1,0,\ldots,0)} u\|_{1,\infty}\,.$$

(ii) By Leibniz's rule (Lemma II.2), part (i) of this lemma (applied to $\frac{\partial\chi}{\partial k_0}$) and Lemma II.7

$$\sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^d}\frac{t^\delta}{\delta!}\left|\!\left|\!\left|\mathcal{D}_{1,2}^{\delta}\mathcal{D}_{1,2}^{(1,0,\dots,0)}\int d\eta\,\hat\chi(\xi,\eta)u(\eta,\xi')\right|\!\right|\!\right|_{1,\infty}$$

$$\le\sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^d}\ \sum_{\substack{\alpha,\beta\in\mathbb{N}_0\times\mathbb{N}_0^d\\ \alpha+\beta=\delta+(1,0,\dots,0)}}\binom{\delta+(1,0,\dots,0)}{\alpha,\beta}\frac{t^\delta}{\delta!}$$

$$\times\left|\!\left|\!\left|\int d\eta\,\mathcal{D}_{1,2}^{\alpha}\hat\chi(\xi,\eta)\mathcal{D}_{1,2}^{\delta}\mathcal{D}_{1,2}^{\beta}u(\eta,\xi')\right|\!\right|\!\right|_{1,\infty}$$

$$\le\sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^d}\ \sum_{\substack{\alpha',\beta\in\mathbb{N}_0\times\mathbb{N}_0^d\\ \alpha'+\beta=\delta,\ \beta_0=0}}\binom{\delta+(1,0,\dots,0)}{\alpha'+(1,0,\dots,0),\beta}\frac{t^\delta}{\delta!}$$

$$\times\left|\!\left|\!\left|\int d\eta\,\mathcal{D}_{1,2}^{\alpha'}\mathcal{D}_{1,2}^{(1,0,\dots,0)}\hat\chi(\xi,\eta)\mathcal{D}_{1,2}^{\beta}u(\eta,\xi')\right|\!\right|\!\right|_{1,\infty}$$

$$+\sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^d}\ \sum_{\substack{\alpha,\beta'\in\mathbb{N}_0\times\mathbb{N}_0^d\\ \alpha+\beta'=\delta}}\binom{\delta+(1,0,\dots,0)}{\alpha,\beta'+(1,0,\dots,0)}\frac{t^\delta}{\delta!}$$

$$\times\left|\!\left|\!\left|\int d\eta\,\mathcal{D}_{1,2}^{\alpha}\hat\chi(\xi,\eta)\mathcal{D}_{1,2}^{\beta'}\mathcal{D}_{1,2}^{(1,0,\dots,0)}u(\eta,\xi')\right|\!\right|\!\right|_{1,\infty}$$

$$\le\sum_{\substack{\alpha',\beta\in\mathbb{N}_0\times\mathbb{N}_0^d\\ \beta_0=0}}(\alpha'_0+1)\frac{t^{\alpha'}}{\alpha'!}\frac{t^\beta}{\beta!}$$

$$\times\left|\!\left|\!\left|\int d\eta\,\mathcal{D}_{1,2}^{\alpha'}\mathcal{D}_{1,2}^{(1,0,\dots,0)}\hat\chi(\xi,\eta)\mathcal{D}_{1,2}^{\beta}u(\eta,\xi')\right|\!\right|\!\right|_{1,\infty}$$

$$+\sum_{\alpha,\beta'\in\mathbb{N}_0\times\mathbb{N}_0^d}(\alpha_0+\beta'_0+1)\frac{t^{\alpha}}{\alpha!}\frac{t^{\beta'}}{\beta'!}$$

$$\times\left|\!\left|\!\left|\int d\eta\,\mathcal{D}_{1,2}^{\alpha}\hat\chi(\xi,\eta)\mathcal{D}_{1,2}^{\beta'}\mathcal{D}_{1,2}^{(1,0,\dots,0)}u(\eta,\xi')\right|\!\right|\!\right|_{1,\infty}$$

$$\leq \sum_{\substack{\alpha',\beta\in\mathbb{N}_0\times\mathbb{N}_0^d\\ \beta_0=0}} (\alpha'_0+1)\frac{t^{\alpha'}}{\alpha'!}\frac{t^{\beta}}{\beta!}\left\|\frac{\partial}{\partial x_0}x^{\alpha'+(1,0,\dots,0)}\chi'\right\|_{L^1}\bigg|_{t=0}$$

$$\times \left|\!\left|\!\left|\mathcal{D}_{1,2}^{(1,0,\dots,0)}\mathcal{D}_{1,2}^{\beta}u\right|\!\right|\!\right|_{1,\infty}$$

$$+\sum_{\alpha,\beta'\in\mathbb{N}_0\times\mathbb{N}_0^d}(\alpha_0+\beta'_0+1)\frac{t^{\alpha}}{\alpha!}\frac{t^{\beta'}}{\beta'!}\left|\!\left|\!\left|\mathcal{D}_{1,2}^{\alpha}\hat{\chi}\right|\!\right|\!\right|_{1,\infty}$$

$$\times \left|\!\left|\!\left|\mathcal{D}_{1,2}^{\beta'}\mathcal{D}_{1,2}^{(1,0,\dots,0)}u\right|\!\right|\!\right|_{1,\infty}$$

$$\leq (r_0+1)\left((r_0+1)\|\hat{\chi}\|_{1,\infty}+\left\|x_0\frac{\partial\chi'}{\partial x_0}\right\|_{L^1}+\|\hat{\chi}\|_{1,\infty}\right)$$

$$\times\|\mathcal{D}_{1,2}^{(1,0,\dots,0)}u\|_{1,\infty}+\sum_{\substack{\delta\in\mathbb{N}_0\times\mathbb{N}_0^d\\ \delta_0>r_0}}\infty t^{\delta}$$

since $\frac{\partial}{\partial x_0}(x^{\alpha'+(1,0,\dots,0)}\chi'(x)) = (\alpha'_0+1)x^{\alpha'}\chi'(x)+x^{\alpha'}x_0\frac{\partial}{\partial x_0}\chi'(x)$. □

## X. Momentum Space Norms

In this section, we introduce momentum space norms designed to control amputated Green's functions in momentum space. The set of momentum conserving $m$-tuples of momenta is

$$\check{\mathcal{B}}_m = \{(\check{\eta}_1,\dots,\check{\eta}_m)\in\check{\mathcal{B}}^m \,|\, \check{\eta}_1+\cdots+\check{\eta}_m=0\}$$

(we use the addition introduced before Definition IX.1). We are particularly interested in the two and four point functions. In the renormalization group analysis we shall control the external fields in momentum space, while the fields that are going to be integrated out are still treated in position space. That is, we will estimate partial Fourier transforms of functions on $\mathcal{B}^m\times\mathcal{B}^n$ as in Definition IX.1(ii). Motivated by Remark IX.2 we define

**Definition X.1.** A function $f$ on $\check{\mathcal{B}}^m\times\mathcal{B}^n$ is called translation invariant, if for all $t\in\mathbb{R}\times\mathbb{R}^d$

$$f(\check{\eta}_1,\dots,\check{\eta}_m;\xi_1+t,\dots,\xi_n+t) = e^{\imath\langle\check{\eta}_1+\cdots+\check{\eta}_m,t\rangle_-}f(\check{\eta}_1,\dots,\check{\eta}_m;\xi_1,\dots,\xi_n)\,.$$

Generalizing Definition II.1 we set

**Definition X.2 (Differential-decay operators).** Let $m, n \ge 0$. If $n \ge 1$, let $f$ be a function on $\check{\mathcal{B}}^m \times \mathcal{B}^n$. If $n = 0$, let $f$ be a function on $\check{\mathcal{B}}_m$.

(i) For $1 \le j \le m$ and a multiindex $\delta$ set

$$\mathrm{D}_j^\delta f((p_1,\tau_1,b_1),\dots,(p_m,\tau_m,b_m);\xi_1,\dots,\xi_n)$$
$$= [\imath(-1)^{b_j}]^{\delta_0} \prod_{\ell=1}^{d} [-\imath(-1)^{b_j}]^{\delta_\ell} \frac{\partial^{\delta_0}}{\partial p_{j,0}^{\delta_0}} \frac{\partial^{\delta_1}}{\partial \mathbf{p}_{j,1}^{\delta_1}} \cdots \frac{\partial^{\delta_d}}{\partial \mathbf{p}_{j,d}^{\delta_d}}$$
$$\times f((p_1,\tau_1,b_1),\dots,(p_m,\tau_m,b_m);\xi_1,\dots,\xi_n)\,.$$

(ii) Let $1 \le i \ne j \le m+n$ and $\delta$ a multiindex. Set

$$\begin{aligned}
&\mathrm{D}_{i;j}^\delta f = (\mathrm{D}_i - \mathrm{D}_j)^\delta f && \text{if } 1 \le i < j \le m\\
&\mathrm{D}_{i;j}^\delta f = (\mathrm{D}_i - \xi_{j-m})^\delta f && \text{if } 1 \le i \le m,\ m+1 \le j \le m+n\\
&\mathrm{D}_{i;j}^\delta f = (\xi_i - \mathrm{D}_{j-m})^\delta f && \text{if } m+1 \le i \le m+n,\ 1 \le j \le m\\
&\mathrm{D}_{i;j}^\delta f = (\xi_{i-m} - \xi_{j-m})^\delta f = \mathcal{D}_{i-m,j-m}^\delta f && \text{if } m+1 \le i < j \le m+n\,.
\end{aligned}$$

(iii) A differential-decay operator (dd-operator) of type $(m,n)$, with $m+n \ge 2$, is an operator D of the form

$$\mathrm{D} = \mathrm{D}_{i_1;j_1}^{\delta^{(1)}} \cdots \mathrm{D}_{i_r;j_r}^{\delta^{(r)}}$$

with $1 \le i_\ell \ne j_\ell \le m+n$ for all $1 \le \ell \le r$. A dd-operator of type $(1,0)$ is an operator of the form $\mathrm{D} = \mathrm{D}_1^{\delta^{(1)}} \cdots \mathrm{D}_1^{\delta^{(r)}} = \mathrm{D}_1^{\delta^{(1)}+\cdots+\delta^{(r)}}$. The total order of D is $\delta(\mathrm{D}) = \delta^{(1)} + \cdots + \delta^{(r)}$.

**Remark X.3.** (i) Let D be a differential-decay operator. If $f$ is a translation invariant function on $\check{\mathcal{B}}_m \times \mathcal{B}^n$, then $\mathrm{D}f$ is again translation invariant.

(ii) For a translation invariant function $\varphi$ on $\mathcal{B}^m \times \mathcal{B}^n$

$$\mathrm{D}_{i;j}(\varphi^\sim) = (\mathcal{D}_{i,j}\varphi)^\sim\,.$$

In particular, Leibniz's rule also applies for differential-decay operators.

(iii) Let $f$ be a translation invariant function on $\check{\mathcal{B}}^m \times \mathcal{B}$. Then, for $\xi = (x_0,\mathbf{x},\sigma,a) \in \mathcal{B}$,

$$f(\check{\eta}_1,\dots,\check{\eta}_m;\xi) = e^{\imath\langle \check{\eta}_1+\cdots+\check{\eta}_m,(x_0,\mathbf{x})\rangle_-} f(\check{\eta}_1,\dots,\check{\eta}_m;(0,\sigma,a))\,.$$

Consequently, for $1 \le i \le m$ and a multiindex $\delta$

$$\mathrm{D}_{i;m+1}^\delta f(\check{\eta}_1,\dots,\check{\eta}_m;\xi) = e^{\imath\langle \check{\eta}_1+\cdots+\check{\eta}_m,(x_0,\mathbf{x})\rangle_-} \mathrm{D}_i^\delta f(\check{\eta}_1,\dots,\check{\eta}_m;(0,\sigma,a))\,.$$

**Definition X.4.** For a function $f$ on $\check{\mathcal{B}}_m$, set

$$\|f\|^\sim = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^2} \frac{1}{\delta!} \max_{\substack{\mathrm{D}\text{ dd-operator}\\ \text{with } \delta(\mathrm{D})=\delta}} \sup_{\check{\eta}_1,\dots,\check{\eta}_m \in \check{\mathcal{B}}} |\mathrm{D}f(\check{\eta}_1,\dots,\check{\eta}_m)| t^\delta\,.$$

Let $f$ be a function on $\check{\mathcal{B}}^m \times \mathcal{B}^n$ with $n \ge 1$. Set

$$\|f\|^{\sim} = \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \frac{1}{\delta!} \max_{\substack{\mathrm{D}\ \text{dd-operator}\\ \text{with } \delta(\mathrm{D})=\delta}} \sup_{\check\eta_1,\dots,\check\eta_m\in\check{\mathcal{B}}} \left|\left|\left|\mathrm{D}f(\check\eta_1,\dots,\check\eta_m;\xi_1,\dots,\xi_n)\right|\right|\right|_{1,\infty} t^{\delta}$$

when $m \le p \le m+n$. The norm $|||\cdot|||_{1,\infty}$ of Example II.6 refers to the variables $\xi_1,\dots,\xi_n$. That is,

$$|||\mathrm{D}f(\check\eta_1,\dots,\check\eta_m;\xi_1,\dots,\xi_n)|||_{1,\infty}$$

$$= \max_{1\le j_0\le n} \sup_{\xi_{j_0}\in\mathcal{B}} \int \prod_{\substack{j=1,\dots,n\\ j\ne j_0}} d\xi_j \, |\mathrm{D}f(\check\eta_1,\dots,\check\eta_m;\xi_1,\dots,\xi_n)|\,.$$

**Remark X.5.** In the case $m=0$ the norm $\|\cdot\|_{1,\infty}$ of Example II.6 and the norm $\|\cdot\|^{\sim}$ of Definition X.4 agree.

In analogy to Lemma II.7, we have

**Lemma X.6.** *Let $f$ be a translation invariant function on $\check{\mathcal{B}}^m\times\mathcal{B}^n$, $f'$ a translation invariant function on $\check{\mathcal{B}}^{m'}\times\mathcal{B}^{n'}$ and $1\le\mu\le n$, $1\le\nu\le n'$.*

*If $n\ge 2$ or $n'\ge 2$ define the function $g$ on $\check{\mathcal{B}}^{m+m'}\times\mathcal{B}^{n+n'-2}$ by*

$$g(\check\eta_1,\dots,\check\eta_{m+m'};\xi_1,\dots,\xi_{\mu-1},\xi_{\mu+1},\dots,\xi_n,\xi_{n+1},\dots,\xi_{n+\nu-1},\xi_{n+\nu+1},\dots,\xi_{n+n'})$$

$$= \int_{\mathcal{B}} d\zeta\, f(\check\eta_1,\dots,\check\eta_m;\xi_1,\dots,\xi_{\mu-1},\zeta,\xi_{\mu+1},\dots,\xi_n)$$

$$\times f'(\check\eta_{m+1},\dots,\check\eta_{m+m'};\xi_{n+1},\dots,\xi_{n+\nu-1},\zeta,\xi_{n+\nu+1},\dots,\xi_{n+n'})\,.$$

*If $n=n'=1$, define the function $g$ on $\check{\mathcal{B}}_{m+m'}$ by*

$$g(\check\eta_1,\dots,\check\eta_{m+m'})(2\pi)^{d+1}\delta(\check\eta_1+\cdots+\check\eta_{m+m'})$$

$$= \int_{\mathcal{B}} d\zeta\, f(\check\eta_1,\dots,\check\eta_m;\zeta) f'(\check\eta_{m+1},\dots,\check\eta_{m+m'};\zeta)\,.$$

*Then*

$$\|g\|^{\sim} \le \|f\|^{\sim}\,\|f'\|^{\sim} \begin{cases} 4 & \text{if } n=n'=1 \\ 1 & \text{otherwise} \end{cases}\,.$$

**Proof.** If $n\ge 2$ or $n'\ge 2$, the proof is analogous to that of Lemma II.7. Therefore we only discuss the case $n=n'=1$. In this case, by Remark X.3(iii)

$$\int_{\mathcal{B}} d\xi\, f(\check\eta_1,\dots,\check\eta_m;\xi) f'(\check\eta_{m+1},\dots,\check\eta_{m+m'};\xi)$$

$$= \int dx_0 \int d\mathbf{x} \sum_{\substack{\sigma\in\{\uparrow,\downarrow\}\\ b\in\{0,1\}}} f(\check\eta_1,\dots,\check\eta_m;(0,\sigma,b))$$

$$\times e^{\imath\langle\check\eta_1+\cdots+\check\eta_{m+m'},(x_0,\mathbf{x})\rangle} - f'(\check\eta_{m+1},\ldots,\check\eta_{m+m'};(0,\sigma,b))$$

$$= \sum_{\substack{\sigma\in\{\uparrow,\downarrow\}\\ b\in\{0,1\}}} f(\check\eta_1,\ldots,\check\eta_m;(0,\sigma,b))f'(\check\eta_{m+1},\ldots,\check\eta_{m+m'};(0,\sigma,b))$$

$$\times (2\pi)^{d+1}\delta(\check\eta_1+\cdots+\check\eta_{m+m'})\,.$$

Consequently

$$g(\check\eta_1,\ldots,\check\eta_{m+m'}) = \sum_{\substack{\sigma\in\{\uparrow,\downarrow\}\\ b\in\{0,1\}}} f(\check\eta_1,\ldots,\check\eta_m;(0,\sigma,b))f'(\check\eta_{m+1},\ldots,\check\eta_{m+m'};(0,\sigma,b))\,.$$

The claim now follows by iterated application of the product rule for derivatives and Remark X.3(iii). The factor of 4 comes from the sum over $\sigma$ and $b$ and is required only when $n = n' = 1$. □

**Remark X.7.**

$$F(\check\eta_1,\ldots,\check\eta_m) = \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \frac{1}{\delta!} \max_{\substack{\mathrm{D}\ \text{dd-operator}\\ \text{with }\delta(\mathrm{D})=\delta}} |||\mathrm{D}f(\check\eta_1,\ldots,\check\eta_m;\cdot,\ldots,\cdot)|||_{1,\infty} t^\delta$$

so that

$$\|f\|^\sim = \sup_{\check\eta_1,\ldots,\check\eta_m\in\check{\mathcal{B}}} F(\check\eta_1,\ldots,\check\eta_m)$$

(with the supremum of the formal power series $F$ taken componentwise) and define $G(\check\eta_1,\ldots,\check\eta_{m+m'})$ and $F'(\check\eta_{m+1},\ldots,\check\eta_{m+m'})$ similarly. The proof of Lemma X.6 actually shows that

$$G(\check\eta_1,\ldots,\check\eta_{m+m'}) \le F(\check\eta_1,\ldots,\check\eta_m)F'(\check\eta_{m+1},\ldots,\check\eta_{m+m'}) \quad \begin{cases} 4 & \text{if } n=n'=1\\ 1 & \text{otherwise}\end{cases}$$

for all $(\check\eta_1,\ldots,\check\eta_{m+m'}) \in \check{\mathcal{B}}^{m+m'}$.

**Definition X.8.** (i) For $n \ge 1$, denote by $\check{\mathcal{F}}_m(n)$ the space of all translation invariant, complex valued functions $f(\check\eta_1,\ldots,\check\eta_m;\xi_1,\ldots,\xi_n)$ on $\check{\mathcal{B}}^m\times\mathcal{B}^n$ that are antisymmetric in their external ($=\check\eta$) variables. Let $\check{\mathcal{F}}_m(0)$ be the space of all antisymmetric, complex valued functions $f(\check\eta_1,\ldots,\check\eta_m)$ on $\check{\mathcal{B}}_m$.

(ii) Let $C(\xi,\xi')$ be any skew symmetric function on $\mathcal{B}^2$. Let $f\in\check{\mathcal{F}}_m(n)$ and $1\le i<j\le n$. We define "contraction", for $n\ge 2$, by

$$\mathop{\mathcal{C}\mathrm{on}_C}_{i\to j} f(\check\eta_1,\ldots,\check\eta_m;\xi_1,\ldots,\xi_{i-1},\xi_{i+1},\ldots,\xi_{j-1},\xi_{j+1},\ldots,\xi_n)$$

$$= (-1)^{j-i+1}\int d\xi_i d\xi_j C(\xi_i,\xi_j) f(\check\eta_1,\ldots,\check\eta_m;\xi_1,\ldots,\xi_n)$$

and, for $n = 2$, by

$$\underset{1\to 2}{\mathcal{C}\mathrm{on}_C}\, f(\check\eta_1,\dots,\check\eta_m)(2\pi)^{d+1}\delta(\check\eta_1+\cdots+\check\eta_m)$$

$$= \int d\xi_1 d\xi_2 C(\xi_1,\xi_2) f(\check\eta_1,\dots,\check\eta_m;\xi_1,\xi_2)\,.$$

$\check{\mathcal{F}}_m(n)$ consists of the partial Fourier transforms $\varphi^\sim$ [as in Definition IX.1(ii)] of translation invariant functions $\varphi \in \mathcal{F}_m(n)$ as in Definition II.9. Also, $\underset{i\to j}{\mathcal{C}\mathrm{on}_C}\,\varphi^\sim = (\underset{i\to j}{\mathcal{C}\mathrm{on}_C}\,\varphi)^\sim$, where $\underset{i\to j}{\mathcal{C}\mathrm{on}_C}\,\varphi$ is defined in Definition III.1.

**Corollary X.9.** *Let $C(\xi,\xi') \in \mathcal{F}_0(2)$ be an antisymmetric function. Let $m, m' \geq 0$, $n, n' \geq 1$ and $f \in \check{\mathcal{F}}_m(n)$, $f' \in \check{\mathcal{F}}_{m'}(n')$. Then*

$$\left\| \underset{1\to n+1}{\mathcal{C}\mathrm{on}_C}\, \mathrm{Ant}_{\mathrm{ext}}(f\otimes f') \right\|^\sim \leq 4\|C\|_{1,\infty}\|f\|^\sim\,\|f'\|^\sim\,.$$

**Proof.** The claim follows by iterated application of Lemma X.6 and the observation that $\|C\|^\sim = \|C\|_{1,\infty}$ by Remark X.5. □

We shall prove an analog of Theorem VIII.6, for the momentum space norms of Definition X.4. By way of preparation, we first formulate the following variant of Lemma V.1.

**Lemma X.10.** *Let $\rho_{m;n}$ be a sequence of nonnegative real numbers such that $\rho_{m;n'} \leq \rho_{m;n}$ for $n' \leq n$. Define (locally) for $f \in \check{\mathcal{F}}_m(n)$*

$$\|f\| = \rho_{m;n}\|f\|^\sim$$

*where $\|f\|^\sim$ is the norm of Definition* X.4.

(i) *The seminorms $\|\cdot\|$ are symmetric.*
(ii) *For a covariance $C$, let $S(C)$ be the quantity introduced in Definition* IV.1. *Then $2S(C)$ is an integral bound for the covariance $C$ with respect to the family of seminorms $\|\cdot\|$.*
(iii) *Let $C$ be a covariance. Assume that for all $m, m' \geq 0$ and $n, n' \geq 1$*

$$\rho_{m+m';n+n'-2} \leq \rho_{m;n}\rho_{m';n'}$$

*and let $\mathfrak{c}$ obey*

$$\mathfrak{c} \geq 4\|C\|_{1,\infty}\,.$$

*Then $\mathfrak{c}$ is a contraction bound for the covariance $C$ with respect to the family of seminorms $\|\cdot\|$.*

**Proof.** Parts (i) and (ii) are trivial. To prove part (iii), let $f \in \check{\mathcal{F}}_m(n)$, $f' \in \check{\mathcal{F}}_{m'}(n')$ and $1 \leq i \leq n$, $1 \leq j \leq n'$. If $n \geq 2$ or $n' \geq 2$ define the function $g$ on

$\check{\mathcal{B}}^{m+m'} \times \mathcal{B}^{n+n'-2}$ by

$$g(\check\eta_1, \ldots, \check\eta_{m+m'}; \xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_n, \xi_{n+1}, \ldots, \xi_{n+j-1}, \xi_{n+j+1}, \ldots, \xi_{n+n'})$$

$$= \int d\zeta d\zeta' f(\check\eta_1, \ldots, \check\eta_m; \xi_1, \ldots, \xi_{i-1}, \zeta, \xi_{i+1}, \ldots, \xi_n) C(\zeta, \zeta')$$

$$\times f'(\check\eta_{m+1}, \ldots, \check\eta_{m+m'}; \xi_{n+1}, \ldots, \xi_{n+j-1}, \zeta', \xi_{n+j+1}, \ldots, \xi_{n+n'}) .$$

If $n = n' = 1$, define the function $g$ on $\check{\mathcal{B}}_{m+m'}$ by

$$g(\check\eta_1, \ldots, \check\eta_{m+m'})(2\pi)^{d+1}\delta(\check\eta_1 + \cdots + \check\eta_{m+m'})$$

$$= \int d\zeta d\zeta' f(\check\eta_1, \ldots, \check\eta_m; \zeta) C(\zeta, \zeta') f'(\check\eta_{m+1}, \ldots, \check\eta_{m+m'}; \zeta')$$

or equivalently, by

$$g(\check\eta_1, \ldots, \check\eta_{m+m'})$$

$$= \sum_{\substack{\sigma \in \{\uparrow,\downarrow\} \\ b \in \{0,1\}}} \int d\zeta' f(\check\eta_1, \ldots, \check\eta_m; (0, \sigma, b)) C((0, \sigma, b), \zeta') f'(\check\eta_{m+1}, \ldots, \check\eta_{m+m'}; \zeta') .$$

Then

$$\underset{i \to n+j}{\mathcal{C}\mathrm{on}_C} \, \mathrm{Ant_{ext}}(f \otimes f') = \mathrm{Ant_{ext}}\, g$$

and therefore

$$\left\| \underset{i \to n+j}{\mathcal{C}\mathrm{on}_C} \, \mathrm{Ant_{ext}}(f \otimes f') \right\| \le \|g\| .$$

As

$$\|g\|^{\sim} \le 4\|f\|^{\sim}\, \|C\|_{1,\infty} \|f'\|^{\sim}$$

and consequently

$$\begin{aligned} \left\| \underset{i \to n+j}{\mathcal{C}\mathrm{on}_C} \, \mathrm{Ant_{ext}}(f \otimes f') \right\| &\le 4\rho_{m+m';n+n'-2} \|C\|_{1,\infty} \|f\|^{\sim}\, \|f'\|^{\sim} \\ &\le \mathfrak{c}\rho_{m;n} \|f\|^{\sim} \rho_{m';n'} \|f'\|^{\sim} \\ &= \mathfrak{c}\|f\|\, \|f'\| . \end{aligned}$$

□

The analog of “external improving” for the current setting is the following lemma. We shall later, in [4, Lemma XVII.5], prove a general scale version. Let $(\rho_{m;n})_{m,n \in \mathbb{N}_0}$ be a system of positive real numbers and $X \in \mathfrak{N}_{d+1}$ with $X_{\mathbf{0}} < 1$.

For an even Grassmann function

$$\mathcal{W}(\phi,\psi) = \sum_{\substack{m,n\ge 0\\ m+n \text{ even}}} \int_{\mathcal{B}^{m+n}} d\eta_1\cdots d\eta_m d\xi_1\cdots d\xi_n$$

$$\times W_{m,n}(\eta_1\cdots\eta_m,\xi_1,\dots,\xi_n)\phi(\eta_1)\cdots\phi(\eta_m)\psi(\xi_1)\cdots\psi(\xi_n)$$

with kernels $W_{m,n}$ that are separately antisymmetric under permutations of their $\eta$ and $\xi$ arguments, define

$$N_0^{\sim}(\mathcal{W};\beta;X,\boldsymbol{\rho}) = \mathfrak{e}_0(X) \sum_{\substack{m+n\ge 2\\ m+n \text{ even}}} \beta^{m+n}\rho_{m;n}\|W_{m,n}^{\sim}\|^{\sim}$$

where $\|\cdot\|^{\sim}$ is the norm of Definition X.4.

**Lemma X.11.** *Let $(\rho_{m;n})_{m,n\in\mathbb{N}_0}$ be a system of positive real numbers and $X\in\mathfrak{N}_{d+1}$ with $X_{\mathbf{0}}<1$. There are constants* $_{const}$ *and $\Gamma_0$, independent of $M$, $X$ and $\boldsymbol{\rho}$ such that the following holds for all $\Gamma\le\Gamma_0$. Let $C(k)$ be a function that obeys $\|C(k)\|^{\sim}\le\Gamma\frac{\rho_{m;n}}{\rho_{m+1;n-1}}\mathfrak{e}_0(X)$ for all $m\ge 0$, $n\ge 1$ and let $C(\xi,\xi')$ be the covariance associated to it by Definition* IX.3.

(i) *Let $\mathcal{W}(\phi,\psi)$ be a Grassmann function and set*

$$\mathcal{W}'(\phi,\psi) = \mathcal{W}(\phi,\psi+CJ\phi)\,.$$

*Then*

$$N_0^{\sim}(\mathcal{W}'-\mathcal{W};\beta;X,\boldsymbol{\rho}) \le {\scriptstyle const}\,\Gamma N_0^{\sim}(\mathcal{W};2\beta;X,\boldsymbol{\rho})\,.$$

(ii) *Assume that there is a $Y\in\mathfrak{N}_{d+1}$ such that $\|C'(k)\|^{\sim}\le\frac{\rho_{m;n}}{\rho_{m+1;n-1}}Y\mathfrak{e}_0(X)$ for all $m\ge 0$, $n\ge 1$. Set*

$$\mathcal{W}'_s(\phi,\psi) = \mathcal{W}(\phi,\psi+CJ\phi+sC'J\phi)\,.$$

*Then*

$$N_0^{\sim}\left(\frac{d}{ds}\mathcal{W}'\bigg|_{s=0};\beta;X,\boldsymbol{\rho}\right) \le {\scriptstyle const}\,Y N_0^{\sim}(\mathcal{W};2\beta;X,\boldsymbol{\rho})\,.$$

**Proof.** Let

$$J^{\sim}((k_0,\mathbf{k},\sigma,a),(x_0,\mathbf{x},\sigma',a')) = \delta_{\sigma,\sigma'}e^{i(-1)^a\langle k,x\rangle_-}\begin{cases}1 & \text{if } a=1,\ a'=0\\ -1 & \text{if } a=0,\ a'=1\\ 0 & \text{otherwise}\end{cases}$$

be the partial Fourier transform of $J(\eta,\xi)$ with respect to its first argument. Observe that

$$\int d\zeta J^{\sim}(\check\eta,\zeta)C(\zeta,\zeta') = C(k)E_+(\check\eta,\zeta') \quad \text{where } \check\eta=(k_0,\mathbf{k},\sigma,a)\,.$$

Let $f \in \check{\mathcal{F}}_m(n)$, $1 \le i \le n$ and set, for $\check{\eta}_{m+1} = (k_{m+1}, \sigma_{m+1}, a_{m+1})$,

$$g(\check{\eta}_1, \dots, \check{\eta}_{m+1}; \xi_1, \dots, \xi_{n-1})$$

$$= \mathrm{Ant}_{\mathrm{ext}} \int d\zeta d\zeta' J^{\sim}(\check{\eta}_{m+1}, \zeta) C(\zeta, \zeta') f(\check{\eta}_1, \dots, \check{\eta}_m; \xi_1 \dots, \xi_{i-1}, \zeta', \xi_i, \dots, \xi_{n-1})$$

$$= \mathrm{Ant}_{\mathrm{ext}} \int d\zeta C(k_{m+1}) E_+(\check{\eta}_{m+1}, \zeta) f(\check{\eta}_1, \dots, \check{\eta}_m; \xi_1 \dots, \xi_{i-1}, \zeta, \xi_i, \dots, \xi_{n-1})$$

for $n = 2$ and

$$g(\check{\eta}_1, \dots, \check{\eta}_{m+1})(2\pi)^{d+1} \delta(\check{\eta}_1 + \dots + \check{\eta}_{m+1})$$

$$= \mathrm{Ant}_{\mathrm{ext}} \int d\zeta d\zeta' J^{\sim}(\check{\eta}_{m+1}, \zeta) C(\zeta, \zeta') f(\check{\eta}_1, \dots, \check{\eta}_m; \zeta')$$

or equivalently

$$g(\check{\eta}_1, \dots, \check{\eta}_{m+1}) = \mathrm{Ant}_{\mathrm{ext}}\, C(k_{m+1}) f(\check{\eta}_1, \dots, \check{\eta}_m; (0, \mathbf{0}, \sigma_{m+1}, a_{m+1}))$$

for $n = 1$. In both cases, since $\mathrm{D}^{\delta}_{m+1} E_+(\check{\eta}_{m+1}, \zeta) = \zeta^{\delta} E_+(\check{\eta}_{m+1}, \zeta)$,

$$\|g\|^{\sim} \le \|f\|^{\sim} \, \|C(k)\|^{\sim}$$

so that

$$\rho_{m+1;n-1} \|g\|^{\sim} \le \Gamma \rho_{m;n} \|f\|^{\sim} \mathfrak{e}_0(X)\,. \tag{X.1}$$

(i) Write $\mathcal{W}(\phi, \psi) = \sum_{m,n} \mathcal{W}_{m,n}(\phi, \psi)$, with $\mathcal{W}_{m,n}$ of degree $m$ in $\phi$ and degree $n$ in $\psi$, and

$$\mathcal{W}(\phi, \psi + \zeta) = \sum_{m,n} \mathcal{W}_{m,n}(\phi, \psi + \zeta) = \sum_{m,n} \sum_{\ell=0}^{n} \mathcal{W}_{m,n-\ell,\ell}(\phi, \psi, \zeta)$$

with $\mathcal{W}_{m,n-\ell,\ell}$ of degrees $m$ in $\phi$, $n - \ell$ in $\psi$ and $\ell$ in $\zeta$. Let $w_{m,n}$ and $w_{m,n-\ell,\ell}$ be the kernels of $\mathcal{W}_{m,n}(\phi, \psi)$ and $\mathcal{W}_{m,n-\ell,\ell}(\phi, \psi, CJ\phi)$ respectively. By the binomial theorem and repeated application of (X.1),

$$\mathfrak{e}_0(X) \rho_{m+\ell;n-\ell} \|w^{\sim}_{m,n-\ell,\ell}\|^{\sim} \le ({\scriptstyle const}\, \Gamma)^{\ell} \binom{n}{\ell} \mathfrak{e}_0(X) \rho_{m;n} \|w^{\sim}_{m,n}\|^{\sim}$$

if $\ell \ge 1$. Then,

$$\mathcal{W}'(\phi, \psi) - \mathcal{W}(\phi, \psi) = \mathcal{W}(\phi, \psi + CJ\phi) - \mathcal{W}(\phi, \psi) = \sum_{m,n \ge 0} \sum_{\ell=1}^{n} \mathcal{W}_{m,n-\ell,\ell}(\phi, \psi, CJ\phi)$$

and

$$\begin{aligned} N_0^\sim(\mathcal{W}'-\mathcal{W};\beta;X,\boldsymbol{\rho}) &\le \mathfrak{e}_0(X)\sum_{m,n\ge 0}\sum_{\ell=1}^{n}\beta^{m+n}\rho_{m+\ell;n-\ell}\|w^\sim_{m,n-\ell,\ell}\|^\sim \\ &\le \mathfrak{e}_0(X)\sum_{m,n\ge 0}\sum_{\ell=1}^{n}\binom{n}{\ell}({\scriptstyle const}\,\Gamma)^\ell\beta^{m+n}\rho_{m;n}\|w^\sim_{m,n}\|^\sim \\ &= \mathfrak{e}_0(X)\sum_{m,n\ge 0}[(1+{\scriptstyle const}\,\Gamma)^n-1]\beta^{m+n}\rho_{m;n}\|w^\sim_{m,n}\|^\sim\,. \end{aligned}$$

If ${\scriptstyle const}\,\Gamma \le \frac{1}{3}$,

$$\begin{aligned} (1+{\scriptstyle const}\,\Gamma)^n-1 &\le {\scriptstyle const}\,\Gamma n(1+{\scriptstyle const}\,\Gamma)^{n-1} \\ &\le {\scriptstyle const}\,\Gamma\left(\frac{3}{2}\right)^n(1+{\scriptstyle const}\,\Gamma)^{n-1} \\ &\le {\scriptstyle const}\,\Gamma 2^n \end{aligned}$$

and

$$N_0^\sim(\mathcal{W}'-\mathcal{W};\beta;X,\boldsymbol{\rho}) \le {\scriptstyle const}\,\Gamma N_0^\sim(\mathcal{W};2\beta;X,\boldsymbol{\rho})\,.$$

(ii) Write

$$\mathcal{W}(\phi,\psi+\zeta+\eta) = \sum_{m,n}\sum_{\substack{n_1,n_2,n_3\ge 0\\ n_1+n_2+n_3=n}}\mathcal{W}_{m,n_1,n_2,n_3}(\phi,\psi,\zeta,\eta)$$

with $\mathcal{W}_{m,n_1,n_2,n_3}$ of degrees $m$ in $\phi$, $n_1$ in $\psi$, $n_2$ in $\zeta$ and $n_3$ in $\eta$. Let $w_{m,n_1,n_2,n_3}$ be the kernel of $\mathcal{W}_{m,n_1,n_2,n_3}(\phi,\psi,CJ\phi,C'J\phi)$. By the binomial theorem, repeated application of (X.1) and the obvious analog of (X.1) with $C$ replaced by $C'$,

$$\begin{aligned} &\mathfrak{e}_0(X)\rho_{m+\ell+1;n-\ell-1}\|w^\sim_{m,n-\ell-1,\ell,1}\|^\sim \\ &\quad\le ({\scriptstyle const}\,\Gamma)^\ell({\scriptstyle const}\,Y)\binom{n}{n-\ell-1,\ell,1}\mathfrak{e}_0(X)\rho_{m;n}\|w^\sim_{m,n}\|^\sim\,. \end{aligned}$$

Then,

$$\begin{aligned} \frac{d}{ds}\mathcal{W}'_s(\phi,\psi)\bigg|_{s=0} &= \frac{d}{ds}\mathcal{W}(\phi,\psi+CJ\phi+sC'J\phi)\bigg|_{s=0} \\ &= \sum_{m,n\ge 0}\sum_{\ell=0}^{n-1}\mathcal{W}_{m,n-\ell-1,\ell,1}(\phi,\psi,CJ\phi,C'J\phi) \end{aligned}$$

and

$$
\begin{aligned}
N_0^{\sim}\left(\frac{d}{ds}\mathcal{W}'_s\Big|_{s=0};\beta;X,\boldsymbol{\rho}\right)
&\le \mathfrak{e}_0(X)\sum_{m,n\ge 0}\sum_{\ell=0}^{n-1}\beta^{m+n}\rho_{m+\ell+1;n-\ell-1}\|w^{\sim}_{m,n-\ell-1,\ell,1}\|^{\sim}\\
&\le \mathfrak{e}_0(X)\sum_{m,n\ge 0}\sum_{\ell=0}^{n-1}\binom{n}{n-\ell-1,\ell,1}({\scriptstyle const}\,\Gamma)^{\ell}({\scriptstyle const}\,Y)\beta^{m+n}\rho_{m;n}\|w^{\sim}_{m,n}\|^{\sim}\\
&= {\scriptstyle const}\,Y\mathfrak{e}_0(X)\sum_{m,n\ge 0}\sum_{\ell=0}^{n-1}n\binom{n-1}{\ell}({\scriptstyle const}\,\Gamma)^{\ell}\beta^{m+n}\rho_{m;n}\|w^{\sim}_{m,n}\|^{\sim}\\
&= {\scriptstyle const}\,Y\mathfrak{e}_0(X)\sum_{m,n\ge 0}n(1+{\scriptstyle const}\,\Gamma)^{n-1}\beta^{m+n}\rho_{m;n}\|w^{\sim}_{m,n}\|^{\sim}\,.
\end{aligned}
$$

If ${\scriptstyle const}\,\Gamma\le\frac{1}{3}$,

$$
n(1+{\scriptstyle const}\,\Gamma)^{n-1}\le\left(\frac{3}{2}\right)^n(1+{\scriptstyle const}\,\Gamma)^{n-1}\le 2^n
$$

and

$$
N_0^{\sim}\left(\frac{d}{ds}\mathcal{W}'_s\Big|_{s=0};\beta;X,\boldsymbol{\rho}\right)\le {\scriptstyle const}\,Y N_0^{\sim}(\mathcal{W};2\beta;X,\boldsymbol{\rho})\,. \qquad\square
$$

In [7, Definition XIII.9], we shall amputate a Grassmann function by applying the Fourier transform $\hat{A}$, in the sense of Definition IX.4, of $A(k)=ik_0-e(\mathbf{k})$ to its external arguments. Precisely, if $\mathcal{W}(\phi,\psi)$ is a Grassmann function, then

$$
\mathcal{W}^a(\phi,\psi)=\mathcal{W}(\hat{A}\phi,\psi)
$$

where

$$
(\hat{A}\phi)(\xi)=\int d\xi'\,\hat{A}(\xi,\xi')\phi(\xi')\,.
$$

If $C(\xi,\xi')$ is the covariance associated to $C(k)$ in the sense of Definition IX.3 and $J$ is the particle hole swap operator of (VI.1), then, by parts (i) and (ii) of Lemma IX.5,

$$
\int d\eta d\eta'\,C(\xi,\eta)J(\eta,\eta')\hat{A}(\eta',\xi)=\hat{E}(\xi,\xi')
$$

where $\hat{E}$ is the Fourier transform of $(ik_0-e(\mathbf{k}))C(k)$ in the sense of Definition IX.4.

**Theorem X.12.** *Fix $j_0\ge 1$ and set, for $\delta e\in\mathcal{E}_\mu$,*

$$
C_0(k;\delta e)=\frac{U(\mathbf{k})-\nu^{(>j_0)}(k)}{\imath k_0-e(\mathbf{k})+\delta e(\mathbf{k})}\,.
$$

*Let $C_0(\delta e)$ be the Fourier transform of $C_0(k;\delta e)$ in the sense of Definition* IX.3. *Then there are* ($M$ *and* $j_0$*-dependent*) *constants* $\beta_0$, $\varepsilon_0$, const *and* $\mu > 0$ *such that, for all* $\beta \geq \beta_0$ *and* $\varepsilon, \varepsilon' \leq \varepsilon_0$, *the following holds*:

> *Choose a system* $(\rho_{m;n})_{m,n\in\mathbb{N}_0}$, *of positive real numbers obeying* $\rho_{m;n-1} \leq \rho_{m;n}$, $\rho_{m+1;n-1} \leq \varepsilon' \rho_{m;n}$ *and* $\rho_{m+m';n+n'-2} \leq \rho_{m;n}\rho_{m';n'}$. *Let* $X \in \mathfrak{N}_{d+1}$ *with* $X_{\mathbf{0}} < \frac{1}{4}$. *For all* $\delta e \in \mathcal{E}_\mu$ *with* $\|\delta\hat{e}\|_{1,\infty} \leq X$ *and all even Grassmann functions* $\mathcal{V}(\psi)$ *with* $N_0^\sim(\mathcal{V}; 32\beta; X, \boldsymbol{\rho}) \leq \varepsilon \mathfrak{e}_0(X)$,
>
> $$\tilde{\mathcal{W}}(\phi,\psi;\delta e) = \tilde{\Omega}_{C_0(\delta e)}(\mathcal{V})(\phi,\psi) - \mathcal{V}(\psi) - \frac{1}{2}\phi J C_0(\delta e) J\phi$$
>
> *obeys*
>
> $$N_0^\sim(\tilde{\mathcal{W}}^a(\phi,\psi;\delta e); \beta; X, \boldsymbol{\rho}) \leq \varepsilon\left(\frac{1}{\beta} + \sqrt{\varepsilon'}\right)\mathfrak{e}_0(X)$$
>
> *and*
>
> $$N_0^\sim\left(\frac{d}{ds}\tilde{\mathcal{W}}^a(\phi,\psi;\delta e + s\delta e')\Big|_{s=0}; \beta; X, \boldsymbol{\rho}\right) \leq \varepsilon\left(\frac{1}{\beta} + \sqrt{\varepsilon'}\right)\mathfrak{e}_0(X)\|\delta\hat{e}'\|_{1,\infty}\,.$$

**Proof.** The proof of this theorem is similar that of Theorem VIII.6. Let $V_{\mathrm{ext}}$ be the vector space generated by $\phi(\eta)$, $\eta \in \mathcal{B}$, and recall from Sec. II that $V$ is the vector space generated by $\psi(\xi)$, $\xi \in \mathcal{B}$. Set $\tilde{V} = V_{\mathrm{ext}} \oplus V$ and $\tilde{A} = \mathbb{C}$. Then

$$\tilde{A} \otimes \tilde{V}^{\otimes n} = \tilde{V}^{\otimes n} = \bigoplus_{\substack{m_1+n_1+\cdots+m_r+n_r=n\\ n_1,m_2,\ldots,n_{r-1},m_r\geq 1}} V_{\mathrm{ext}}^{\otimes m_1} \otimes V^{\otimes n_1} \otimes \cdots \otimes V_{\mathrm{ext}}^{\otimes m_r} \otimes V^{\otimes n_r}\,.$$

Every element $F_{m_1,n_1,\ldots,m_r,n_r}$ of $V_{\mathrm{ext}}^{\otimes m_1} \otimes V^{\otimes n_1} \otimes \cdots \otimes V_{\mathrm{ext}}^{\otimes m_r} \otimes V^{\otimes n_r}$ can be uniquely written in the form

$$\begin{aligned} F_{m_1,n_1,\ldots,m_r,n_r} = \int & d\eta_1 \cdots d\eta_{m_1+\cdots+m_r} d\xi_1 \cdots d\xi_{n_1+\cdots+n_r} \\ & \times f_{m_1,n_1,\ldots,m_r,n_r}(\eta_1,\ldots,\eta_{m_1+\cdots+m_r};\xi_1,\ldots,\xi_{n_1+\cdots+n_r}) \\ & \times \phi(\eta_1)\otimes\cdots\otimes\phi(\eta_{m_1})\otimes\psi(\xi_1)\otimes\cdots\otimes\psi(\xi_{n_1}) \\ & \otimes\cdots\otimes\psi(\xi_{n_1+\cdots+n_{r-1}+1})\otimes\cdots\otimes\psi(\xi_{n_1+\cdots+n_r})\,. \end{aligned}$$

We define

$$|F_{m_1,n_1,\ldots,m_r,n_r}|^\sim = \rho_{m_1+\cdots+m_r;n_1+\cdots n_r}\|f^\sim_{m_1,n_1,\ldots,m_r,n_r}\|^\sim$$

and for

$$F = \sum_{\substack{m_1+n_1+\cdots+m_r+n_r=n\\ n_1,m_2,\ldots,n_{r-1},m_r\geq 1}} F_{m_1,n_1,\ldots,m_r,n_r} \in \tilde{V}^{\otimes n}$$

with $F_{m_1,n_1,\dots,m_r,n_r} \in V_{\rm ext}^{\otimes m_1} \otimes V^{\otimes n_1} \otimes \cdots \otimes V_{\rm ext}^{\otimes m_r} \otimes V^{\otimes n_r}$

$$|F|^\sim = \sum_{m_1,n_1,\dots,m_r,n_r} |F_{m_1,n_1,\dots,m_r,n_r}|^\sim .$$

Define the covariance $\tilde C$ on $\tilde V$ by

$$\tilde C(\phi(\eta),\phi(\eta')) = 0$$

$$\tilde C(\phi(\eta),\psi(\xi)) = 0$$

$$\tilde C(\psi(\xi),\psi(\xi')) = C_0(\xi,\xi';\delta e) .$$

The restriction of $\tilde C$ to $V$ coincides with the covariance on $V$ determined by $C_0(\delta e)$ as at the beginning of Sec. II, while $V_{\rm ext}$ is isotropic and perpendicular to $V$ with respect to $\tilde C$.

We have already observed, in the proof of Theorem VIII.6, that there is a constant $\mathrm{const}_1$ such that $S(C_0(\delta e)) \le \mathrm{const}_1$ and $\|C_0(\delta e)\|_{1,\infty} \le \frac14 \mathrm{const}_1\, \mathfrak{e}_0(\|\delta\hat e\|_{1,\infty})$. Let $C_0^a(\delta e)$ be the covariance associated with $\frac{U(\mathbf{k})-\nu^{(>j_0)}(k)}{\imath k_0 - e(\mathbf{k}) + \delta e(\mathbf{k})}(\imath k_0 - e(\mathbf{k}))$. Then

$$\|C_0^a(k;\delta e)\|^\sim \le \mathrm{const}\,\|C_0(\delta e)\|_{1,\infty} \le \mathrm{const}_1\, \mathfrak{e}_0(\|\delta\hat e\|_{1,\infty}) .$$

By Lemma X.10, $\mathrm{b} = 4\,\mathrm{const}_1$ is an integral bound for $C_0(\delta e)$, and hence for $\tilde C$, and $\mathfrak{c} = \mathrm{const}_1\, \mathfrak{e}_0(X)$ is a contraction bound for $C_0(\delta e)$, and hence for $\tilde C$. Furthermore, Lemma X.11(i), with $C = C_0^a(\delta e)$, $X = \|\delta\hat e\|_{1,\infty}$ and $\Gamma = \varepsilon'\mathrm{b}$ is applicable.

Set $\alpha = \frac{\beta}{\mathrm{b}}$. For any Grassmann function $\mathcal{W}(\phi,\psi) = \sum_{m,n} \mathcal{W}_{m,n}(\phi,\psi)$, with $\mathcal{W}_{m,n}(\phi,\psi)$ of degree $m$ and $n$ in $\phi$ and $\psi$, respectively, let

$$N^\sim(\mathcal{W};\mathfrak{c},\mathrm{b},\alpha) = \frac{1}{\mathrm{b}^2}\mathfrak{c}\sum_{m,n} \alpha^{m+n}\mathrm{b}^{m+n}|W_{m,n}^\sim|^\sim$$

be the norm of Definition III.9 and of [1, Definition II.23], but with $V$ replaced by $\tilde V$ and $A$ replaced by $\tilde A = \mathbb{C}$. Then $N^\sim(\mathcal{W};\mathfrak{c},\mathrm{b},\alpha) = \frac{1}{4\mathrm{b}} N_0^\sim(\mathcal{W};\beta;X,\boldsymbol{\rho})$ and, if $\mathcal{V} = {:}\mathcal{V}'{:}_{C_0(\delta e)}$,

$$N^\sim(\mathcal{V}';\mathfrak{c},\mathrm{b},\alpha) \le \frac{1}{4\mathrm{b}} N_0^\sim(\mathcal{V};2\beta;X,\boldsymbol{\rho})$$

$$N^\sim(\mathcal{V}'-\mathcal{V};\mathfrak{c},\mathrm{b},\alpha) \le \frac{\mathrm{b}}{2\beta^2} N_0^\sim(\mathcal{V};2\beta;X,\boldsymbol{\rho})$$

by [1, Corollary II.32].

To prove the first part of the theorem, set $\mathcal{V} = {:}\mathcal{V}'{:}_{C_0(\delta e)}$. Then the hypotheses of Theorem III.10 with $C = C_0(\delta e)$ and $\mathcal{W} = \mathcal{V}'$ are fulfilled. Therefore

$$\begin{aligned} &\frac{1}{4\mathrm{b}} N_0^\sim(\Omega_{C_0(\delta e)}(\mathcal{V}) - \mathcal{V};\beta;X,\boldsymbol{\rho}) \\ &\quad \le N^\sim(\Omega_{C_0(\delta e)}({:}\mathcal{V}'{:}_{C_0(\delta e)}) - \mathcal{V}';\mathfrak{c},\mathrm{b},\alpha) + N^\sim(\mathcal{V}'-\mathcal{V};\mathfrak{c},\mathrm{b},\alpha) \end{aligned}$$

$$\le \frac{2}{\alpha^2}\frac{N^\sim(\mathcal{V}';\mathfrak{c},\mathrm{b},8\alpha)^2}{1-\frac{4}{\alpha^2}N^\sim(\mathcal{V}';\mathfrak{c},\mathrm{b},8\alpha)} + N^\sim(\mathcal{V}'-\mathcal{V};\mathfrak{c},\mathrm{b},\alpha)$$

$$\le \frac{2}{\alpha^2}\frac{\frac{1}{16\mathrm{b}^2}N_0^\sim(\mathcal{V};16\beta;X,\boldsymbol{\rho})^2}{1-\frac{\mathrm{b}}{\beta^2}N_0^\sim(\mathcal{V};16\beta;X,\boldsymbol{\rho})} + \frac{\mathrm{b}}{2\beta^2}N_0^\sim(\mathcal{V};2\beta;X,\boldsymbol{\rho})$$

$$\le \frac{\varepsilon^2}{8\beta^2}\frac{\mathfrak{e}_0(X)^2}{1-\frac{\varepsilon\mathrm{b}}{\beta^2}\mathfrak{e}_0(X)} + \frac{\varepsilon\mathrm{b}}{2\beta^2}\mathfrak{e}_0(X)$$

$$\le \frac{\varepsilon\mathrm{b}}{\beta^2}\mathfrak{e}_0(X)$$

since $\varepsilon_0$ is chosen so that $\frac{\varepsilon\mathrm{b}}{\beta^2} < \frac{1}{4}$ and, by Corollary A.5(ii), $\frac{\mathfrak{e}_0^2(X)}{1-\frac{1}{4}\mathfrak{e}_0(X)} \le \mathit{const}\,\mathfrak{e}_0(X)$. Observe that $\mathcal{V}$ and hence $\Omega_{C_0(\delta e)}(\mathcal{V})$ are independent of $\phi$ and consequently are not affected by amputation. So, by Lemmas VII.3 and X.11,

$$\begin{aligned}
&N_0^\sim(\tilde{\mathcal{W}}^a(\phi,\psi;\delta e);\beta;X,\boldsymbol{\rho})\\
&\quad= N_0^\sim(\Omega_{C_0(\delta e)}(\mathcal{V})(\phi,\psi+C_0^a(\delta e)J\phi)-\mathcal{V}(\psi);\beta;X,\boldsymbol{\rho})\\
&\quad\le N_0^\sim(\Omega_{C_0(\delta e)}(\mathcal{V})(\phi,\psi+C_0^a(\delta e)J\phi)-\Omega_{C_0(\delta e)}(\mathcal{V})(\phi,\psi);\beta;X,\boldsymbol{\rho})\\
&\qquad+N_0^\sim(\Omega_{C_0(\delta e)}(\mathcal{V})(\phi,\psi)-\mathcal{V}(\psi);\beta;X,\boldsymbol{\rho})\\
&\quad\le \mathit{const}\,\varepsilon'\mathrm{b}N_0^\sim(\Omega_{C_0(\delta e)}(\mathcal{V});2\beta;X,\boldsymbol{\rho}) + \frac{4\varepsilon\mathrm{b}^2}{\beta^2}\mathfrak{e}_0(X)\\
&\quad\le \mathit{const}\,\varepsilon'\mathrm{b}[N_0^\sim(\Omega_{C_0(\delta e)}(\mathcal{V})-\mathcal{V};2\beta;X,\boldsymbol{\rho})+N_0^\sim(\mathcal{V};2\beta;X,\boldsymbol{\rho})] + \frac{4\varepsilon\mathrm{b}^2}{\beta^2}\mathfrak{e}_0(X)\\
&\quad\le \mathit{const}\,\varepsilon'\mathrm{b}\left[\frac{\varepsilon\mathrm{b}^2}{\beta^2}\mathfrak{e}_0(X)+\varepsilon\mathfrak{e}_0(X)\right] + \frac{4\varepsilon\mathrm{b}^2}{\beta^2}\mathfrak{e}_0(X)\\
&\quad\le \mathit{const}\,(1+\mathrm{b})^3\varepsilon\left(\frac{1}{\beta^2}+\varepsilon'\right)\mathfrak{e}_0(X) \le \varepsilon\left(\frac{1}{\beta}+\sqrt{\varepsilon'}\right)\mathfrak{e}_0(X)\,.
\end{aligned}$$

Finally, we prove the bound on $\frac{d}{ds}\tilde{\mathcal{W}}^a(\phi,\psi;\delta e+s\delta e')|_{s=0}$. As

$$\frac{d}{ds}C_0(k;\delta e+s\delta e')\Big|_{s=0} = -\frac{U(\mathbf{k})-\nu^{(>j_0)}(k)}{[\imath k_0-e(\mathbf{k})+\delta e(\mathbf{k})]^2}\delta e'(\mathbf{k})$$

$$\frac{d}{ds}C_0^a(k;\delta e+s\delta e')\Big|_{s=0} = -\frac{U(\mathbf{k})-\nu^{(>j_0)}(k)}{[\imath k_0-e(\mathbf{k})+\delta e(\mathbf{k})]^2}\delta e'(\mathbf{k})(\imath k_0-e(\mathbf{k}))\,.$$

Propositions IV.3(i) and IV.11 give that

$$S\left(\frac{d}{ds}C_0(\delta e+s\delta e')\Big|_{s=0}\right)\le \mathrm{const}_1\sqrt{|||\delta\hat e'|||_{1,\infty}}$$

$$\left|\left|\left|\frac{d}{ds}C_0(\delta e+s\delta e')\Big|_{s=0}\right|\right|\right|_\infty \le \mathrm{const}_1|||\delta\hat e'|||_{1,\infty}$$

$$\left\|\frac{d}{ds}C_0(\delta e+s\delta e')\Big|_{s=0}\right\|_{1,\infty}\le \frac{1}{4}\,\mathrm{const}_1\,\mathfrak{e}_0(\|\delta\hat e\|_{1,\infty})\|\delta\hat e'\|_{1,\infty}$$

$$\left\|\frac{d}{ds}C_0^a(k;\delta e+s\delta e')\Big|_{s=0}\right\|^\sim \le \mathrm{const}_1\,\mathfrak{e}_0(\|\delta\hat e\|_{1,\infty})\|\delta\hat e'\|_{1,\infty}\,.$$

Set $\mathrm{b}'=4\,\mathrm{const}_1\sqrt{|||\delta\hat e'|||_{1,\infty}}$ and $\mathfrak{c}'=\mathrm{const}_1\,\mathfrak{e}_0(X)\|\delta\hat e'\|_{1,\infty}$. By Lemma X.10, $\frac12\mathrm{b}'$ is an integral bound for $\frac{d}{ds}C_0(\delta e+s\delta e')|_{s=0}$ and $\mathfrak{c}'$ is a contraction bound for $\frac{d}{ds}C_0(\delta e+s\delta e')|_{s=0}$. Furthermore, Lemma X.11(ii), with $C=C_0^a(\delta e)$, $C'=\frac{d}{ds}C_0^a(\delta e+s\delta e')|_{s=0}$, $X=\|\delta\hat e\|_{1,\infty}$, $\Gamma=\mathrm{const}_1\,\varepsilon'$ and $Y=\varepsilon'\mathrm{b}\|\delta\hat e'\|_{1,\infty}$ is applicable.

Define $C_\kappa=C_0(\delta e+\kappa\delta e')$ and $\mathcal{W}_\kappa$ by $:\mathcal{W}_\kappa:_{C_\kappa}=\mathcal{V}$. Even when $\alpha$ is replaced by $2\alpha$, the hypotheses of [1, Lemmas IV.5(i) and IV.7(i)], with $\mu=1$, are satisfied. By these two lemmas, followed by [1, Corollary II.32(iii)],

$$\begin{aligned}
&N^\sim\left(\frac{d}{ds}\Omega_{C_s}(\mathcal{V})\Big|_{s=0};\mathfrak{c},\mathrm{b},\alpha\right)\\
&\quad= N^\sim\left(\frac{d}{ds}\Omega_{C_s}(:\mathcal{W}_s:_{C_s})\Big|_{s=0};\mathfrak{c},\mathrm{b},\alpha\right)\\
&\quad\le N^\sim\left(\frac{d}{ds}\Omega_{C_s}(:\mathcal{V}':_{C_s})\Big|_{s=0};\mathfrak{c},\mathrm{b},\alpha\right)+N^\sim\left(\frac{d}{ds}\Omega_{C_0}(:\mathcal{W}_s:_{C_0})\Big|_{s=0};\mathfrak{c},\mathrm{b},\alpha\right)\\
&\quad\le \frac{1}{2\alpha^2}\frac{N^\sim(\mathcal{V}';\mathfrak{c},\mathrm{b},8\alpha)^2}{1-\frac{4}{\alpha^2}N^\sim(\mathcal{V}';\mathfrak{c},\mathrm{b},8\alpha)}\,\mathrm{const}_1\,\mathfrak{e}_0(X)\|\delta\hat e'\|_{1,\infty}\\
&\qquad+\left\{1+\frac{2}{\alpha^2}\frac{N^\sim(\mathcal{V}';\mathfrak{c},\mathrm{b},8\alpha)}{1-\frac{4}{\alpha^2}N^\sim(\mathcal{V}';\mathfrak{c},\mathrm{b},8\alpha)}\right\}N^\sim\left(\frac{d}{ds}\mathcal{W}_s\Big|_{s=0};\mathfrak{c},\mathrm{b},2\alpha\right)\\
&\quad\le \frac{1}{2\alpha^2}\frac{N^\sim(\mathcal{V}';\mathfrak{c},\mathrm{b},8\alpha)^2}{1-\frac{4}{\alpha^2}N^\sim(\mathcal{V}';\mathfrak{c},\mathrm{b},8\alpha)}\,\mathrm{const}_1\,\mathfrak{e}_0(X)\|\delta\hat e'\|_{1,\infty}\\
&\qquad+\left\{1+\frac{2}{\alpha^2}\frac{N^\sim(\mathcal{V}';\mathfrak{c},\mathrm{b},8\alpha)}{1-\frac{4}{\alpha^2}N^\sim(\mathcal{V}';\mathfrak{c},\mathrm{b},8\alpha)}\right\}\frac{|||\delta\hat e'|||_{1,\infty}}{(2\alpha-1)^2}N^\sim(\mathcal{V};\mathfrak{c},\mathrm{b},4\alpha)\\
&\quad\le \mathrm{const}\,\frac{\varepsilon}{\alpha^2}\mathfrak{e}_0(X)\|\delta\hat e'\|_{1,\infty}
\end{aligned}$$

as above. By Lemmas VII.3 and X.11,

$$N_0^{\sim}\left(\frac{d}{ds}\tilde{\mathcal{W}}^a(\phi,\psi;\delta e+s\delta e')\Big|_{s=0};\beta;X,\boldsymbol{\rho}\right)$$

$$= N_0^{\sim}\left(\frac{d}{ds}\Omega_{C_0(\delta e+s\delta e')}(\mathcal{V})(\phi,\psi+C_0^a(\delta e+s\delta e')J\phi)\Big|_{s=0};\beta;X,\boldsymbol{\rho}\right)$$

$$\le N_0^{\sim}\left(\frac{d}{ds}\Omega_{C_0(\delta e+s\delta e')}(\mathcal{V})(\phi,\psi+C_0^a(\delta e)J\phi)\Big|_{s=0};\beta;X,\boldsymbol{\rho}\right)$$

$$+ N_0^{\sim}\left(\frac{d}{ds}\Omega_{C_0(\delta e)}(\mathcal{V})(\phi,\psi+C_0^a(\delta e+s\delta e')J\phi)\Big|_{s=0};\beta;X,\boldsymbol{\rho}\right)$$

$$\le (1+{\scriptstyle const}\,\varepsilon'\mathrm{b})N_0^{\sim}\left(\frac{d}{ds}\Omega_{C_0(\delta e+s\delta e')}(\mathcal{V})\Big|_{s=0};2\beta;X,\boldsymbol{\rho}\right)$$

$$+ {\scriptstyle const}\,\varepsilon'\mathrm{b}N_0^{\sim}(\Omega_{C_0(\delta e)}(\mathcal{V});2\beta;X,\boldsymbol{\rho})\|\delta\hat{e}'\|_{1,\infty}$$

$$\le \mathrm{const}\,\frac{\varepsilon}{\beta^2}\mathfrak{e}_0(X)\|\delta\hat{e}'\|_{1,\infty}+\mathrm{const}\,\varepsilon\varepsilon'\mathfrak{e}_0(X)\|\delta\hat{e}'\|_{1,\infty}$$

$$\le \varepsilon\left(\frac{1}{\beta}+\sqrt{\varepsilon'}\right)\mathfrak{e}_0(X)\|\delta\hat{e}'\|_{1,\infty}$$

as above. □

## Appendices

### B. *Symmetries*

**Definition B.1 (Symmetries).** Let $x=(x_0,\mathbf{x},\sigma)\in\mathbb{R}\times\mathbb{R}^d\times\{\uparrow,\downarrow\}$, $\xi=(x,a)\in\mathcal{B}=(\mathbb{R}\times\mathbb{R}^d\times\{\uparrow,\downarrow\})\times\{0,1\}$ and $t=(t_0,\mathbf{t})\in\mathbb{R}\times\mathbb{R}^d$. We set

$$\begin{aligned} x+t &= (x_0+t_0,\mathbf{x}+\mathbf{t},\sigma) & \xi+t &= (x+t,a)\\ R_0x &= (-x_0,\mathbf{x},\sigma) & R_0\xi &= (R_0x,a)\\ -x &= (-x_0,-\mathbf{x},\sigma) & -\xi &= (-x,a)\,. \end{aligned}$$

(T) A function $f(\xi_1,\ldots,\xi_n)$ on $\mathcal{B}^n$ is called **translation invariant** if, for all $t\in\mathbb{R}\times\mathbb{R}^d$,

$$f(\xi_1+t,\ldots,\xi_n+t)=f(\xi_1,\ldots,\xi_n)\,.$$

In the same way, one defines translation invariance for functions on $(\mathbb{R}\times\mathbb{R}^d\times\{\uparrow,\downarrow\})^n$.

(N) A function $f$ on $\mathcal{B}^n$ **conserves particle number** if $f((x_1,a_1),\ldots,(x_n,a_n))=0$ unless

$$\#\{j|a_j=0\}=\#\{j|a_j=1\}=\frac{n}{2}\,.$$

(S) Let $f$ be a function on $\mathcal{B}^n$. Set, for each $A \in SU(2)$,

$$f^A((\cdot,\sigma_1,b_1),\ldots,(\cdot,\sigma_n,b_n)) = \sum_{\tau_1,\ldots,\tau_n} f((\cdot,\tau_1,b_1),\ldots,(\cdot,\tau_n,b_n)) \prod_{j=1}^n A^{(b_j)}_{\tau_j,\sigma_j}$$

where $A^{(0)} = A$ and $A^{(1)} = \bar{A}$. $f$ is called **spin independent** if $f = f^A$ for all $A \in SU(2)$.

(R) A function $f(\xi_1,\ldots,\xi_n)$ on $\mathcal{B}^n$ is called $k_0$**-reversal real** if

$$f(R_0\xi_1,\ldots,R_0\xi_n) = \overline{f(-\xi_1,\ldots,-\xi_n)}$$

or, equivalently, if its Fourier transform obeys

$$\check{f}(R_0\check{\xi}_1,\ldots,R_0\check{\xi}_n) = \overline{\check{f}(\check{\xi}_1,\ldots,\check{\xi}_n)}$$

where $R_0(k_0,\mathbf{k},\sigma,a) = (-k_0,\mathbf{k},\sigma,a)$.

(B) A function $f(\xi_1,\ldots,\xi_n)$ on $\mathcal{B}^n$ is called **bar/unbar exchange invariant** if

$$f((x_1,1-b_1),\ldots,(x_n,1-b_n)) = i^n f((-x_1,b_1),\ldots,(-x_n,b_n))$$

or, equivalently, if its Fourier transform obeys

$$\check{f}((k_1,\sigma_1,1-b_1),\ldots,(k_n,\sigma_n,1-b_n)) = i^n \check{f}((k_1,\sigma_1,b_1),\ldots,(k_n,\sigma_n,b_n))\,.$$

(vi) Let $m \in \mathbb{N}$ and $\Sigma_1,\ldots,\Sigma_m \in \{\mathrm{B},\mathrm{N},\mathrm{R},\mathrm{S},\mathrm{T}\}$. Then $f$ is $\Sigma_1\cdots\Sigma_m$-symmetric if $f$ satisfies part $(\Sigma_i)$ of this definition for $1 \le i \le m$.

(vii) Let $\Sigma \in \{\mathrm{B},\mathrm{N},\mathrm{R},\mathrm{S},\mathrm{T}\}$. A Grassmann function

$$\mathcal{W}(\phi,\psi) = \sum_{m,n} W_{m,n}\phi^m\psi^n$$

is $\Sigma$-symmetric if all of the coefficient functions $W_{m,n}$ are.

**Remark B.2.** Let $\mathcal{W}(\phi,\psi) = \sum_{m,n} W_{m,n}\phi^m\psi^n$ be a Grassmann function. $\mathcal{W}(\phi,\psi)$ is translation invariant if and only if it is invariant under

$$\phi(\xi) \to \phi(\xi+t)\,,\quad \psi(\xi) \to \psi(\xi+t)\quad \text{for all } t \in \mathbb{R}\times\mathbb{R}^d\,.$$

$\mathcal{W}(\phi,\psi)$ conserves particle number if and only if it is invariant under

$$\phi(x,a) \to e^{i(-1)^a\theta}\phi(x,a)\,,\quad \psi(x,a) \to e^{i(-1)^a\theta}\psi(x,a)\quad \text{for all } \theta \in \mathbb{R}\,.$$

$\mathcal{W}(\phi,\psi)$ is spin independent if and only if it is invariant under

$$\phi(\cdot,\sigma,a) \to \sum_{\tau\in\{\uparrow,\downarrow\}} A^{(a)}_{\sigma,\tau}\phi(\cdot,\tau,a)\,,$$

$$\psi(\cdot,\sigma,a) \to \sum_{\tau\in\{\uparrow,\downarrow\}} A^{(a)}_{\sigma,\tau}\psi(\cdot,\tau,a)\quad \text{for all } A \in SU(2)\,.$$

$\mathcal{W}(\phi,\psi)$ is bar/unbar exchange invariant if and only if it is invariant under

$$\phi(x,a) \to i\phi(-x,1-a)\,,\quad \psi(x,a) \to i\psi(-x,1-a)$$

or, equivalently, under

$$\check\phi(k,\sigma,a)\to i\check\phi(k,\sigma,1-a)\,,\quad \check\psi(k,\sigma,a)\to i\check\psi(k,\sigma,1-a)\,.$$

Define $\bar{\mathcal{W}}(\phi,\psi)=\sum_{m,n}\overline{W_{m,n}}\phi^m\psi^n$ and

$$(R_0\psi)(\xi)=\psi(R_0\xi)\qquad (R\psi)(\xi)=\psi(-\xi)\,.$$

$\mathcal{W}(\phi,\psi)$ is $k_0$-reversal real if and only if

$$\mathcal{W}(R_0\phi,R_0\psi)=\bar{\mathcal{W}}(R\phi,R\psi)\,.$$

**Remark B.3.** (i) If the function $f(\xi_1,\xi_2)$ on $\mathcal{B}^2$ is NS-symmetric, then it is of the form

$$f((x,\sigma,a),(y,\tau,b))=\delta_{\sigma,\tau}\tilde f((x,a),(y,b))$$

for some $\tilde f$.

(ii) If the function $f(\xi_1,\xi_2)$ on $\mathcal{B}^2$ is antisymmetric and NST-symmetric, then, translating by $-x-y$,

$$f((x,\sigma,0),(y,\sigma,1))=f((-y,\sigma,0),(-x,\sigma,1))=-f((-x,\sigma,1),(-y,\sigma,0))$$

so that $f$ is also B-symmetric.

(iii) If the function $f(\xi_1,\xi_2)$ on $\mathcal{B}^2$ is antisymmetric and NST-symmetric and if $\check f(k)$ is its Fourier transform specified at the end of Definition IX.1(i), then

$$f=J\hat{\check f}\,.$$

The operator $J$ was defined in (VI.1) and for a function $\chi(k)$, the Fourier transform $\hat\chi(\xi,\xi')$ was defined in Definition IX.4. If, in addition, $f$ is $k_0$-reversal real, then $\check f(-k_0,\mathbf{k})=\overline{\check f(k_0,\mathbf{k})}$.

(iv) If

$$\mathcal{V}(\psi,\bar\psi)=\int_{(\mathbb{R}\times\mathbb{R}^d\times\{\uparrow,\downarrow\})^4}V_0(x_1,x_2,x_3,x_4)\bar\psi(x_1)\psi(x_2)\bar\psi(x_3)\psi(x_4)\,dx_1dx_2dx_3dx_4$$

with

$$V_0((x_{1,0},\mathbf{x}_1,\sigma_1),\ldots,(x_{4,0},\mathbf{x}_4,\sigma_4))=-\frac{1}{2}\delta(x_1,x_2)\delta(x_3,x_4)v(x_{1,0}-x_{3,0},\mathbf{x}_1-\mathbf{x}_3)$$

where $\delta((x_0,\mathbf{x},\sigma),(x_0',\mathbf{x}',\sigma'))=\delta(x_0-x_0')\delta(\mathbf{x}-\mathbf{x}')\delta_{\sigma,\sigma'}$, then $\mathcal{V}$ is BNST-symmetric. If in addition $v(x_0,-\mathbf{x})=\overline{v(x_0,\mathbf{x})}$, then $\mathcal{V}$ is also R-symmetric. This is the case if $v(x_1-x_3)=\delta(x_{1,0}-x_{3,0})\mathbf{v}(\mathbf{x}_1-\mathbf{x}_3)$ with $\mathbf{v}$ having a real-valued Fourier transform.

**Remark B.4.** $\phi J\psi$ is BNRST-symmetric. If $C(\xi,\xi')$ is the covariance associated to a function $C(k)$ as in Definition IX.3, then $C(\xi,\xi')$ is antisymmetric and BNST-symmetric. If $C(-k_0,\mathbf{k})=\overline{C(k_0,\mathbf{k})}$ then $C(\xi,\xi')$ is R-symmetric.

**Remark B.5.** Assume that $C(\xi,\xi')$ is the covariance associated to the function $C(k)$ as in Definition IX.3 and that $\mathcal{W}(\phi,\psi)$ is a Grassmann function. Let

$\Sigma \in \{\text{BNST}\}$. If $\mathcal{W}$ is $\Sigma$-symmetric, then $\int \mathcal{W}(\phi,\psi)d\mu_C(\psi)$, $\tilde{\Omega}_C(\mathcal{W})$ and $\Omega_C(\mathcal{W})$ are too. If $C(-k_0,\mathbf{k}) = \overline{C(k_0,\mathbf{k})}$, then $\int \mathcal{W}(\phi,\psi)d\mu_C(\psi)$, $\Omega_C(\mathcal{W})$ and $\tilde{\Omega}_C(\mathcal{W})$ are R-symmetric.

**Lemma B.6.** *Let* $W(\eta,\xi)$ *be* BNST-*symmetric. Then*

$$\int d\xi d\eta \psi(\xi)(J(\check{W})\hat{\ })(\xi,\eta)\phi(\eta) = \int d\eta d\xi \phi(\eta) W(\eta,\xi)\psi(\xi)\,.$$

**Proof.** The bar/unbar invariance implies

$$\begin{aligned} &\check{W}((k,\sigma,a),(k,\sigma,1-a)) = -\check{W}((k,\sigma,1-a),(k,\sigma,a)) \\ &\quad \Longrightarrow \check{W}((k,\sigma,a),(k,\sigma,1-a)) = -(-1)^a \check{W}((k,\sigma,1),(k,\sigma,0)) \\ &\quad = -(-1)^a \check{W}(k) \end{aligned}$$

for an arbitrary spin $\sigma$. Hence, if $x$ and $y$ both have spin component $\sigma$,

$$\begin{aligned} (J(\check{W})\hat{\ })&((x,1-a),(y,a)) = (-1)^a (\check{W})\hat{\ }((x,a),(y,a)) \\ &= (-1)^a \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{(-1)^a \imath\langle k,x-y\rangle_-} \check{W}(k) \\ &= -\int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{(-1)^a \imath\langle k,x-y\rangle_-} \check{W}((k,\sigma,a),(k,\sigma,1-a)) \\ &= -\int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{d^{d+1}k'}{(2\pi)^{d+1}} e^{-(-1)^a \imath\langle k,y\rangle_-} e^{-(-1)^{1-a} \imath\langle k',x\rangle_-} \\ &\quad \times \check{W}((k,\sigma,a),(k',\sigma,1-a))(2\pi)^{d+1}\delta(k-k') \\ &= -W((y,a),(x,1-a))\,. \end{aligned}$$

The lemma follows. ∎

## C. *Some standard Grassmann integral formulae*

For a function $C(\xi,\xi')$ on $\mathcal{B}\times\mathcal{B}$ we set, as in Sec. VII

$$\psi\phi = \int \psi(\xi)\phi(\xi)d\xi$$

$$C\phi = \int C(\xi,\xi')\phi(\xi')d\xi'$$

$$\phi C\psi = \int \phi(\xi)C(\xi,\xi')\psi(\xi')d\xi d\xi'\,.$$

**Lemma C.1.** *Let $f(\psi)$ be a Grassmann function and let $C$ be an arbitrary antisymmetric covariance. Then*

$$\int f(\psi)e^{\psi\phi}d\mu_C(\psi) = e^{-\frac{1}{2}\phi C\phi}\int f(\psi + C\phi)d\mu_C(\psi)\,.$$

**Proof.** It suffices to consider $f(\psi) = e^{\psi\zeta}$ with $\zeta$ another Grassmann field. Then comparing

$$\int f(\psi)e^{\psi\phi}d\mu_C(\psi) = \int e^{\psi(\zeta+\phi)}d\mu_C(\psi) = e^{-\frac{1}{2}(\zeta+\phi)C(\zeta+\phi)}$$

$$\int f(\psi + C\phi)d\mu_C(\psi) = \int e^{(\psi+C\phi)\zeta}d\mu_C(\psi) = e^{-\phi C\zeta}e^{-\frac{1}{2}\zeta C\zeta}$$

gives the desired result. □

**Lemma C.2.** *Let $C(\xi,\xi')$ and $U(\xi,\xi')$ be antisymmetric functions such that the norm of the integral operator with kernel $\int d\xi'' C(\xi,\xi'')U(\xi'',\xi')$ is strictly smaller than one. Let $C'(\xi,\xi')$ be the kernel of the integral operator $[\mathbb{1} - CU]^{-1}C$ and set*

$$\mathcal{U} = \int U(\xi,\xi)\psi(\xi)\psi(\xi)d\xi d\xi'$$

*and $Z = \int e^{\frac{1}{2}\mathcal{U}(\psi)}d\mu_C(\psi)$. Then,*

(a) *For all Grassmann functions $f(\psi)$*

$$\frac{1}{Z}\int f(\psi)e^{\frac{1}{2}\mathcal{U}(\psi)}d\mu_C(\psi) = \int f(\psi)d\mu_{C'}(\psi)\,.$$

(b) *For all Grassmann functions $\mathcal{W}(\psi)$*

$$\frac{1}{Z}\int e^{\mathcal{W}(\psi+\phi)}d\mu_C(\psi) = e^{\frac{1}{2}\phi U[\mathbb{1}+C'U]\phi}\int e^{(\mathcal{W}-\frac{1}{2}\mathcal{U})(\psi+[\mathbb{1}+C'U]\phi)}d\mu_{C'}(\psi)\,.$$

**Proof.** We give the proof for the case that the Grassmann algebra is finite dimensional. The general case then follows by approximation.

(a) It suffices to consider the generating functional $f(\psi) = e^{\psi\phi}$. By definition

$$\begin{aligned}\int e^{\psi\phi}e^{\frac{1}{2}\mathcal{U}(\psi)}d\mu_C(\psi) &= \int e^{\psi\phi}e^{\frac{1}{2}\psi U\psi}d\mu_C(\psi)\\ &= \mathrm{Pf}(C)\int e^{\psi\phi}e^{\frac{1}{2}\psi U\psi}e^{-\frac{1}{2}\psi C^{-1}\psi}d\psi\\ &= \mathrm{Pf}(C)\int e^{\psi\phi}e^{-\frac{1}{2}\psi C'^{-1}\psi}d\psi\\ &= \frac{\mathrm{Pf}(C)}{\mathrm{Pf}(C')}\int e^{\psi\phi}d\mu_{C'}(\psi)\end{aligned}$$

where $\mathrm{Pf}(C)$ is the Pfaffian of $C$. In particular, setting $\phi = 0$, $\frac{\mathrm{Pf}(C)}{\mathrm{Pf}(C')} = \int e^{\frac{1}{2}\mathcal{U}(\psi)} d\mu_C(\psi) = Z$.

(b) By part (a)

$$\begin{aligned}\int e^{\mathcal{W}(\psi+\phi)} d\mu_C(\psi) &= e^{\frac{1}{2}\mathcal{U}(\phi)} \int e^{\mathcal{W}(\psi+\phi)-\frac{1}{2}\mathcal{U}(\psi+\phi)+\psi U\phi+\frac{1}{2}\mathcal{U}(\psi)} d\mu_C(\psi) \\ &= \frac{\mathrm{Pf}(C)}{\mathrm{Pf}(C')} e^{\frac{1}{2}\mathcal{U}(\phi)} \int e^{\mathcal{W}(\psi+\phi)-\frac{1}{2}\mathcal{U}(\psi+\phi)+\psi U\phi} d\mu_{C'}(\psi) \\ &= \frac{\mathrm{Pf}(C)}{\mathrm{Pf}(C')} e^{\frac{1}{2}\mathcal{U}(\phi)} e^{\frac{1}{2}\phi U C' U\phi} \int e^{(\mathcal{W}-\frac{1}{2}\mathcal{U})(\psi+\phi+C'U\phi)} d\mu_{C'}(\psi)\end{aligned}$$

by Lemma C.1, with the replacements $C \to C'$ and $\phi \to U\phi = -\phi U$. □

**Remark C.3.** Recall from Lemma IX.5 that, if $C(k)$ is a function on $\mathbb{R} \times \mathbb{R}^d$ and $C(\xi, \xi')$ the associated covariance in the sense of Definition IX.3, then $C = -\widehat{C(k)}J$. Also recall from Remark B.3(iii) that if $U$ is antisymmetric, particle number conserving, translation invariant and spin independent, then $U = J\widehat{\check{U}(k)}$. In this case $C' = -\widehat{C'(k)}J$ where $C'(k) = \frac{C(k)}{1-C(k)\check{U}(k)}$.

## *Notation*

*Norms*

| Norm | Characteristics | Reference |
|---|---|---|
| $\vert\vert\vert \cdot \vert\vert\vert_{1,\infty}$ | no derivatives, external positions, acts on functions | Example II.6 |
| $\Vert \cdot \Vert_{1,\infty}$ | derivatives, external positions, acts on functions | Example II.6 |
| $\Vert \cdot \Vert_{\infty}^{\check{}}$ | derivatives, external momenta, acts on functions | Definition IV.6 |
| $\vert\vert\vert \cdot \vert\vert\vert_{\infty}$ | no derivatives, external positions, acts on functions | Example III.4 |
| $\Vert \cdot \Vert_{1}^{\check{}}$ | derivatives, external momenta, acts on functions | Definition IV.6 |
| $\Vert \cdot \Vert_{\infty,B}^{\check{}}$ | derivatives, external momenta, $B \subset \mathbb{R} \times \mathbb{R}^d$ | Definition IV.6 |
| $\Vert \cdot \Vert_{1,B}^{\check{}}$ | derivatives, external momenta, $B \subset \mathbb{R} \times \mathbb{R}^d$ | Definition IV.6 |
| $\Vert \cdot \Vert$ | $\rho_{m;n} \Vert \cdot \Vert_{1,\infty}$ | Lemma V.1 |
| $N(\mathcal{W}; \mathfrak{c}, \mathrm{b}, \alpha)$ | $\frac{1}{\mathrm{b}^2} \mathfrak{c} \sum_{m,n\geq 0} \alpha^n \mathrm{b}^n \Vert \mathcal{W}_{m,n} \Vert$ | Definition III.9<br>Theorem V.2 |
| $N_0(\mathcal{W}; \beta; X, \boldsymbol{\rho})$ | $\mathfrak{e}_0(X) \sum_{m+n\in 2\mathbb{N}} \beta^n \rho_{m;n} \Vert \mathcal{W}_{m,n} \Vert_{1,\infty}$ | Theorem VIII.6 |
| $\Vert \cdot \Vert_{L^1}$ | derivatives, acts on functions on $\mathbb{R} \times \mathbb{R}^d$ | before Lemma IX.6 |
| $\Vert \cdot \Vert^{\sim}$ | derivatives, external momenta, acts on functions | Definition X.4 |
| $N_0^{\sim}(\mathcal{W}; \beta; X, \boldsymbol{\rho})$ | $\mathfrak{e}_0(X) \sum_{m+n\in 2\mathbb{N}} \beta^{m+n} \rho_{m;n} \Vert W_{m,n}^{\sim} \Vert^{\sim}$ | before Lemma X.11 |

*Other notation*

| Notation | Description | Reference |
|---|---|---|
| $\Omega_S(\mathcal{W})(\phi,\psi)$ | $\log \frac{1}{Z}\int e^{\mathcal{W}(\phi,\psi+\zeta)} d\mu_S(\zeta)$ | before (I.6) |
| $J$ | particle/hole swap operator | (VI.1) |
| $\tilde{\Omega}_C(\mathcal{W})(\phi,\psi)$ | $\log \frac{1}{Z}\int e^{\phi J\zeta} e^{\mathcal{W}(\phi,\psi+\zeta)} d\mu_C(\zeta)$ | Definition VII.1 |
| $r_0$ | number of $k_0$ derivatives tracked | Sec. VI |
| $r$ | number of $\mathbf{k}$ derivatives tracked | Sec. VI |
| $M$ | scale parameter, $M > 1$ | before Definition VIII.1 |
| const | generic constant, independent of scale | |
| $\scriptstyle const$ | generic constant, independent of scale and $M$ | |
| $\nu^{(j)}(k)$ | $j$th scale function | Definition VIII.1 |
| $\tilde{\nu}^{(j)}(k)$ | $j$th extended scale function | Definition VIII.4(i) |
| $\nu^{(\geq j)}(k)$ | $\varphi(M^{2j-1}(k_0^2 + e(\mathbf{k})^2))$ | Definition VIII.1 |
| $\tilde{\nu}^{(\geq j)}(k)$ | $\varphi(M^{2j-2}(k_0^2 + e(\mathbf{k})^2))$ | Definition VIII.4(ii) |
| $\bar{\nu}^{(\geq j)}(k)$ | $\varphi(M^{2j-3}(k_0^2 + e(\mathbf{k})^2))$ | Definition VIII.4(iii) |
| $S(C)$ | $\sup_m \sup_{\xi_1,\ldots,\xi_m\in\mathcal{B}} \left(\left\lvert\int \psi(\xi_1)\cdots\psi(\xi_m) d\mu_C(\psi)\right\rvert\right)^{1/m}$ | Definition IV.1 |
| $\check{f}$ | Fourier transform | Definition IX.1(i) |
| $f^{\sim}$ | partial Fourier transform | Definition IX.1(ii) |
| $\hat{\chi}$ | Fourier transform | Definition IX.4 |
| $\mathcal{B}$ | $\mathbb{R}\times\mathbb{R}^d\times\{\uparrow,\downarrow\}\times\{0,1\}$ viewed as position space | beginning of Sec. II |
| $\check{\mathcal{B}}$ | $\mathbb{R}\times\mathbb{R}^d\times\{\uparrow,\downarrow\}\times\{0,1\}$ viewed as momentum space | beginning of Sec. IX |
| $\check{\mathcal{B}}_m$ | $\{(\check{\eta}_1,\ldots,\check{\eta}_m)\in\check{\mathcal{B}}^m \mid \check{\eta}_1+\cdots+\check{\eta}_m = 0\}$ | before Definition X.1 |
| $\mathcal{F}_m(n)$ | functions on $\mathcal{B}^m\times\mathcal{B}^n$, antisymmetric in $\mathcal{B}^m$ arguments | Definition II.9 |
| $\check{\mathcal{F}}_m(n)$ | functions on $\check{\mathcal{B}}^m\times\mathcal{B}^n$, antisymmetric in $\check{\mathcal{B}}^m$ arguments | Definition X.8 |

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