Singular Fermi Surfaces I. General Power Counting and Higher Dimensional Cases

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Abstract

We prove regularity properties of the self–energy, to all orders in perturbation theory, for systems with singular Fermi surfaces which contain *Van Hove points* where the gradient of the dispersion relation vanishes. In this paper, we show for spatial dimensions $d \geq 3$ that despite the Van Hove singularity, the overlapping loop bounds we proved together with E. Trubowitz for regular non–nested Fermi surfaces [J. Stat. Phys. 84 (1996) 1209] still hold, provided that the Fermi surface satisfies a no–nesting condition. This implies that for a fixed interacting Fermi surface, the self-energy is a continuously differentiable function of frequency and momentum, so that the quasiparticle weight and the Fermi velocity remain close to their values in the noninteracting system to all orders in perturbation theory. In a companion paper, we treat the more singular two–dimensional case.

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1 Introduction

In 1953, Van Hove published a general argument implying the occurrence of singularities in the phonon and electron spectrum of crystals [1]. The core of his argument is an application of Morse theory [2] — a sufficiently smooth function defined on the torus and having only nondegenerate critical points must have saddle points.

In the independent–electron approximation, the dispersion relation $k \mapsto \epsilon(k)$ of the electrons plays the role of the Morse function, and the *Van Hove singularities* (VHS) manifest themselves in the electronic density of states

$$\rho(E) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \delta(E - \epsilon(k)) \tag{1}$$

at those values of the energy where the level set $\{k : \epsilon(k) = E\}$ contains one (or more) of the saddle points, the *Van Hove points*. The nature of these singularities in ρ depends on the dimension. In two dimensions, ρ has a logarithmic singularity. In three dimensions, ρ is continuous but its derivative has singularities. In all dimensions, these singularities have observable consequences, although they occur only at discrete values of the energy.

In mean–field theories for symmetry breaking, the density of states plays an important role because it enters the self–consistency equations for the order parameter. For instance, in BCS theory, the superconducting gap Δ is determined as a function of the temperature $T = \beta^{-1}$ as the solution to the equation

$$\Delta = g\, \Delta \int \mathrm{d}E\, \frac{\rho(E)}{2\sqrt{(E - E_F)^2 + \Delta^2}}\, \tanh \frac{\beta\sqrt{(E - E_F)^2 + \Delta^2}}{2} \tag{2}$$

where $g > 0$ is the coupling constant that determines the strength of the mean–field interaction between Cooper pairs and E_F is the Fermi energy determined by the electron density. (We have written the equation for an s–wave superconductor.) The properties of $\rho(E)$ for E near to E_F obviously influence the temperature–dependence of Δ, as well as the value of the critical temperature T_c, defined as the largest value of T below which (2) has a nonzero solution. If ρ is smooth, the small–g asymptotics of T_c is $T_c \sim \mathrm{e}^{-\rho(E_F)/g}$. A logarithmic divergence in ρ of the form $\rho(E) = K \ln \frac{W}{|E - E_{VH}|}$ (with fixed constants K and W) enhances the critical temperature to $T_c \sim \mathrm{e}^{-K/\sqrt{g}}$ if $E_F = E_{VH}$. Similarly, van Hove singularities cause ferromagnetism in mean–field theory at arbitrarily small couplings $g \ll 1$.

In a true many–body theory, all this becomes much less clear–cut. Besides the obvious remark that in two dimensions, there is no long–range order at positive temperatures [4], hence the above discussion is restricted to mean–field theory, the question whether Van Hove singularities indeed occur in interacting systems and if so, what their influence on observable quantities is, remains open and important. The theoretical quantity related to the electron spectrum and the density of states of the interacting system is the interacting dispersion relation or the spectral function, obtained from the full propagator, hence ultimately from the electron self–energy. The VHS might cease to exist in the interacting system for various reasons. The interacting Fermi surface may turn out to avoid the saddle points, or the singularity caused by the saddle points of the dispersion relation may be smoothed out by more drastic effects, such as the opening of gaps in the vicinity of the saddle points. On the other hand, the VHS might also become more generic because the Fermi surface may get pinned at the Van Hove points, and the singularity might also get stronger due to interaction effects. A lot of research has gone into these questions because Van Hove singularities were invoked as a possible explanation of high–temperature superconductivity (see, e.g. [5] and references therein). In particular, there are competition effects between superconductivity, ferromagnetism and antiferromagnetism [6, 7, 8], as well as interesting phenomena connected to Fermi surface fluctuations [9, 10], to mention but a few results. The above speculations as to the fate of the Fermi surface and the VHS have been discussed widely in the literature [5].

In this paper, we begin a mathematical study of Fermi surfaces that contain Van Hove points, but that satisfy a no–nesting condition away from these points, with the aim of understanding some of the above questions. We prove regularity properties of the electron self–energy to all orders of perturbation theory using the multiscale techniques of [12, 13, 14, 15], which are closely related to the renormalization group techniques used in [6, 7]. In the present paper, we give bounds that apply in all dimensions $d \geq 2$ and then consider the case $d \geq 3$ in more detail. In a companion paper [11], we focus on the two–dimensional case, and in particular on the question of the renormalization of the quasiparticle weight and the Fermi velocity.

Our motivation for imposing the no–nesting condition is twofold. First, an example of a dispersion relation in $d = 2$ with a Fermi surface that contains Van Hove points and satisfies our no–nesting condition is the (t, t) Hubbard model with $t' \neq 0$ and $t\ t' < 0$ at the Van Hove density. For $t' = 0$, the Van Hove density is at half–filling, and the Fermi surface becomes flat, hence nested under our definition. However, there is ample evidence that in the Hubbard model it is the parameter range $t' \neq 0$ and electron density near to the van Hove density that is relevant for high–T_c superconductivity (see, e.g. [5, 6, 7]). Second, nesting causes additional singularities, and to get a clear picture of which property of the Fermi surface causes what kind of phenomena, it is useful to disentangle the effects of the VHS from those of nesting.

We now give an overview of the technical parts of the present paper and state our main result about the self–energy and the correlation functions. In Section 2, we prove bounds for volumes of thin shells in momentum space close to the Fermi surface. These volume bounds are the essential ingredient for power counting bounds. In Lemma 2.3, we show that these volume bounds are not changed by the introduction of the most common singularities in $d \geq 3$ and increase by a logarithm of the scale in $d = 2$. This implies by the general bounds of [12] that the superficial power counting of the model is unchanged for $d \geq 3$ and changes "only" by logarithms in $d = 2$. Lemma 2.4 contains a refinement of these bounds in which one restricts to small balls near the singular points. In Section 3, we turn to the finer aspects of power counting that are necessary to understand the regularity of the self–energy, for spatial dimensions $d \geq 3$. We define a weak no–nesting condition which is essentially identical to that of [12] and prove that the volume improvement estimate (1.34) of [12] carries over unchanged (Proposition 3.6). By Theorem 2.40 of [12], this implies that the bulk of the conclusions of Theorems 1.2 – 1.8 of [12] carry over to the situation with VHS in $d \geq 3$. Namely,

Theorem 1.1 *Let $d \geq 3$, and let the dispersion relation $\mathbf{k} \mapsto e(\mathbf{k})$ satisfy*

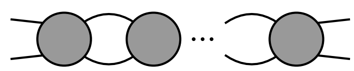
- $\mathcal{F} = \{ \mathbf{k} \in \mathbb{R}^d \mid e(\mathbf{k}) = 0 \}$ *is compact*
- $e(\mathbf{k})$ *is* C^3
- $\nabla e(\mathbf{k})$ *vanishes only at isolated points of* $\mathcal{F}$. *We shall call them singular points.*
- *if* $e(\mathbf{k}) = 0$ *and* $\nabla e(\mathbf{k}) = \mathbf{0}$, *then* $[\frac{\partial^2}{\partial \mathbf{k}_i \partial \mathbf{k}_j} e(\mathbf{k})]_{1 \leq i,j \leq d}$ *is nonsingular and has at least one positive eigenvalue and at least one negative eigenvalue.*
- *There is no nesting, in the precise sense of Hypothesis NN in Section 3.1*

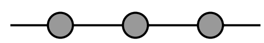
Let the interaction be short–range in the sense that the Fourier transform $k \mapsto \hat{v}(k)$ of the two–body interaction is twice continuously differentiable in k. Introduce the counterterm function $\mathbf{k} \mapsto K(\mathbf{k})$ as in Section 2 of [12], but using the localization operator $(\ell T)(q_0, \mathbf{q}) = T(0, \mathbf{q})$ in place of the localization operator of Definition 2.6 of [12], to renormalize the perturbation expansion. Then

1. *To any fixed order in renormalized perturbation theory, the electronic self–energy (i.e. the sum of the values of all two–legged one–particle–irreducible Feynman graphs) is continuously differentiable in the frequency and momentum variables. There is an $\varepsilon > 0$ so that all first derivatives of the self–energy are Hölder continuous of degree ε. The counterterm function K has the same regularity properties.*

2. *To any fixed order in renormalized perturbation theory, all correlation functions are well–defined, locally integrable functions of the external momenta.*

3. *To any fixed order in renormalized perturbation theory, the only contributions to the four–point function that fail to be bounded and continuous are the (generalized) ladder diagrams*

   Here each vertex  *is an arbitrary connected four–legged subdiagram and each line* *is a string*  *whose vertices (if any) are arbitrary one particle irreducible two–legged subdiagrams.*

4. *For each natural number r, denote by $\lambda^r K_r(\mathbf{k})$ the order r contribution to the renormalized perturbation expansion of the counterterm function $K(\mathbf{k})$. For each natural number R, the map $e \mapsto e + \sum_{r=1}^{R} \lambda^r K_r$ is locally injective. The precise meaning of and hypotheses for this statement are given in the paragraph containing (21).*

The above statements are proven in Section 3.3 at temperature $T = 0$. However, the same methods show that they extend to small $T \geq 0$, with the change that for $T > 0$, singularities are replaced by finite values that, however, diverge as $T \to 0$.

As explained in detail in [12], the counterterm K fixes the Fermi surface, so that all our results are about the model with a fixed interacting Fermi surface. Whether the situation that the Fermi surface contains zeroes of the gradient of e can indeed be achieved is related to the question whether there is an inversion theorem generalizing that of [15] to the situation with VHS, i.e. which provides existence of a solution of the equation $e + K(e) = E$ for the present situation (item 4 of the above theorem only gives local uniqueness). This is a difficult question which is still under investigation (see also [11]).

A natural question is the relation between these statements to all orders in perturbation theory and results obtained from truncated renormalization group flows in applied studies, which are often claimed to be "nonperturbative". The all–order results are statements about an iterative solution to a full renormalization group flow. The solution of renormalization group flows obtained by truncating the infinite hierarchy to a finite hierarchy creates scale–dependent approximations to the Green functions. These approximations give the leading order behaviour if the truncation has been done appropriately. Often, the results indicate instabilities of the flow, which signal that the true state of the system is not well–described by an action of the form assumed in the flow. A true divergence in the solution occurs only when the regime of validity of the truncation is left. (In the simplest situations, such singularities coincide with the divergence of a geometric series.) In careful studies, the equations are never integrated to the point where anything diverges. In that case, the regularity bounds obtained by all–order estimates are more accurate than those obtained from the solution of the flow equations truncated at finite order. There is one case where the integration of the renormalization group equations gives an effect within the validity of the truncation, but qualitatively different from all–order theory: this is when the flow satisfies infrared asymptotic freedom, i.e. the coupling function becomes screened at low scales. For instance, in the repulsive Hubbard model, the ladders with the bare interaction lead to a screening of the superconducting interaction, corresponding to $g < 0$ and hence to no solution in the BCS gap equation. (However, in this case, an attractive Cooper interaction is generated in second order, and it then grows in the flow to lower scales.) Such screening effects can only make terms smaller. Hence the upper bounds provided by the all–order analysis are still as good as the integration of truncations to the same order, as far as regularity properties are concerned. In practice, the truncations done in the RG equations are of very low order, so that the all–order analysis includes many contributions that are not taken into account in these truncations.

A *nonperturbative* mathematical proof involves bounding the remainders created in the expansion (or truncation). This is possible in $d = 2$ using the sector method of [17] (see [18, 19, 20]), but a full construction has not yet been achieved in $d \geq 3$. Because the graphical structures used in our arguments only require one overlap of loops, we expect that a suitable adaptation will be possible in constructive studies. In addition to the above–mentioned problem with constructive arguments in $d \geq 3$, the important question of the inversion theorem should also be addressed.

2 General power counting bounds in $d \geq 2$

2.1 Analytic structure of the one–body problem

Here we discuss briefly the properties of the one–body problem, to show that the Fermi surface of the noninteracting system is given as the zero set of an analytic function, hence no-nesting in a polynomial sense is a generic condition. For lattice models, analyticity of the dispersion relation e is obvious for hopping amplitudes that decay exponentially with distance (or are even of finite range). For continuum Schrödinger operators, it follows from the statements below, which even hold for the case with a magnetic field.

Let $d \geq 2$ and Γ be a lattice in $\mathbb{R}^d$ of maximal rank. Let $r > d$. Define

$$\begin{aligned} \mathcal{A} &= \{ A = (A_1, \dots, A_d) \in (L^r_{\mathbb{R}}(\mathbb{R}^d/\Gamma))^d \mid \textstyle\int_{\mathbb{R}^d/\Gamma} A(x) dx = 0 \} \\ \mathcal{V} &= \{ V \in L^{r/2}_{\mathbb{R}}(\mathbb{R}^d/\Gamma) \mid \textstyle\int_{\mathbb{R}^d/\Gamma} V(x) dx = 0 \} \end{aligned}$$

For $(A, V) \in \mathcal{A} \times \mathcal{V}$ set

$$H_{\mathbf{k}}(A, V) = (i\nabla + A(x) - \mathbf{k})^2 + V(x)$$

When $d = 2, 3$, the operator $H_{\mathbf{k}}(A, V)$ describes an electron in $\mathbb{R}^d$ with quasi-momentum $\mathbf{k}$ moving under the influence of the magnetic field with periodic vector potential $A(x) = (A_1(x), \dots A_d(x))$ and electric field with periodic potential $V(x)$. The conditions $\int_{\mathbb{R}^d/\Gamma} A(x) dx = 0$ and $\int_{\mathbb{R}^d/\Gamma} V(x) dx = 0$ are included purely for convenience and can always be achieved by translating $\mathbf{k}$ and shifting the zero point of the energy scale. The following theorem is proven in [16].

Theorem 2.1 *Let*

$$\begin{aligned} \mathcal{A}_{\mathbb{C}} &= \{ A = (A_1, \dots, A_d) \in (L^r(\mathbb{R}^d/\Gamma))^d \mid \textstyle\int_{\mathbb{R}^d/\Gamma} A(x) dx = 0 \} \\ \mathcal{V}_{\mathbb{C}} &= \{ V \in L^{r/2}(\mathbb{R}^d/\Gamma) \mid \textstyle\int_{\mathbb{R}^d/\Gamma} V(x) dx = 0 \} \end{aligned}$$

be the complexifications of $\mathcal{A}$ and $\mathcal{V}$ respectively. There exists an analytic function F on $\mathbb{C}^d \times \mathbb{C} \times \mathcal{A}_{\mathbb{C}} \times \mathcal{V}_{\mathbb{C}}$ such that, for $\mathbf{k}, A, V$ real,

$$\lambda \in \operatorname{Spec}(H_{\mathbf{k}}(A, V)) \qquad \Longleftrightarrow \qquad F(\mathbf{k}, \lambda, A, V) = 0$$

The theorem is proven by providing a formula for F. Write

$$(i\nabla + A(x) - \mathbf{k})^2 + V(x) - \lambda = \mathbb{1} - \Delta + u(\mathbf{k}, \lambda) + w(\mathbf{k}, A, V)$$

with

$$
\begin{aligned}
u(\mathbf{k},\lambda) &= -2i\mathbf{k}\cdot\nabla + \mathbf{k}^2 - \lambda - \mathbb{1} \\
w(\mathbf{k},A,V) &= i\nabla\cdot A + iA\cdot\nabla - 2\mathbf{k}\cdot A + A^2 + V
\end{aligned}
$$

Then the function $F(\mathbf{k},\lambda,A,V)$ of the above theorem is a suitably regularized determinant of $\mathbb{1} + \frac{1}{\sqrt{\mathbb{1}-\Delta}}u(\mathbf{k},\lambda)\frac{1}{\sqrt{\mathbb{1}-\Delta}} + \frac{1}{\sqrt{\mathbb{1}-\Delta}}w(\mathbf{k},A,V)\frac{1}{\sqrt{\mathbb{1}-\Delta}}$.

2.2 Volumes of Shells around Singular Fermi Surfaces

Suppose that the energy eigenvalues for the one–body problem with quasimomentum $\mathbf{k}$ are the solutions of an equation $F(\mathbf{k},\lambda) = 0$. That is, the bands $e_1(\mathbf{k}) \le e_2(\mathbf{k}) \le e_3(\mathbf{k}) \le \cdots$ all obey $F(\mathbf{k}, e_n(\mathbf{k})) = 0$. Our analysis of the regularity properties of the self–energy and correlation functions depends on having good bounds on the volume of the set of all quasimomenta $\mathbf{k}$ for which there are very low energy bands. More precisely, fi x any $M > 1$ and let $j \le 0$. We need to know the volume of the set of all quasimomenta $\mathbf{k}$ for which there is at least one band with $|e_n(\mathbf{k})| \le M^j$. The following lemma provides a useful simplifi cation.

Lemma 2.2 *Let $\mathcal{K}$ be a compact subset of $\mathbb{R}^d$ and $F : \mathcal{K}\times[-1,1] \to \mathbb{R}$ be C^1. Then there is a constant C such that*

$$
\begin{aligned}
&\mathrm{Vol}\{\ \mathbf{k}\in\mathcal{K} \mid F(\mathbf{k},\lambda) = 0 \textit{ for some } |\lambda| \le M^j\ \} \\
&\qquad\qquad \le\ \mathrm{Vol}\{\ \mathbf{k}\in\mathcal{K} \mid |F(\mathbf{k},0)| \le CM^j\ \}
\end{aligned}
$$

for all $j \le 0$. In particular, if all bands $e_n(\mathbf{k})$ obey $F(\mathbf{k}, e_n(\mathbf{k})) = 0$ then,

$$
\mathrm{Vol}\{\ \mathbf{k}\in\mathcal{K} \mid |e_n(\mathbf{k})| \le M^j \textit{ for some } n\ \} \le\ \mathrm{Vol}\{\ \mathbf{k}\in\mathcal{K} \mid |F(\mathbf{k},0)| \le CM^j\ \}
$$

Proof: Since F is C^1 on the compact set $\mathcal{K}\times[-1,1]$,

$$
C \equiv \sup_{(\mathbf{k},\lambda)\in\mathcal{K}\times[-1,1]} \left|\frac{\partial F}{\partial\lambda}(\mathbf{k},\lambda)\right| < \infty
$$

Hence, if for some $\mathbf{k}\in\mathcal{K}$ and some $|\lambda| \le M^j$, we have $F(\mathbf{k},\lambda) = 0$, then, for that same $\mathbf{k}$,

$$
|F(\mathbf{k},0)| = |F(\mathbf{k},\lambda) - F(\mathbf{k},0)| \le C|\lambda| \le CM^j
$$

Hence

$$
\{\ \mathbf{k}\in\mathcal{K} \mid F(\mathbf{k},\lambda) = 0 \text{ for some } |\lambda| \le M^j\ \} \subset \{\ \mathbf{k}\in\mathcal{K} \mid |F(\mathbf{k},0)| \le CM^j\ \}
$$

■

We now, and for the rest of this paper, focus on a single band $k \mapsto \epsilon(k)$, and assume that the chemical potential μ, used to fix the density, is such that $e(k) = \epsilon(k) - \mu$ has a nonempty zero set, the Fermi surface, which has also not degenerated to a point.

In the scale analysis, momentum space is cut up in shells around the Fermi surface. Here we take the convention of labelling these shells by negative integers $j \le 0$. The shell number j contains momenta k and Matsubara frequencies k_0 with $\frac{1}{2}M^j \le |\mathrm{i}k_0 - e(k)| \le M^j$. Here $M > 1$ is fixed once and for all. For the (standard) details about the scale decomposition and the corresponding renormalization group flow, obtained by integrating over degrees of freedom in the shell number j successively, downwards from $j = 0$, see, e.g. [12], Section 2.

The next lemma contains the basic volume bound for the scale analysis. In the case without VHS, the bound is of order M^j. The lemma implies that this bound remains unchanged for $d \ge 3$, and that there is an extra logarithm in $d = 2$.

Lemma 2.3 *Let $\mathcal{K}$ be a compact subset of $\mathbb{R}^d$ and $e : \mathcal{K} \to \mathbb{R}$ be C^2. Assume that for every point $\mathbf{p} \in \mathcal{K}$ at least one of*

∘ $e(\mathbf{p}) \neq 0$

∘ $\nabla e(\mathbf{p}) \neq 0$

∘ $\det \left[\frac{\partial^2}{\partial \mathbf{k}_i \partial \mathbf{k}_j} e(\mathbf{p})\right]_{1 \le i,j \le d} \neq 0$

is true. Then there is a constant C such that

$$\mathrm{Vol}\{\ \mathbf{k} \in \mathcal{K} \mid |e(\mathbf{k})| \le M^j\ \} \le CM^j \begin{cases} |j| & \text{if } d = 2 \\ 1 & \text{if } d > 2 \end{cases}$$

for all $j \le -1$.

Proof: Since $\mathcal{K}$ is compact, it suffices to prove that, for each $\mathbf{p} \in \mathcal{K}$ there are constants $R > 0$ and C (depending on $\mathbf{p}$) such that for all $j \le -1$,

$$\begin{aligned} \mathcal{V}_{R,j}(\mathbf{p}) &= \mathrm{Vol}\{\ \mathbf{k} \in \mathcal{K} \mid |e(\mathbf{k})| \le M^j, |\mathbf{k} - \mathbf{p}| \le R\ \} \\ &\le CM^j \begin{cases} |j| & \text{if } d = 2 \\ 1 & \text{if } d > 2 \end{cases} \end{aligned}$$

Case 1: $e(\mathbf{p}) \neq 0$. We are free to choose R sufficiently small that $|e(\mathbf{k})| \ge \frac{1}{2}|e(\mathbf{p})|$ for all $\mathbf{k} \in \mathcal{K}$ with $|\mathbf{k} - \mathbf{p}| \le R$. Then $\{\ \mathbf{k} \in \mathcal{K} \mid |e(\mathbf{k})| \le M^j, |\mathbf{k} - \mathbf{p}| \le R\ \}$ is empty unless $M^j \ge \frac{1}{2}|e(\mathbf{p})|$ and it suffices to take

$$C = \frac{2}{|e(\mathbf{p})|}\ \mathrm{Vol}\{\ \mathbf{k} \in \mathcal{K} \mid |\mathbf{k} - \mathbf{p}| \le R\ \}$$

Case 2: $e(\mathbf{p}) = 0$, $\nabla e(\mathbf{p}) \neq 0$. By translating and permuting indices, we may assume that $\mathbf{p} = 0$ and that $\frac{\partial e}{\partial \mathbf{k}_1}(\mathbf{p}) \neq 0$. Then, if R is small enough,

$$\mathbf{x} = X(\mathbf{k}) = (e(\mathbf{k}), \mathbf{k}_2, \ldots, \mathbf{k}_d)$$

is a C^2 diffeomorphism from $\mathcal{K}_R = \{ \mathbf{k} \in \mathcal{K} \mid |\mathbf{k} - \mathbf{p}| \leq R \}$ to some bounded subset $\mathcal{X}$ of $\mathbb{R}^d$. The Jacobian of this diffeomorphism is $\frac{\partial e}{\partial \mathbf{k}_1}(\mathbf{k})$ and is bounded away from zero, say by c_1. Then

$$\begin{aligned}
\mathcal{V}_{R,j}(\mathbf{p}) &= \mathrm{Vol}\{ \mathbf{k} \in \mathcal{K} \mid |e(\mathbf{k})| \leq M^j, |\mathbf{k} - \mathbf{p}| \leq R \} \\
&= \int_{\mathcal{K}_R} 1(|e(\mathbf{k})| \leq M^j)\, d^d\mathbf{k} \\
&= \int_{\mathcal{X}} 1(|\mathbf{x}_1| \leq M^j)\, |\tfrac{\partial e}{\partial \mathbf{k}_1}(X^{-1}(\mathbf{x}))|^{-1}\, d^d\mathbf{x} \\
&\leq c_1^{-1} \int_{\mathcal{X}} 1(|\mathbf{x}_1| \leq M^j)\, d^d\mathbf{x} \\
&\leq c_1^{-1} c_2 M^j
\end{aligned}$$

Here $1(E)$ denotes the indicator function of the event E, i.e. $1(E) = 1$ if E is true and $1(E) = 0$ otherwise.

Case 3: $e(\mathbf{p}) = 0$, $\nabla e(\mathbf{p}) = 0$, $\det\big[\frac{\partial^2}{\partial \mathbf{k}_i \partial \mathbf{k}_j} e(\mathbf{p})\big]_{1 \leq i,j \leq d} \neq 0$. By translating, we may assume that $\mathbf{p} = 0$. Then, if R is small enough, the Morse lemma [3, Theorem 8.3bis] implies that there exists a C^1 diffeomorphism, $X(\mathbf{k})$, from $\mathcal{K}_R$ to some bounded subset $\mathcal{X}$ of $\mathbb{R}^d$ such that

$$e(X^{-1}(\mathbf{x})) = Q_m(\mathbf{x}) = x_1^2 + \ldots + x_m^2 - x_{m+1}^2 - \ldots - x_d^2$$

for some $0 \leq m \leq d$. Then

$$\begin{aligned}
\mathcal{V}_{R,j}(\mathbf{p}) &= \int_{\mathcal{K}_R} 1(|e(\mathbf{k})| \leq M^j)\, d^d\mathbf{k} \\
&= \int_{\mathcal{X}} 1(|Q_m(\mathbf{x})| \leq M^j) \left| \det\big[\tfrac{\partial X_i}{\partial \mathbf{k}_j}(X^{-1}(\mathbf{x}))\big]_{1 \leq i,j \leq d} \right|^{-1} d^d\mathbf{x} \\
&\leq c_1^{-1} \int_{\mathcal{X}} 1(|Q_m(\mathbf{x})| \leq M^j)\, d^d\mathbf{x}
\end{aligned}$$

If $m = 0$ or $m = d$,

$$\int_{\mathcal{X}} 1(|Q_m(\mathbf{x})| \leq M^j)\, d^d\mathbf{x} = \int_{\mathcal{X}} 1(|x_1^2 + \ldots + x_d^2| \leq M^j)\, d^d\mathbf{x}$$

$$
\begin{aligned}
&\le \int_{\mathbb{R}^d} 1(|x_1^2+\ldots+x_d^2| \le M^j)\; d^d\mathbf{x} \\
&= c_d M^{dj/2}
\end{aligned}
$$

so it suffices to consider $1 \le m \le d-1$. Go to spherical coordinates separately in $x_1,\ldots,x_m$ and $x_{m+1},\ldots,x_d$, using

$$
u = \sqrt{x_1^2+\ldots+x_m^2} \qquad v = \sqrt{x_{m+1}^2+\ldots+x_d^2}
$$

If R is small enough

$$
\mathcal{V}_{R,j}(\mathbf{p}) \le c_1^{-1} c_{m,d} \int_{0\le u,v\le 1} 1(|u^2-v^2| \le M^j) u^{m-1} v^{d-m-1}\; du dv
$$

Now make the change of variables $x = u+v$, $y = u-v$. Then

$$
\begin{aligned}
&\int_{0\le v\le u\le 1} 1(|u^2-v^2| \le M^j) u^{m-1} v^{d-m-1}\; du dv \\
&\qquad \le \int_0^2 dx \int_0^1 dy\; 1(xy \le M^j)(x+y)^{m-1}|x-y|^{d-m-1} \\
&\qquad \le \int_0^2 dx \int_0^1 dy\; 1(xy \le M^j)(x+y)^{d-2}
\end{aligned}
$$

and the lemma follows from

$$
\int_0^2 dx \int_0^{\min\{1,M^j/x\}} dy = M^j + M^j \ln \tfrac{2}{M^j}
$$

and, for $n \ge 1$,

$$
\int_0^2 dx \int_0^{\min\{1,M^j/x\}} dy\; x^n \le \tfrac{1}{n+1} M^{(n+1)j} + 2^n M^j
$$

and

$$
\int_0^2 dx \int_0^{\min\{1,M^j/x\}} dy\; y^n \le M^j
$$

∎

That $\mathrm{Vol}\{\, \mathbf{k} \in \mathcal{K} \mid |e(\mathbf{k})| \le M^j \,\} \le C M^j |j|$ and that this bound suffices to yield a well–defined counterterm and well–defined correlation functions, to all orders of perturbation theory, was also proven in [21]. We now refine Lemma 2.3 a little.

Lemma 2.4 *Let $e : \mathbb{R}^d \to \mathbb{R}$ be C^2. Assume that*

- $e(\mathbf{0}) = 0$
- $\nabla e(\mathbf{0}) = \mathbf{0}$
- $\det\left[\frac{\partial^2}{\partial \mathbf{k}_i \partial \mathbf{k}_j} e(\mathbf{0})\right]_{1 \le i,j \le d} \neq 0$
- $\left[\frac{\partial^2}{\partial \mathbf{k}_i \partial \mathbf{k}_j} e(\mathbf{0})\right]_{1 \le i,j \le d}$ *has at least one positive eigenvalue and at least one negative eigenvalue.*

Then there are $C,\ C' > 0$ such that for all $\mathbf{q} \in \mathbb{R}^d$, $j \le 0$ and $0 < \varepsilon < \frac{1}{2}$,

$$\begin{aligned} &\mathrm{Vol}\{\ \mathbf{k} \in \mathbb{R}^d \mid |e(\mathbf{k})| \le M^j, |\mathbf{k} - \mathbf{q}| \le M^{\varepsilon j}, |\mathbf{k}| \le C'\ \} \\ \le\ & CM^j \begin{cases} 1 + (1 - 2\varepsilon)|j| & \text{if } d = 2 \\ M^{(d-2)\varepsilon j} & \text{if } d > 2 \end{cases} \end{aligned}$$

Proof: By the Morse lemma, we can assume without loss of generality that

$$e(\mathbf{k}) = k_1^2 + \ldots + k_m^2 - k_{m+1}^2 - \ldots - k_d^2$$

for some $1 \le m \le d - 1$. Go to spherical coordinates separately in $k_1, \ldots, k_m$ and $k_{m+1}, \ldots, k_d$, using

$$u = \sqrt{k_1^2 + \ldots + k_m^2} \qquad v = \sqrt{k_{m+1}^2 + \ldots + k_d^2}$$

For any fixed $u > 0$, the condition $|\mathbf{k} - \mathbf{q}| \le M^{\varepsilon j}$ restricts $(k_1, \ldots, k_m)$ to lie on a spherical cap of diameter at most $2M^{\varepsilon j}$ on the sphere of radius u. This cap has an area of at most an m–dependent constant times $\min\{u, M^{\varepsilon j}\}^{m-1}$. Similarly, for any fixed $v > 0$, the condition $|\mathbf{k} - \mathbf{q}| \le M^{\varepsilon j}$ restricts $(k_{m+1}, \ldots, k_d)$ to run over an area of at most a constant times $\min\{v, M^{\varepsilon j}\}^{d-m-1}$. The condition $|\mathbf{k} - \mathbf{q}| \le M^{\varepsilon j}$ also restricts u and v to run over intervals I_1, I_2 of length at most $2M^{\varepsilon j}$. Thus

$$\begin{aligned} &\mathrm{Vol}\{\ \mathbf{k} \mid |e(\mathbf{k})| \le M^j, |\mathbf{k} - \mathbf{q}| \le M^{\varepsilon j}\ \} \\ &\le c_{m,d} \int_{I_1 \times I_2} 1(|u^2 - v^2| \le M^j) \min\{u, M^{\varepsilon j}\}^{m-1} \min\{v, M^{\varepsilon j}\}^{d-m-1}\, du dv \\ &\le c_{m,d} \int_{I_1 \times I_2} 1(|u^2 - v^2| \le M^j) \min\Big\{ \max\{u, v\}, M^{\varepsilon j} \Big\}^{d-2}\, du dv \end{aligned}$$

It suffices to consider the case $0 \le v \le u$. Make the change of variables $x = u+v$, $y = u - v$. Then x and y are restricted to run over intervals J_1, J_2 of length at most $4M^{\varepsilon j}$ and

$$\begin{aligned}
&\mathrm{Vol}\{\ \mathbf{k} \mid |e(\mathbf{k})| \le M^j, |\mathbf{k}-\mathbf{q}| \le M^{\varepsilon j}\ \} \\
&\le 2c_{m,d} \int_{\substack{J_1\times J_2 \\ 0\le y\le x}} 1(xy \le M^j) \min\{x, M^{\varepsilon j}\}^{d-2} dx dy
\end{aligned}$$

In the event that $J_1 \subset [M^{\varepsilon j}, \infty]$, then on the domain of integration, $x \ge M^{\varepsilon j}$ and the condition $xy \le M^j$ forces $y \le M^j/M^{\varepsilon j}$, so that

$$\begin{aligned}
&\mathrm{Vol}\{\ \mathbf{k} \mid |e(\mathbf{k})| \le M^j, |\mathbf{k}-\mathbf{q}| \le M^{\varepsilon j}\ \} \\
&\le 2c_{m,d} \int_0^{M^{(1-\varepsilon)j}} dy \int_{J_1} dx\ M^{(d-2)\varepsilon j} \\
&\le 2c_{m,d} M^{(1-\varepsilon)j} 4M^{\varepsilon j} M^{(d-2)\varepsilon j} \\
&= 8c_{m,d} M^j M^{(d-2)\varepsilon j}
\end{aligned}$$

If $J_1 \cap [0, M^{\varepsilon j}] \ne \emptyset$, the domain of integration is contained in $0 \le y \le x \le 5M^{\varepsilon j}$ and

$$\begin{aligned}
&\mathrm{Vol}\{\ \mathbf{k} \mid |e(\mathbf{k})| \le M^j, |\mathbf{k}-\mathbf{q}| \le M^{\varepsilon j}\ \} \\
&\le 2c_{m,d} \int_0^{5M^{\varepsilon j}} dx \int_0^{5M^{\varepsilon j}} dy\, 1(xy \le M^j) x^{d-2}
\end{aligned}$$

For $d = 2$, the lemma follows from

$$\int_0^{5M^{\varepsilon j}} dx \int_0^{\min\{5M^{\varepsilon j}, M^j/x\}} dy = M^j(1 + \ln 25 + (1-2\varepsilon)|j| \ln M)$$

For $d > 2$,

$$\int_0^{5M^{\varepsilon j}} dx \int_0^{\min\{5M^{\varepsilon j}, M^j/x\}} dy\ x^{d-2} \le 5^{d-1} M^j M^{(d-2)\varepsilon j}.$$

■

3 Improved power counting

From now on we assume that $d \geq 3$, and that

- $\mathcal{F} = \{\ \mathbf{k} \in \mathbb{R}^d \mid e(\mathbf{k}) = 0\ \}$ is compact
- $e(\mathbf{k})$ is C^3
- $\nabla e(\mathbf{k})$ vanishes only at isolated points of $\mathcal{F}$. We shall call them singular points.
- if $e(\mathbf{k}) = 0$ and $\nabla e(\mathbf{k}) = \mathbf{0}$, then $[\frac{\partial^2}{\partial \mathbf{k}_i \partial \mathbf{k}_j} e(\mathbf{k})]_{1 \leq i,j \leq d}$ is nonsingular and has at least one positive eigenvalue and at least one negative eigenvalue.

In addition, we make an assumption that there is no nesting. In general, this means that any nontrivial translate of $\mathcal{F}$ or $-\mathcal{F}$ only has intersections with $\mathcal{F}$ of at most some fixed finite degree. Here we only require a weak form of no–nesting – namely that there is only polynomial flatness. This assumption, which is essentially the same as Hypothesis **A3** in [12], is introduced and discussed in detail in the following.

3.1 A no–nesting hypothesis and its consequences

To make precise the "only polynomial flatness" hypotheses, let

$$n : \mathcal{F} \to \mathbb{R}^d, \qquad \omega \mapsto n(\omega) = \tfrac{\nabla e}{|\nabla e|}(\omega)$$

be the unit normal to the Fermi surface. It is defined except at singular points, which are isolated. For $\omega, \omega' \in \mathcal{F}$, define the angle between $n(\omega)$ and $n(\omega')$ by

$$\theta(\omega, \omega') = \arccos(n(\omega) \cdot n(\omega'))$$

Let

$$\mathcal{D}(\omega) = \{\ \omega' \in \mathcal{F} \mid |n(\omega) \cdot n(\omega')| = 1\ \} = \{\ \omega' \in \mathcal{F} \mid n(\omega) = \pm n(\omega')\ \} \tag{3}$$

and denote the $(d-1)$–dimensional measure of $A \subset \mathcal{F}$ by $\mathrm{Vol}_{d-1} A$. Also, for any $A \subset \mathbb{R}^d$ and $\beta > 0$ denote by $U_\beta(A) = \{\ \mathbf{p} \in \mathbb{R}^d \mid \mathrm{distance}(\mathbf{p}, A) < \beta\ \}$ the open β-neighbourhood of A. We assume:

Hypothesis NN. *There are strictly positive numbers* $Z_0,\ \beta_0$ *and* κ *such that for all* $\beta \le \beta_0$ *and all* $\omega \in \mathcal{F}$*,*

$$\mathrm{Vol}_{d-1}\{\ \omega' \in \mathcal{F} \mid |\sin\theta(\omega,\omega')| = \sqrt{1-(n(\omega')\cdot n(\omega))^2} \le \beta\ \} \le Z_0\beta^\kappa$$

To verify this hypothesis, it suffices to find strictly positive numbers z_0, z_1, ρ', β_0 *and* κ' *such that for all for all* $\beta \le \beta_0'$ *and all* $\omega \in \mathcal{F}$*,*

(i) $\mathrm{Vol}_{d-1}\left(U_\beta(\mathcal{D}(\omega)) \cap \mathcal{F}\right) \le z_0\beta^{\kappa'}$

(ii) if $\omega' \notin U_\beta(\mathcal{D}(\omega)) \cap \mathcal{F}$*, then* $|\sin\theta(\omega,\omega')| = \sqrt{1-(n(\omega)\cdot n(\omega'))^2} \ge z_1\beta^{\rho'}$.

Then $\kappa = \frac{\kappa'}{\rho'}$*,* $Z_0 = z_0 z_1^{-\kappa'/\rho'}$ *and* $\beta_0 = z_1{\beta_0'}^{\rho'}$.

Example. As an example, take $d \ge 3$, $1 \le m < d$ and $e(\mathbf{k}) = k_1^2 + \ldots + k_m^2 - k_{m+1}^2 - \ldots - k_d^2$, say with $|\mathbf{k}| \le \sqrt{2}$. The corresponding Fermi surface, $\mathcal{F}$, is the (truncated) cone $k_1^2 + \ldots + k_m^2 = k_{m+1}^2 + \ldots + k_d^2$, which we may parametrize by $\mathbf{k} = (r\boldsymbol{\theta}, r\boldsymbol{\phi})$ with $0 \le r \le 1$, $\boldsymbol{\theta} \in S^{m-1}$ and $\boldsymbol{\phi} \in S^{d-m-1}$. The volume element on $\mathcal{F}$ in this parametrization is $\sqrt{2}r^{d-2}\,dr\,d^{m-1}\boldsymbol{\theta}\,d^{d-m-1}\boldsymbol{\phi}$, where $d^{m-1}\boldsymbol{\theta}$ and $d^{d-m-1}\boldsymbol{\phi}$ are the volume elements on S^{m-1} and S^{d-m-1} respectively. The unit normals to $\mathcal{F}$ at $\mathbf{k} = (r\boldsymbol{\theta}, r\boldsymbol{\phi})$ are $\pm\frac{1}{\sqrt{2}}(\boldsymbol{\theta}, -\boldsymbol{\phi})$.

Now fi x any $\omega = (r\boldsymbol{\theta}, r\boldsymbol{\phi})$ with $0 < r \le 1$. Then

$$D(\omega) = \{\ (t\boldsymbol{\theta}, t\boldsymbol{\phi}) \mid 0 < |t| \le 1\ \}$$

If $(t'\boldsymbol{\theta}', t'\boldsymbol{\phi}') \in U_\beta(D(\omega)) \cap \mathcal{F}$ the there is a t such that

$$\begin{aligned}
|(t'\boldsymbol{\theta}', t'\boldsymbol{\phi}') - (t\boldsymbol{\theta}, t\boldsymbol{\phi})| < \beta \quad &\Longrightarrow \quad \sqrt{|t'\boldsymbol{\theta}' - t\boldsymbol{\theta}|^2 + |t'\boldsymbol{\phi}' - t\boldsymbol{\phi}|^2} < \beta \\
&\Longrightarrow \quad |t'\boldsymbol{\theta}' - t\boldsymbol{\theta}| < \beta,\ |t'\boldsymbol{\phi}' - t\boldsymbol{\phi}| < \beta,\ |t - t'| < \beta \\
&\Longrightarrow \quad |t'\boldsymbol{\theta}' - t'\boldsymbol{\theta}| < 2\beta,\ |t'\boldsymbol{\phi}' - t'\boldsymbol{\phi}| < 2\beta
\end{aligned}$$

For each fi xed t the volume of the $t'\boldsymbol{\theta}'$s in $t'S^{m-1}$ for which $|\boldsymbol{\theta}' - \boldsymbol{\theta}| < 2\beta/|t'|$ is at most a constant, depending only on m, times $|t'|^{m-1}\min\{1, (\frac{\beta}{|t'|})^{m-1}\} \le \beta^{m-1}$ and the volume of the $t'\boldsymbol{\phi}'$s in $t'S^{d-m-1}$ for which $|\boldsymbol{\phi}' - \boldsymbol{\phi}| < 2\beta/|t'|$ is at most a constant, depending only on $d-m-1$, times $|t'|^{d-m-1}\min\{1, (\frac{\beta}{|t'|})^{d-m-1}\} \le \beta^{d-m-1}$. Hence

$$\mathrm{Vol}_{d-1}(U_\beta(D(\omega)) \cap \mathcal{F}) \le c_{d,m}\int_0^1 dt'\ \beta^{d-2} = c_{d,m}\beta^{d-2}$$

Thus condition (i) of Hypothesis NN is satisfied with $\acute{\kappa} = d - 2$.

If $\omega' = (t'\boldsymbol{\theta}', t'\boldsymbol{\phi}') \notin U_\beta(D(\omega)) \cap \mathcal{F}$ then, for every $|t| \leq 1$,

$$|(t'\boldsymbol{\theta}', t'\boldsymbol{\phi}') - (t\boldsymbol{\theta}, t\boldsymbol{\phi})| \geq \beta$$

In particular,

$$|(t'\boldsymbol{\theta}', t'\boldsymbol{\phi}') \pm (t'\boldsymbol{\theta}, t'\boldsymbol{\phi})| \geq \beta \Longrightarrow |(\boldsymbol{\theta}', \boldsymbol{\phi}') \pm (\boldsymbol{\theta}, \boldsymbol{\phi})| \geq \beta$$

The angle between $n(t'\boldsymbol{\theta}', t'\boldsymbol{\phi}') = \pm\frac{1}{\sqrt{2}}(\boldsymbol{\theta}', -\boldsymbol{\phi}')$ and $n(r\boldsymbol{\theta}, r\boldsymbol{\phi}) = \pm\frac{1}{\sqrt{2}}(\boldsymbol{\theta}, -\boldsymbol{\phi})$ is the same ($\pm\pi$) as the angle between $(\boldsymbol{\theta}', \boldsymbol{\phi}')$ and $(\boldsymbol{\theta}, \boldsymbol{\phi})$ (measured at the origin). By picking signs appropriately, we may assume that $0 \leq \theta(\omega, \omega') \leq \frac{\pi}{2}$. Thus

$$|\sin\theta(\omega, \omega')| \geq |\sin\tfrac{1}{2}\theta(\omega, \omega')| = \tfrac{1}{2\sqrt{2}}|(\boldsymbol{\theta}', \boldsymbol{\phi}') \pm (\boldsymbol{\theta}, \boldsymbol{\phi})| \geq \tfrac{1}{2\sqrt{2}}\beta$$

and condition (ii) of Hypothesis NN is satisfied with $\acute{\rho} = 1$. So $\kappa = d - 2$.

Proposition 3.1 *Let $d \geq 3$ and let $e : \mathbb{R}^d \to \mathbb{R}$ be C^3. Assume that*

- $e(\mathbf{0}) = 0$
- $\nabla e(\mathbf{0}) = \mathbf{0}$
- $\det\,[\frac{\partial^2}{\partial \mathbf{k}_i \partial \mathbf{k}_j} e(\mathbf{0})]_{1 \leq i,j \leq d} \neq 0$
- $[\frac{\partial^2}{\partial \mathbf{k}_i \partial \mathbf{k}_j} e(\mathbf{0})]_{1 \leq i,j \leq d}$ *has $m \geq 1$ positive eigenvalues and $d - m \geq 1$ negative eigenvalues.*

Then there is a $c > 0$ and constants $\beta_0 > 0$ and Z_0 such that for every unit vector $\mathbf{a} \in \mathbb{R}^d$,

$$\mathrm{Vol}_{d-1}\{\ \mathbf{k} \in \mathcal{F} \mid \sqrt{1 - (n(\mathbf{k}) \cdot \mathbf{a})^2} \leq \beta\ \} \leq Z_0 \beta^{\max\{m-1, d-m-1\}}$$
$$\textit{where } \mathcal{F} = \{\ \mathbf{k} \in \mathbb{R}^d \mid |\mathbf{k}| < c,\ e(\mathbf{k}) = 0\ \}$$

for all $0 < \beta < \beta_0$.

Proof: By a rotation, followed by a permutation of indices, we may assume that $[\frac{\partial^2}{\partial \mathbf{k}_i \partial \mathbf{k}_j} e(\mathbf{0})]_{1 \leq i,j \leq d}$ is a diagonal matrix, with diagonal entries $2\lambda_1$, $2\lambda_2$, $\cdots$, $2\lambda_d$ that are in decreasing order. By hypothesis, $\lambda_j > 0$ for $1 \leq j \leq m$ and $\lambda_j < 0$ for $m + 1 \leq j \leq d$. Replace λ_j by $-\lambda_j$ for $j > m$. Then,

$$e(\mathbf{k}) = \lambda_1 k_1^2 + \cdots + \lambda_m k_m^2 - \lambda_{m+1} k_{m+1}^2 - \cdots - \lambda_d k_d^2 + G(\mathbf{k})$$

with $G(\mathbf{k})$ a C^3 function having a third order zero at $\mathbf{0}$. Define

$$\begin{aligned} R_1(\mathbf{k}) &= \sqrt{\lambda_1 k_1^2 + \cdots + \lambda_m k_m^2} \\ R_2(\mathbf{k}) &= \sqrt{\lambda_{m+1} k_{m+1}^2 + \cdots + \lambda_d k_d^2} \\ R(\mathbf{k}) &= \sqrt{\lambda_1 k_1^2 + \cdots + \lambda_d k_d^2} \end{aligned}$$

Also use

$$\begin{aligned} \tilde{S}_1^{m-1} &= \{\, (k_1, \cdots, k_m) \mid \lambda_1 k_1^2 + \cdots + \lambda_m k_m^2 = 1 \,\} \\ \tilde{S}_2^{d-m-1} &= \{\, (k_{m+1}, \cdots, k_d) \mid \lambda_{m+1} k_{m+1}^2 + \cdots + \lambda_d k_d^2 = 1 \,\} \end{aligned}$$

to denote "unit" $(m-1)$–dimensional and $(d-m-1)$–dimensional ellipsoids, respectively. For each $r > 0$, the surface $R(\mathbf{k}) = r$ is a $d-1$ dimensional ellipsoid in $\mathbb{R}^d$ with smallest semi–axis $r/\max_j \sqrt{\lambda_j}$ and largest semi–axis $r/\min_j \sqrt{\lambda_j}$. We now concentrate on the intersection of $\mathcal{F}$ and that ellipsoid. The proof of Proposition 3.1 will continue following the proof of Lemma 3.4.

Lemma 3.2 *Suppose that*

$$|G(\mathbf{k})| \le g_0 R(\mathbf{k})^3 \qquad |\nabla G(\mathbf{k})| \le g_1 R(\mathbf{k})^2$$

and that c is small enough (depending only on g_0, g_1 and the λ_i's).

- *(a) For each $\boldsymbol{\theta}_1 \in \tilde{S}^{m-1}$, $\boldsymbol{\theta}_2 \in \tilde{S}^{d-m-1}$ and $r \ge 0$ such that the ellipsoid $\{\, \mathbf{k} \in \mathbb{R}^d \mid R(\mathbf{k}) = r \,\}$ is contained in the sphere $\{\, \mathbf{k} \mid |\mathbf{k}| < c \,\}$, there is a unique (r_1, r_2) such that*

  $$r_1, r_2 \ge 0 \qquad r_1^2 + r_2^2 = r^2 \qquad \textit{and} \qquad (r_1\boldsymbol{\theta}_1, r_2\boldsymbol{\theta}_2) \in \mathcal{F}$$

  Furthermore $|r_1 - r_2| \le g_0 r^2$.

- *(b) $\mathcal{F}$ is a C^3 manifold, except for a singularity at $\mathbf{k} = 0$.*

Proof: (a) The point $(r_1\boldsymbol{\theta}_1, r_2\boldsymbol{\theta}_2)$ is on $\mathcal{F}$ if and only if

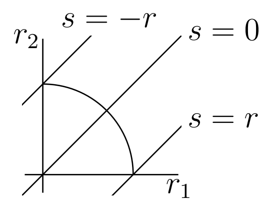
$$0 = r_1^2 - r_2^2 + G(r_1\boldsymbol{\theta}_1, r_2\boldsymbol{\theta}_2) = [r_1 - r_2][r_1 + r_2] + G(r_1\boldsymbol{\theta}_1, r_2\boldsymbol{\theta}_2)$$

or

$$r_1 - r_2 = -\tfrac{G(r_1\boldsymbol{\theta}_1, r_2\boldsymbol{\theta}_2)}{r_1 + r_2} \tag{4}$$

For each $-r \le s \le r$ there are unique $r_1(s) \ge 0$ and $r_2(s) \ge 0$ such that

$$r_1(s) - r_2(s) = s \qquad r_1(s)^2 + r_2(s)^2 = r^2$$



Furthermore $r_1'(s) - r_2'(s) = 1$ and $r_1(s)r_1'(s) + r_2(s)r_2'(s) = 0$ gives that $r_1'(s) = \frac{r_2(s)}{r_1(s)+r_2(s)}$ and $r_2'(s) = -\frac{r_1(s)}{r_1(s)+r_2(s)}$ have magnitude at most 1. Since $r_1(s) + r_2(s) \ge r$, $H(s) = -\frac{G(r_1(s)\boldsymbol{\theta}_1, r_2(s)\boldsymbol{\theta}_2)}{r_1(s)+r_2(s)}$ obeys

$$|H(s)| \le g_0 r^2$$

and

$$\begin{aligned}
|H'(s)| &= \left| \tfrac{[r_1(s)+r_2(s)]\nabla G(r_1(s)\boldsymbol{\theta}_1, r_2(s)\boldsymbol{\theta}_2)\cdot(r_1'(s)\boldsymbol{\theta}_1, r_2'(s)\boldsymbol{\theta}_2) - [r_1'(s)+r_2'(s)]G(r_1(s)\boldsymbol{\theta}_1, r_2(s)\boldsymbol{\theta}_2)}{[r_1(s)+r_2(s)]^2} \right| \\
&\le \tfrac{1}{r} g_1 r^2 |(r_1'(s)\boldsymbol{\theta}_1, r_2'(s)\boldsymbol{\theta}_2)| + \tfrac{1}{r^2} 2 g_0 r^3 \\
&\le r[g_1 |(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)| + 2g_0] \\
&\le r\Big[g_1 \max_{1\le i\le d} \sqrt{\tfrac{2}{\lambda_i}} + 2g_0\Big] \\
&< 1
\end{aligned}$$

provided c is small enough. Consequently the function $s - H(s)$ increases strictly monotonically from $-r - H(-r) \le -r + g_0 r^2$ to $r - H(r) \ge r - g_0 r^2$ as s increases from $-r$ to r. So this function has a unique zero and (4) has a unique solution and the solution obeys $|r_1 - r_2| \le g_0 r^2$.

(b) Since

$$\nabla e(\mathbf{k}) = 2(\lambda_1 k_1, \cdots, \lambda_m k_m, -\lambda_{m+1} k_{m+1}, \cdots, -\lambda_d k_d) + \nabla G(\mathbf{k})$$

and $|\nabla G(\mathbf{k})| \le g_1 R(\mathbf{k})^2 \le g_1(\max_i \lambda_i)\, |\mathbf{k}|^2$, the only zero of $\nabla e(\mathbf{k})$ is at $\mathbf{k} = 0$, assuming that c has been chosen small enough. ■

Lemma 3.3 *Define, for each* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, $\theta(\mathbf{a}, \mathbf{b}) \in [0, \pi]$ *to be the angle between* $\mathbf{a}$ *and* $\mathbf{b}$. *Let* $\mathrm{P}_1 : \mathbb{R}^d \to \mathbb{R}^m$ *and* $\mathrm{P}_2 : \mathbb{R}^d \to \mathbb{R}^{d-m}$ *be the orthogonal projections onto the first* m *and last* $d-m$ *components of* $\mathbb{R}^d$, *respectively. Assume that the hypotheses of Lemma 3.2 are satisfied. There is a constant* g_2 *(depending* *only on the λ_i's) such that if $\mathbf{0} \neq \omega \in \mathcal{F}$, $\mathbf{0} \neq \mathbf{a} \in \mathbb{R}^d$ with $|\sin\theta(n(\omega), \mathbf{a})| \leq \beta$, then*

$$|\sin\theta(\mathrm{P}_1 n(\omega), \mathrm{P}_1\mathbf{a})| \leq g_2\beta \qquad |\sin\theta(\mathrm{P}_2 n(\omega), \mathrm{P}_2\mathbf{a})| \leq g_2\beta \tag{5}$$

Proof: We'll prove the first bound of (5). We may assume that $\mathbf{a}$ is a unit vector. Possibly replacing $\mathbf{a}$ by $-\mathbf{a}$, we may also assume that the angle between $\mathbf{a}$ and $n(\omega)$ is at most $\frac{\pi}{2}$. By part (a) of Lemma 3.4, below,

$$|\mathbf{a} - n(\omega)| = 2\sin\tfrac{1}{2}\theta(\mathbf{a}, n(\omega)) \leq 2\sin\theta(\mathbf{a}, n(\omega)) \leq 2\beta$$

So, by part (a) of Lemma 3.4,

$$\sin\theta(\mathrm{P}_1\mathbf{a}, \mathrm{P}_1 n(\omega)) \leq \tfrac{|\mathrm{P}_1\mathbf{a} - \mathrm{P}_1 n(\omega)|}{|\mathrm{P}_1 n(\omega)|} \leq \tfrac{|\mathbf{a} - n(\omega)|}{|\mathrm{P}_1 n(\omega)|} \leq \tfrac{2\beta}{|\mathrm{P}_1 n(\omega)|}$$

Thus it suffices to prove that $|\mathrm{P}_1 n(\omega)|$ is bounded away from zero. Recall that

$$\nabla e(\omega) = n_1(\omega) + \nabla G(\omega)$$

where

$$n_1(\omega) = 2(\lambda_1 k_1, \cdots, \lambda_m k_m, -\lambda_{m+1}k_{m+1}, \cdots, -\lambda_d k_d)$$

Use $\alpha \sim \gamma$ to designate that there are constants $c, C > 0$, depending only on the λ_i's, such that $c|\gamma| \leq |\alpha| \leq C|\gamma|$. In this notation

$$|n_1(\omega)| \sim |\omega| \quad |\mathrm{P}_1 n_1(\omega)| \sim |\mathrm{P}_1\omega| \sim |R_1(\omega)| \quad |\mathrm{P}_2 n_1(\omega)| \sim |\mathrm{P}_2\omega| \sim |R_2(\omega)|$$

By part (a) of Lemma 3.2, since $\omega \in \mathcal{F}$,

$$|R_1(\omega) - R_2(\omega)| \leq g_0 R(\omega)^2 \qquad R_1(\omega)^2 + R_2(\omega)^2 = R(\omega)^2$$

As the maximum of $R_1(\omega)$ and $R_2(\omega)$ must be at least $\frac{1}{\sqrt{2}}R(\omega)$, we have

$$R(\omega) \geq R_1(\omega), R_2(\omega) \geq \tfrac{1}{\sqrt{2}}R(\omega) - g_0 R(\omega)^2 \geq \tfrac{1}{2}R(\omega)$$

if c is small enough. So

$$|\mathrm{P}_1 n_1(\omega)|, |\mathrm{P}_2 n_1(\omega)| \sim |R(\omega)| \sim |\omega|$$

As

$$|\nabla G(\omega)| \leq g_1 R(\omega)^2 \leq g_1(\max_i \lambda_i)\ |\omega|^2$$

we have that

$$|\mathrm{P}_1\nabla e(\omega)|, |\mathrm{P}_2\nabla e(\omega)| \sim |\omega|$$

and hence that

$$|\mathrm{P}_1 n(\omega)| = \tfrac{|\mathrm{P}_1\nabla e(\omega)|}{\sqrt{|\mathrm{P}_1\nabla e(\omega)|^2+|\mathrm{P}_2\nabla e(\omega)|^2}}$$

is bounded away from zero. ■

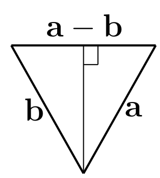
Lemma 3.4 *Let* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$.

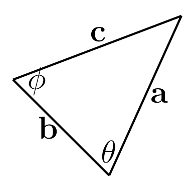
(a) If $|\mathbf{a}| = |\mathbf{b}|$, *then* $\sin \frac{1}{2}\theta(\mathbf{a}, \mathbf{b}) = \frac{1}{2}\frac{|\mathbf{a}-\mathbf{b}|}{|\mathbf{a}|}$.

(b) For all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, $\sin\theta(\mathbf{a}, \mathbf{b}) \le \frac{|\mathbf{a}-\mathbf{b}|}{|\mathbf{a}|}$.

Proof: Part (a) is obvious from the figure on the left below. For part (b), in the notation of the figure on the right below, we have, by the sin law

$$\tfrac{\sin\theta}{|\mathbf{c}|} = \tfrac{\sin\phi}{|\mathbf{a}|} \Longrightarrow \sin\theta = \tfrac{|\mathbf{c}|}{|\mathbf{a}|}\sin\phi \le \tfrac{|\mathbf{b}-\mathbf{a}|}{|\mathbf{a}|}$$





■

Proof: of Proposition 3.1 (continued). Fix $\mathbf{k}_2 \in \mathbb{R}^{d-m-1}$. If $\mathbf{k} = (\mathbf{k}_1, \mathbf{k}_2) \in \mathcal{F}$, then $\mathrm{P}_1 n(\mathbf{k})$ is normal to $\mathcal{F}_{\mathbf{k}_2} = \{\ \mathbf{k}_1 \in \mathbb{R}^m \mid (\mathbf{k}_1, \mathbf{k}_2) \in \mathcal{F}\ \}$ because both $n(\mathbf{k})$ and $\mathrm{P}_2 n(\mathbf{k})$ are perpendicular to any vector $(\mathbf{t}, \mathbf{0})$ that is tangent to $\mathcal{F}$ at $\mathbf{k}$. The matrix

$$\left[\tfrac{\partial^2\ e}{\partial k_i \partial k_j}(\mathbf{k}_1, \mathbf{k}_2)\right]_{1\le i,j\le m} = [2\lambda_i \delta_{i,j}]_{1\le i,j\le m} + \left[\tfrac{\partial^2\ G}{\partial k_i \partial k_j}(\mathbf{k}_1, \mathbf{k}_2)\right]_{1\le i,j\le m}$$

is strictly positive definite (assuming that c is small enough) because $\frac{\partial^2\ G}{\partial k_i \partial k_j}(\mathbf{k}) = O(|\mathbf{k}|)$. So the slice $\mathcal{F}_{\mathbf{k}_2}$ is strictly convex. The solution (r_1, r_2) of Lemma 3.2 depends continuously on $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$ and r, so, assuming that $m > 1$, $\mathcal{F}_{\mathbf{k}_2}$ is connected. Hence, for any fixed nonzero vector $P_1\mathbf{a}$, there are precisely two points of $\mathcal{F}_{\mathbf{k}_2}$ at which $|\sin\theta(\mathrm{P}_1 n(\mathbf{k}_1, \mathbf{k}_2), \mathrm{P}_1\mathbf{a})| = 0$. And at other points $\mathbf{k}_1 \in \mathcal{F}_{\mathbf{k}_2}$,

$|\sin\theta(\mathrm{P}_1 n(\mathbf{k}_1,\mathbf{k}_2),\mathrm{P}_1\mathbf{a})|$ is larger than a constant times the distance from $\mathbf{k}_1$ to the nearest of those two points. So

$$\begin{aligned}\mathrm{Vol}_{d-1}\{\ \mathbf{k}\in\mathcal{F}\ |\ |\sin\theta(n(\mathbf{k}),\mathbf{a})|\le\beta\ \}\\ &\le\ \text{const}\sup_{\mathbf{k}_2}\ \mathrm{Vol}_{m-1}\{\ \mathbf{k}_1\in\mathcal{F}_{\mathbf{k}_2}\ |\ |\sin\theta(\mathrm{P}_1 n(\mathbf{k}),\mathrm{P}_1\mathbf{a})|\le g_2\beta\ \}\\ &\le\ \text{const}\ \beta^{m-1}\end{aligned}$$

The bound

$$\mathrm{Vol}_{d-1}\{\ \mathbf{k}\in\mathcal{F}\ |\ |\sin\theta(n(\mathbf{k}),\mathbf{a})|\le\beta\ \}\le\ \text{const}\ \beta^{d-m-1}$$

is proven similarly. ■

Remark 3.5 *The exponent* $\kappa=\max\{m-1,d-m-1\}$ *of Proposition 3.1 is not optimal, unless* $m=1$ *or* $m=d-1$*. Suppose that* $2\le m\le d-2$*. As we observed in the proof of Proposition 3.1, for each fixed* $\mathbf{k}_2$ *there are precisely two distinct points of* $\mathcal{F}_{\mathbf{k}_2}$ *at which* $\sin\theta(\mathrm{P}_1 n(\mathbf{k}),\mathrm{P}_1\mathbf{a})=0$*. That is, at which* $\mathrm{P}_1 n(\mathbf{k})$ *is parallel or antiparallel to* $\mathrm{P}_1\mathbf{a}$*. Hence*

$$\begin{aligned}\{\ \mathbf{k}\in\mathcal{F}\setminus\{\mathbf{0}\}\ |\ \sin\theta(\mathrm{P}_1 n(\mathbf{k}),\mathrm{P}_1\mathbf{a})=0\ \}\\ =\bigcup_{\mathbf{k}_2\ne\mathbf{0}}\{\ (\mathbf{k}_1,\mathbf{k}_2)\ |\ \mathbf{k}_1\in\mathcal{F}_{\mathbf{k}_2},\ \sin\theta(\mathrm{P}_1 n(\mathbf{k}),\mathrm{P}_1\mathbf{a})=0\ \}\end{aligned}$$

consists of two disjoint $d-m$ *dimensional submanifolds of* $\mathcal{F}$ *and*

$$\{\ \mathbf{k}\in\mathcal{F}\setminus\{\mathbf{0}\}\ |\ |\sin\theta(\mathrm{P}_1 n(\mathbf{k}),\mathrm{P}_1\mathbf{a})|<g_2\beta\ \}$$

consists of two tubes of thickness of order β*, and volume of order* β^{m-1}*, about those submanifolds. Similarly,*

$$\{\ \mathbf{k}\in\mathcal{F}\setminus\{\mathbf{0}\}\ |\ |\sin\theta(\mathrm{P}_2 n(\mathbf{k}),\mathrm{P}_2\mathbf{a})|<g_2\beta\ \}$$

consists of two tubes of thickness of order β*, and volume of order* β^{d-m-1}*, about two disjoint* m *dimensional submanifolds. In the "free" case, when* $G=0$*,*

$$\mathcal{F}=\{\ (r\boldsymbol{\theta}_1,r\boldsymbol{\theta}_2)\ |\ |(r\boldsymbol{\theta}_1,r\boldsymbol{\theta}_2)|\le c,\ \boldsymbol{\theta}_1\in\tilde{S}^{m-1},\boldsymbol{\theta}_2\in\tilde{S}^{d-m-1}\ \}$$

and

$$n(r\boldsymbol{\theta}_1,r\boldsymbol{\theta}_2)\parallel(\Lambda_1\boldsymbol{\theta}_1,-\Lambda_2\boldsymbol{\theta}_2)\quad\textit{where}\ \Lambda_1=[\lambda_i\delta_{i,j}]_{1\le i,j\le m},\ \Lambda_2=[\lambda_i\delta_{i,j}]_{m<i,j\le d}$$

So

$$\begin{aligned}\mathcal{M}_1 &= \{\, \mathbf{k} \in \mathcal{F} \setminus \{\mathbf{0}\} \mid \sin\theta(\mathrm{P}_1 n(\mathbf{k}), \mathrm{P}_1\mathbf{a}) = 0 \,\} \\ &= \{\, (r\boldsymbol{\theta}_1, r\boldsymbol{\theta}_2) \mid 0 < |(r\boldsymbol{\theta}_1, r\boldsymbol{\theta}_2)| \le c,\ \boldsymbol{\theta}_2 \in \tilde{S}^{d-m-1},\ \boldsymbol{\theta}_1 \parallel \Lambda_1^{-1}\mathrm{P}_1\mathbf{a} \,\} \\ \mathcal{M}_2 &= \{\, \mathbf{k} \in \mathcal{F} \setminus \{\mathbf{0}\} \mid \sin\theta(\mathrm{P}_2 n(\mathbf{k}), \mathrm{P}_2\mathbf{a}) = 0 \,\} \\ &= \{\, (r\boldsymbol{\theta}_1, r\boldsymbol{\theta}_2) \mid 0 < |(r\boldsymbol{\theta}_1, r\boldsymbol{\theta}_2)| \le c,\ \boldsymbol{\theta}_1 \in \tilde{S}^{m-1},\ \boldsymbol{\theta}_2 \parallel \Lambda_2^{-1}\mathrm{P}_2\mathbf{a} \,\}\end{aligned}$$

intersect in the lines

$$\mathcal{M}_1 \cap \mathcal{M}_2 = \{\, (r\boldsymbol{\theta}_1, r\boldsymbol{\theta}_2) \mid 0 < |(r\boldsymbol{\theta}_1, r\boldsymbol{\theta}_2)| \le c,\ \boldsymbol{\theta}_1 \parallel \Lambda_1^{-1}\mathrm{P}_1\mathbf{a},\ \boldsymbol{\theta}_2 \parallel \Lambda_2^{-1}\mathrm{P}_2\mathbf{a} \,\}$$

and otherwise cross transversely. (If the λ_i's are all the same, they cross perpendicularly.) So even when G is nonzero, the tubes will cross transversely (for sufficiently small c) and the volume of intersection will be of the order of the product $\beta^{m-1}\beta^{d-m-1} = \beta^{\kappa}$ with $\kappa = d - 2$.

3.2 The overlapping loop bound for $d \ge 3$

In this section we prove the overlapping loop bound. It generalizes the analogous bound of [12, Proposition 1.1] to singular Fermi surfaces in $d \ge 3$. The overlapping loop bound implies [12] that the first order derivatives of Σ are bounded continuous functions of momentum and frequency, to all orders in the renormalized expansion in the interaction, and that the same holds for the counterterm function K.

Proposition 3.6 *Let $d \ge 3$, and let the dispersion relation $\mathbf{k} \mapsto e(\mathbf{k})$ satisfy the generic assumptions stated at the beginning of Section 3 as well as the no–nesting hypothesis NN. Let $K, K_{\mathbf{q}}$ be any compact subsets of $\mathbb{R}^{2d}$ and $\mathbb{R}$, respectively. There are constants $\varepsilon > 0$ and* const *such that for all $j_1, j_2, j_3 < 0$ and all $\mathbf{q} \in K_{\mathbf{q}}$,*

$$\begin{aligned}&\mathrm{Vol}\{(\mathbf{k},\mathbf{p}) \in \mathbb{R}^{2d} \cap K \mid |e(\mathbf{k})| \le M^{j_1}, |e(\mathbf{p})| \le M^{j_2}, |e(\mathbf{q} \pm \mathbf{k} \pm \mathbf{p})| \le M^{j_3}\} \\ &\le\ \mathrm{const}\ M^{j_{\pi(1)}} M^{j_{\pi(2)}} M^{\varepsilon j_{\pi(3)}}\end{aligned}$$

where π is a permutation of $\{1,2,3\}$ with $j_{\pi(3)} = \max\{j_1, j_2, j_3\}$.

Proof: We may assume without loss of generality that $j_3 = \max\{j_1, j_2, j_3\}$. Otherwise make a change of variables with $\mathbf{k}' = \mathbf{q} \pm \mathbf{k} \pm \mathbf{p}$, $\mathbf{p}' = \mathbf{k}$ or $\mathbf{p}$. By compactness, it suffices to show that for any $\tilde{\mathbf{k}}$, $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{q}}$ with $(\tilde{\mathbf{k}}, \tilde{\mathbf{p}}) \in K$ and

$\tilde{\mathbf{q}} \in K_{\mathbf{q}}$, there are constants c and $\varepsilon > 0$ (possibly depending on $\tilde{\mathbf{k}}$, $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{q}}$, but independent of the j_i's) such that

$$\mathrm{Vol}\Big\{ (\mathbf{k},\mathbf{p}) \ \Big| \ |e(\mathbf{k})| \le M^{j_1}, |\mathbf{k}-\tilde{\mathbf{k}}| \le c,\ |e(\mathbf{p})| \le M^{j_2}, |\mathbf{p}-\tilde{\mathbf{p}}| \le c, \\ |e(\mathbf{q}\pm\mathbf{k}\pm\mathbf{p})| \le M^{j_3} \Big\} \le \text{ const } M^{j_1} M^{j_2} M^{\varepsilon j_3} \tag{6}$$

for all $\mathbf{q}$ with $|\mathbf{q}-\tilde{\mathbf{q}}| \le c$ and all $j_1, j_2, j_3 < 0$ with $j_3 = \max\{j_1, j_2, j_3\}$.

If any one of $e(\tilde{\mathbf{k}})$, $e(\tilde{\mathbf{p}})$, $e(\tilde{\mathbf{q}} \pm \tilde{\mathbf{k}} \pm \tilde{\mathbf{p}})$ is nonzero, the left hand side of (6) is exactly zero for all sufficiently small c and sufficiently large $|j_3|$ (which also forces $|j_1|$ and $|j_2|$ to be sufficiently large). On the other hand, for any bounded set of j_3's, (6) follows from

$$\begin{aligned} \mathrm{Vol}\{ \mathbf{k} \in \mathbb{R}^d \mid |e(\mathbf{k})| \le M^{j_1}, |\mathbf{k}-\tilde{\mathbf{k}}| \le c \} &\le \text{ const } M^{j_1} \\ \mathrm{Vol}\{ \mathbf{p} \in \mathbb{R}^d \mid |e(\mathbf{p})| \le M^{j_2}, |\mathbf{p}-\tilde{\mathbf{p}}| \le c \} &\le \text{ const } M^{j_2} \end{aligned}$$

which holds by Lemma 2.3. So it suffices to consider $e(\tilde{\mathbf{k}}) = e(\tilde{\mathbf{p}}) = e(\tilde{\mathbf{q}} \pm \tilde{\mathbf{k}} \pm \tilde{\mathbf{p}}) = 0$.

By Lemma 2.4, if $\tilde{\mathbf{k}}$ is a singular point, then, for any $0 \le \eta < \frac{1}{2}$,

$$\mathrm{Vol}\{ \mathbf{k} \in \mathbb{R}^d \mid |e(\mathbf{k})| \le M^{j}, |\mathbf{k}-\tilde{\mathbf{k}}| \le M^{\eta j}, |\mathbf{k}-\tilde{\mathbf{k}}| \le c \} \le \text{ const } M^{j} M^{(d-2)\eta j}$$

Clearly, the same bound applies when $\tilde{\mathbf{k}}$ is a regular point (that is, if $\nabla e(\tilde{\mathbf{k}}) \ne 0$). By replacing (j, η) with $(j_1, \frac{j_3}{j_1}\eta)$ (observe that $\frac{j_3}{j_1}\eta$ is still between 0 and $\frac{1}{2}$), we have

$$\mathrm{Vol}\{ \mathbf{k} \in \mathbb{R}^d \mid |e(\mathbf{k})| \le M^{j_1}, |\mathbf{k}-\tilde{\mathbf{k}}| \le M^{\eta j_3}, |\mathbf{k}-\tilde{\mathbf{k}}| \le c \} \\ \le \text{ const } M^{j_1} M^{(d-2)\eta j_3}$$

and hence

$$\begin{aligned} &\mathrm{Vol}\{(\mathbf{k},\mathbf{p}) \mid |e(\mathbf{k})| \le M^{j_1}, |\mathbf{k}-\tilde{\mathbf{k}}| \le M^{\eta j_3}, |e(\mathbf{p})| \le M^{j_2}, |\mathbf{p}-\tilde{\mathbf{p}}| \le c, \\ &\qquad |e(\mathbf{q}\pm\mathbf{k}\pm\mathbf{p})| \le M^{j_3}\} \\ &\le \text{ const } M^{j_1} M^{(d-2)\eta j_3}\ \mathrm{Vol}\{ \mathbf{p} \in \mathbb{R}^d \mid |e(\mathbf{p})| \le M^{j_2}, |\mathbf{p}-\tilde{\mathbf{p}}| \le c \} \\ &\le \text{ const } M^{j_1} M^{j_2} M^{(d-2)\eta j_3} \end{aligned}$$

Similarly,

$$\begin{aligned} &\mathrm{Vol}\{(\mathbf{k},\mathbf{p}) \mid |e(\mathbf{k})| \le M^{j_1}, |\mathbf{k}-\tilde{\mathbf{k}}| \le c, |e(\mathbf{p})| \le M^{j_2}, |\mathbf{p}-\tilde{\mathbf{p}}| \le M^{\eta j_3}, \\ &\qquad |e(\mathbf{q}\pm\mathbf{k}\pm\mathbf{p})| \le M^{j_3}\} \\ &\le \text{ const } M^{j_1} M^{j_2} M^{(d-2)\eta j_3} \end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Vol}\Big\{ (\mathbf{k},\mathbf{p}) \Big| \, |e(\mathbf{k})| \le M^{j_1}, |\mathbf{k}-\tilde{\mathbf{k}}| \le c, |e(\mathbf{p})| \le M^{j_2}, |\mathbf{p}-\tilde{\mathbf{p}}| \le c, \\
&\qquad\qquad |e(\mathbf{q}\pm\mathbf{k}\pm\mathbf{p})| \le M^{j_3}, |\mathbf{q}\pm\mathbf{k}\pm\mathbf{p}-\tilde{\mathbf{q}}\mp\tilde{\mathbf{k}}\mp\tilde{\mathbf{p}}| \le M^{\eta j_3} \Big\} \\
&\le \text{ const } M^{j_2} \sup_{\tilde{\mathbf{k}}'} \mathrm{Vol}\{ \mathbf{k} \mid |e(\mathbf{k})| \le M^{j_1}, |\mathbf{k}-\tilde{\mathbf{k}}| \le c, |\mathbf{k}-\tilde{\mathbf{k}}'| \le M^{\eta j_3} \} \\
&\le \text{ const } M^{j_1} M^{j_2} M^{(d-2)\eta j_3}
\end{aligned}$$

Hence it suffices to prove that there is are $\tilde{\varepsilon} > 0$ and $0 < \eta < \frac{1}{2}$ such that

$$\begin{aligned}
&\mathrm{Vol}\Big\{ (\mathbf{k},\mathbf{p}) \Big| \, |e(\mathbf{k})| \le M^{j_1}, M^{\eta j_3} \le |\mathbf{k}-\tilde{\mathbf{k}}| \le c, \\
&\qquad |e(\mathbf{p})| \le M^{j_2}, M^{\eta j_3} \le |\mathbf{p}-\tilde{\mathbf{p}}| \le c \\
&\qquad |e(\mathbf{q}\pm\mathbf{k}\pm\mathbf{p})| \le M^{j_3}, M^{\eta j_3} \le |\mathbf{q}\pm\mathbf{k}\pm\mathbf{p}-\tilde{\mathbf{q}}\mp\tilde{\mathbf{k}}\mp\tilde{\mathbf{p}}| \le 3c \Big\} \\
&\le \text{ const } M^{j_1} M^{j_2} M^{\tilde{\varepsilon} j_3}
\end{aligned} \tag{7}$$

But, by hypothesis, $[\frac{\partial^2}{\partial \mathbf{k}_i \partial \mathbf{k}_j} e(\tilde{\mathbf{k}})]_{1\le i,j\le d}$ is nonsingular for every singular point $\tilde{\mathbf{k}}$. Hence, if $|\mathbf{k}-\tilde{\mathbf{k}}| \ge M^{\eta j_3}$ for all singular points $\tilde{\mathbf{k}}$, then $|\nabla e(\mathbf{k})| \ge CM^{\eta j_3}$ and if $|\mathbf{p}-\tilde{\mathbf{p}}| \ge M^{\eta j_3}$ for all singular points $\tilde{\mathbf{p}}$, then $|\nabla e(\mathbf{p})| \ge CM^{\eta j_3}$ and if $|\mathbf{q}\pm\mathbf{k}\pm\mathbf{p}-\tilde{\mathbf{q}}'| \ge M^{\eta j_3}$ for all singular points $\tilde{\mathbf{q}}'$, then $|\nabla e(\mathbf{q}\pm\mathbf{k}\pm\mathbf{p})| \ge CM^{\eta j_3}$. So, by Proposition 3.7 below, with $\delta = CM^{\eta j_3}$, $\varepsilon_1 = M^{j_1}$, $\varepsilon_2 = M^{j_2}$ and $\varepsilon_3 = M^{j_3}$,

$$\begin{aligned}
&\mathrm{Vol}\Big\{ (\mathbf{k},\mathbf{p}) \Big| \, |e(\mathbf{k})| \le M^{j_1}, M^{\eta j_3} \le |\mathbf{k}-\tilde{\mathbf{k}}| \le c, \\
&\qquad |e(\mathbf{p})| \le M^{j_2}, M^{\eta j_3} \le |\mathbf{p}-\tilde{\mathbf{p}}| \le c \\
&|e(\mathbf{q}\pm\mathbf{k}\pm\mathbf{p})| \le M^{j_3}, M^{\eta j_3} \le |\mathbf{q}\pm\mathbf{k}\pm\mathbf{p}-\tilde{\mathbf{q}}\mp\tilde{\mathbf{k}}\mp\tilde{\mathbf{p}}| \le 3c \Big\} \\
&\le \text{ const } \tfrac{1}{\delta^4} M^{j_1} M^{j_2} M^{\epsilon j_3} \\
&= \text{ const } M^{j_1} M^{j_2} M^{(\epsilon-4\eta) j_3}
\end{aligned}$$

If we choose $\eta = \frac{\epsilon}{d+2}$, then $(d-2)\eta = \epsilon - 4\eta = \frac{d-2}{d+2}\epsilon$ and the proposition follows with

$$\varepsilon = \tfrac{d-2}{d+2}\epsilon = \tfrac{d-2}{d+2}\tfrac{\kappa}{1+\kappa}$$

■

We can now prove the volume improvement estimate that generalizes the one from [12, Proposition 1.1] to our situation.

Proposition 3.7 *Let $K_{\mathbf{k}}$, $K_{\mathbf{p}}$ and $K_{\mathbf{q}}$ be compact subsets of $\mathbb{R}^d$ and $v_1, v_2 \in \{+1, -1\}$. There are constants C_{vol} and C_δ such that the following holds. Assume that there are $\delta, \kappa, \rho > 0$ such that*

(A1) for all $\mathbf{k} \in K_{\mathbf{k}}$, $\mathbf{p} \in K_{\mathbf{p}}$ and $\mathbf{q} \in K_{\mathbf{q}}$: $|\nabla e(\mathbf{k})| \geq \delta$, $|\nabla e(\mathbf{p})| \geq \delta$, and $|\nabla e(v_1\mathbf{k} + v_2\mathbf{p} + \mathbf{q})| \geq \delta$

(A2) the "only polynomial flatness" condition of Hypothesis NN is satisfied.

Set

$$\epsilon = \tfrac{\kappa}{1+\kappa} \tag{8}$$

Let

$$I_2(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \sup_{\mathbf{q} \in K_{\mathbf{q}}} \int_{K_{\mathbf{k}} \times K_{\mathbf{p}}} d^d\mathbf{k} d^d\mathbf{p} \, 1\left(|e(\mathbf{k})| \leq \varepsilon_1\right) 1\left(|e(\mathbf{p})| \leq \varepsilon_2\right) \times 1\left(|e(v_1\mathbf{k} + v_2\mathbf{p} + \mathbf{q})| \leq \varepsilon_3\right) \tag{9}$$

Then, for all $0 < \varepsilon_1 \leq 1$, $0 < \varepsilon_2 \leq 1$, $\max\{\varepsilon_1, \varepsilon_2\} \leq \varepsilon_3 \leq 1$ with $\delta \geq C_\delta \max\{\sqrt{\varepsilon_1}, \sqrt{\varepsilon_2}\}$

$$I_2(\varepsilon_1, \varepsilon_2, \varepsilon_3) \leq C_{vol} \tfrac{1}{\delta^4} \varepsilon_1 \varepsilon_2 \varepsilon_3^{\epsilon}. \tag{10}$$

Proof: By compactness it suffices to assume that $K_{\mathbf{k}}$ is contained either in the ball $\{ \mathbf{k} \in \mathbb{R}^d \mid |\mathbf{k} - \tilde{\mathbf{k}}| \leq c \}$ for some $\tilde{\mathbf{k}} \in \mathcal{F}$ with $\nabla e(\tilde{\mathbf{k}}) \neq 0$ (i.e. $\tilde{\mathbf{k}}$ is a regular point) or in the annulus $\{ \mathbf{k} \in \mathbb{R}^d \mid c'\delta \leq |\mathbf{k} - \tilde{\mathbf{k}}| \leq c \}$ for some $\tilde{\mathbf{k}} \in \mathcal{F}$ with $\nabla e(\tilde{\mathbf{k}}) = 0$ (i.e. $\tilde{\mathbf{k}}$ is a singular point). We are free to choose $c, c' > 0$, depending on $\tilde{\mathbf{k}}$. We may make similar assumptions about $K_{\mathbf{p}}$ and the allowed values of $v_1\mathbf{k} + v_2\mathbf{p} + \mathbf{q}$.

Make a change of variables from $\mathbf{k}$ to (ρ_1, ω_1), with $\rho_1 = e(\mathbf{k})$. We may assume that $K_{\mathbf{k}}$ is covered by a single such coordinate patch, with Jacobian

$$|J_1(\rho_1, \omega_1)| \leq \frac{\text{const}}{\delta}$$

In the case that $\tilde{\mathbf{k}}$ is a singular point, we would use the Morse lemma, to provide a diffeomorphism $\mathbf{k}(\mathbf{x})$ such that

$$e(\mathbf{k}(\mathbf{x})) = x_1^2 + \ldots + x_m^2 - x_{m+1}^2 - \ldots - x_d^2$$

On the inverse image of $K_{\mathbf{k}}$,

$$2|\mathbf{x}| = |\nabla_{\mathbf{x}} e(\mathbf{k}(\mathbf{x}))| = |(\nabla_{\mathbf{k}} e)(\mathbf{k}(\mathbf{x}))^t \tfrac{\partial \mathbf{k}}{\partial \mathbf{x}}(\mathbf{x})| \geq \text{ const } \delta$$

So we may first change variables from $\mathbf{k}$ to $\mathbf{x}$, with Jacobian bounded and bounded away from zero (uniformly in δ) and then, in the region where, for example $|x_1| \geq$ const $\max\{|x_2|,\ldots,|x_d|\}$, change variables from $\mathbf{x}$ to

$$(\rho,\omega) = (x_1^2 + \ldots + x_m^2 - x_{m+1}^2 - \ldots - x_d^2, x_2, \ldots, x_d)$$

The second change of variables has Jacobian $2|x_1| \geq$ const δ. Observe that, under this change of variables, the matrix

$$\frac{\partial\ \mathbf{k}}{\partial(\rho,\omega)} = \frac{\partial\mathbf{k}}{\partial\mathbf{x}}\begin{bmatrix} 2x_1 & 2x_2 & \ldots & -2x_d \\ 0 & & & \\ \vdots & & \mathbb{1} & \\ 0 & & & \end{bmatrix}^{-1} = \frac{\partial\mathbf{k}}{\partial\mathbf{x}}\begin{bmatrix} \frac{1}{2x_1} & -\frac{x_2}{x_1} & \ldots & \frac{x_d}{x_1} \\ 0 & & & \\ \vdots & & \mathbb{1} & \\ 0 & & & \end{bmatrix}$$

has operator norm bounded by $\frac{\text{const}}{\delta}$. So $|\mathbf{k}(\rho,\omega) - \mathbf{k}(0,\omega)| \leq$ const $\frac{|\rho|}{\delta}$.

Make a similar change of variables from $\mathbf{p}$ to (ρ_2,ω_2), with $\rho_2 = e(\mathbf{p})$. Again, we may assume that $K_{\mathbf{p}}$ is covered by a single such coordinate patch, with Jacobian $|J_2(\rho_2,\omega_2)| \leq \frac{\text{const}}{\delta}$. Then

$$\begin{aligned} I_2(\varepsilon_1,\varepsilon_2,\varepsilon_3) \;&\leq\; \sup_{\mathbf{q}\in K_{\mathbf{q}}} \int_{-\varepsilon_1}^{\varepsilon_1} d\rho_1 \int_{S_1} d\omega_1\ J_1(\rho_1,\omega_1) \int_{-\varepsilon_2}^{\varepsilon_2} d\rho_2 \int_{S_2} d\omega_2\ J_2(\rho_2,\omega_2) \\ &\qquad 1(|e(v_1\mathbf{k}(\rho_1,\omega_1) + v_2\mathbf{p}(\rho_2,\omega_2) + \mathbf{q})| \leq \varepsilon_3) \\ &\leq\; \text{const}\ \varepsilon_1\varepsilon_2\tfrac{1}{\delta^2} \sup_{\mathbf{q}\in K_{\mathbf{q}}} \sup_{|\rho_1|,|\rho_2|\leq\varepsilon_3} \int_{S_1} d\omega_1 \int_{S_2} d\omega_2 \\ &\qquad 1(|e(v_1\mathbf{k}(\rho_1,\omega_1) + v_2\mathbf{p}(\rho_2,\omega_2) + \mathbf{q})| \leq \varepsilon_3) \end{aligned}$$

By the mean value theorem

$$|e(v_1\mathbf{k}(\rho_1,\omega_1) + v_2\mathbf{p}(\rho_2,\omega_2) + \mathbf{q}) - e(v_1\mathbf{k}(0,\omega_1) + v_2\mathbf{p}(0,\omega_2) + \mathbf{q})| \leq \text{const}\ \tfrac{\varepsilon_3}{\delta}$$

for all ρ_1,ρ_2 with $|\rho_i| \leq \varepsilon_3$. Thus

$$|e(v_1\mathbf{k}(\rho_1,\omega_1) + v_2\mathbf{p}(\rho_2,\omega_2) + \mathbf{q})| \leq \varepsilon_3$$

implies

$$|e(v_1\mathbf{k}(0,\omega_1) + v_2\mathbf{p}(0,\omega_2) + \mathbf{q})| \leq \text{ const } \tfrac{\varepsilon_3}{\delta}$$

and

$$I_2(\varepsilon_1,\varepsilon_2,\varepsilon_3) \leq \text{ const } \varepsilon_1\varepsilon_2\tfrac{1}{\delta^2}W(\text{ const } \tfrac{\varepsilon_3}{\delta})$$

with

$$W(\zeta) = \sup_{\mathbf{q}\in K_{\mathbf{q}}} \int_{S_1} d\omega_1 \int_{S_2} d\omega_2 \; 1(|e(v_1\mathbf{k}(0,\omega_1) + v_2\mathbf{p}(0,\omega_2) + \mathbf{q})| \le \zeta)$$

We claim that $|\nabla e(v_1\mathbf{k}(0,\omega_1) + v_2\mathbf{p}(0,\omega_2) + \mathbf{q})| \ge$ const δ for all $\omega_1 \in S_1$ and $\omega_2 \in S_2$. This will be used in the proof of the following Lemma, which generalizes [12, Lemma A.1] and which implies the bound (10). We have assumed that $K_{\mathbf{k}}$, $K_{\mathbf{p}}$ and $K_{\mathbf{q}}$ are contained in small balls or annuli centred on $\tilde{\mathbf{k}}$, $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{q}}$ respectively. If $v_1\tilde{\mathbf{k}} + v_2\tilde{\mathbf{p}} + \tilde{\mathbf{q}}$ is a regular point, simple continuity yields that $|\nabla e(v_1\mathbf{k}(0,\omega_1) + v_2\mathbf{p}(0,\omega_2) + \mathbf{q})| \ge$ const provided we chose c small enough. So it suffi ces to consider the case that $\mathbf{r} = v_1\tilde{\mathbf{k}} + v_2\tilde{\mathbf{p}} + \tilde{\mathbf{q}}$ is a singular point.

The constraint $|\rho_1| < \varepsilon_1$ ensures that $|\mathbf{k}(\rho_1,\omega_1) - \mathbf{k}(0,\omega_1)| \le$ const $\frac{\varepsilon_1}{\delta}$ and the constraint $|\rho_2| < \varepsilon_2$ ensures that $|\mathbf{k}(\rho_2,\omega_2) - \mathbf{k}(0,\omega_2)| \le$ const $\frac{\varepsilon_2}{\delta}$. So the original condition that $|\nabla e(v_1\mathbf{k}(\rho_1,\omega_1) + v_2\mathbf{p}(\rho_2,\omega_2) + \mathbf{q})| \ge \delta$ implies that

$$|v_1\mathbf{k}(\rho_1,\omega_1) + v_2\mathbf{p}(\rho_2,\omega_2) + \mathbf{q} - \mathbf{r}| \ge \text{ const } \delta$$

and hence

$$|v_1\mathbf{k}(0,\omega_1) + v_2\mathbf{p}(0,\omega_2) + \mathbf{q} - \mathbf{r}| \ge \text{ const } \delta - \tfrac{\varepsilon_1}{\delta} - \tfrac{\varepsilon_2}{\delta} \ge \text{ const } \delta$$

provided $\delta \ge$ const $\max\{\sqrt{\varepsilon_1}, \sqrt{\varepsilon_2}\}$. So

$$|\nabla e(v_1\mathbf{k}(0,\omega_1) + v_2\mathbf{p}(0,\omega_2) + \mathbf{q})| \ge \text{ const } \delta$$

as desired. ■

Lemma 3.8 $W(\zeta) \le Z_3 \frac{1}{\delta}\zeta^\epsilon$ *where* $\epsilon = \frac{\kappa}{1+\kappa}$.

Proof: Let $\gamma \in (0,1)$,

$$\mathcal{T} = \left\{ (\omega_1,\omega_2) \in \mathcal{F}\times\mathcal{F} \;\middle|\; \sqrt{1 - (n(\omega_1)\cdot n(\omega_2))^2} \ge \zeta^{1-\gamma} \right\}$$

be the set where the intersection is transversal and $\mathcal{E} = \mathcal{F}\times\mathcal{F}\setminus\mathcal{T}$ its complement. We shall choose γ at the end. Split $W(\zeta) = T(\zeta) + E(\zeta)$ into the contributions from these two sets.

The contribution from the set of exceptional momenta $\mathcal{E}$ is bounded using Hypothesis NN. For each $\omega_1 \in S_1$, let

$$\mathcal{E}_{\omega_1} = \left\{ \omega_2 \in S_2 \;\middle|\; \sqrt{1 - (n(\omega_1) \cdot n(\omega_2))^2} < \zeta^{1-\gamma} \right\}$$

Then by Hypothesis NN

$$E(\zeta) \le \int_{S_1} d\omega_1 \int_{\mathcal{E}_{\omega_1} \cap S_2} d\omega_2 \le \int_{S_1} d\omega_1 \; Z_0 \zeta^{\kappa(1-\gamma)} = \text{ const } \zeta^{\kappa(1-\gamma)}$$

Now we bound T. We start by introducing a cover of $\mathcal{F}$ by coordinate patches. Let, for each singular point $\tilde{\mathbf{k}}$ of $\mathcal{F}$, $\mathcal{O}_{\tilde{\mathbf{k}}}$ be the open neighbourhood of $\tilde{\mathbf{k}}$ that is the image of $\{|\mathbf{x}| < 1\}$ under the Morse diffeomorphism $\mathbf{k}(\mathbf{x})$. If

$$e(\mathbf{k}(\mathbf{x})) = x_1^2 + \ldots + x_m^2 - x_{m+1}^2 - \ldots - x_d^2$$

write $\mathbf{x} = (r\boldsymbol{\theta}_1, r\boldsymbol{\theta}_2)$ with $0 \le r \le 1/\sqrt{2}$, $\boldsymbol{\theta}_1 \in S^{m-1}$ and $\boldsymbol{\theta}_2 \in S^{d-m-1}$. Introduce "roughly orthonormal" coordinate patches on S^{m-1}.

Here is what we mean by the statement that $\boldsymbol{\theta}_1(\alpha_1, \ldots, \alpha_{m-1})$ is "roughly orthonormal". Let

$$A(\alpha_1, \ldots, \alpha_{m-1}) = \left[\tfrac{\partial \boldsymbol{\theta}_1}{\partial \alpha_1}(\alpha_1, \ldots, \alpha_{m-1}), \ldots, \tfrac{\partial \;\; \boldsymbol{\theta}_1}{\partial \alpha_{m-1}}(\alpha_1, \ldots, \alpha_{m-1}) \right]$$

be the $m \times m-1$ matrix whose columns are the tangent vectors to the coordinate axes at $\boldsymbol{\theta}_1(\alpha_1, \ldots, \alpha_{m-1})$. The columns of this matrix span the tangent space to S^{m-1} at $\boldsymbol{\theta}_1(\alpha_1, \ldots, \alpha_{m-1})$. Let $V(\alpha_1, \ldots, \alpha_{m-1})$ be an $(m-1) \times (m-1)$ matrix such that the columns of $A(\alpha_1, \ldots, \alpha_{m-1})V(\alpha_1, \ldots, \alpha_{m-1})$ are mutually orthogonal unit vectors. Those columns form an orthonormal basis for the tangent space to S^{m-1} at $\boldsymbol{\theta}_1(\alpha_1, \ldots, \alpha_{m-1})$. "Roughly orthogonal" signifi es that V and its inverse are uniformly bounded on the domain of the coordinate patch. The only consequence of rough orthonormality that we will use is that, if v is any vector in the tangent space to S^{m-1} at $\boldsymbol{\theta}_1(\alpha_1, \ldots, \alpha_{m-1})$, then, because

$$\begin{aligned} \|v\| &= \left\| v^t [A(\alpha_1, \ldots, \alpha_{m-1}) V(\alpha_1, \ldots, \alpha_{m-1})] \right\| \\ &\le \|v^t A(\alpha_1, \ldots, \alpha_{m-1})\| \|V(\alpha_1, \ldots, \alpha_{m-1})\| \end{aligned}$$

implies

$$\|v^t A(\alpha_1, \ldots, \alpha_{m-1})\| \ge \|V(\alpha_1, \ldots, \alpha_{m-1})\|^{-1} \|v\|$$

we have

$$\max_{1\le j\le m-1} |v\cdot \tfrac{\partial \boldsymbol{\theta}_1}{\partial\alpha_j}(\alpha_1,\ldots,\alpha_{m-1})| \ge \tfrac{1}{\sqrt{m-1}}\|V(\alpha_1,\ldots,\alpha_{m-1})\|^{-1}\|v\| \tag{11}$$

Also introduce a “roughly orthonormal” coordinate patch $\boldsymbol{\theta}_2(\alpha_m,\ldots,\alpha_{d-2})$ on S^{d-m-1} and parametrize (a patch on) the cone $x_1^2+\ldots+x_m^2-x_{m+1}^2-\ldots-x_d^2=0$ by

$$\mathbf{x}(\alpha_1,\ldots,\alpha_{d-1}) = (\alpha_{d-1}\boldsymbol{\theta}_1(\alpha_1,\ldots,\alpha_{m-1}),\alpha_{d-1}\boldsymbol{\theta}_2(\alpha_m,\ldots,\alpha_{d-2}))$$

and the corresponding patch on $\mathcal{O}_{\tilde{\mathbf{k}}}$ by $\mathbf{k}(\mathbf{x}(\alpha_1,\ldots,\alpha_{d-1}))$. Denote

$$\omega_1(\alpha_1,\ldots,\alpha_{d-1}) = \mathbf{k}(\mathbf{x}(\alpha_1,\ldots,\alpha_{d-1}))$$

For patches away from the singular points, any roughly orthonormal coordinate systems will do. Observe that, if v is any vector in the tangent space to $\mathcal{F}$ at $\omega_1(\alpha_1,\ldots,\alpha_{d-1})$, then

$$\max_{1\le j\le d-1} |v\cdot \tfrac{\partial\omega}{\partial\alpha_j}(\alpha_1,\ldots,\alpha_{d-1})| \ge \|v\| \begin{cases} \text{const} & \text{regular patch} \\ \text{const}\ \alpha_{d-1} & \text{singular patch}\end{cases} \tag{12}$$

Now fi x any $\mathbf{q}\in K_{\mathbf{q}}$ and consider the contribution to

$$\iint_{S_1\times S_2\cap\mathcal{T}} d\omega_1 d\omega_2\ 1\Big(|e(v_1\mathbf{k}(0,\omega_1)+v_2\mathbf{p}(0,\omega_2)+\mathbf{q})|\le\zeta\Big)$$

from one pair, $\omega_1(\alpha_1,\ldots,\alpha_{d-1})$ and $\omega_2(\beta_1,\ldots,\beta_{d-1})$, of coordinate patches as described above. The Jacobian $\frac{\partial\quad\omega_1}{\partial\alpha_1\ldots\partial\alpha_{d-1}}$ is bounded by a constant, in the regular case, and a constant times α_{d-1}^{d-2}, in the singular case. Denote by $\theta(\omega_1,\omega_2)$ the angle between $n(\omega_1)$ and $n(\omega_2)$. By the transversality condition, $\sin\theta(\omega_1,\omega_2)\ge\zeta^{1-\gamma}$. Consequently, for at least one $i\in\{1,2\}$ the sine of the angle between $n(\omega_i)$ and $\nabla e(v_1\mathbf{k}(0,\omega_1)+v_2\mathbf{p}(0,\omega_2)+\mathbf{q})$ is at least

$$\sin\tfrac{1}{2}\theta(\omega_1,\omega_2)\ge\tfrac{1}{2}\sin\theta(\omega_1,\omega_2)\ge\tfrac{1}{2}\zeta^{1-\gamma}$$

and the length of the projection of $\nabla e(v_1\mathbf{k}(0,\omega_1)+v_2\mathbf{p}(0,\omega_2)+\mathbf{q})$ on $T_{\omega_i}\mathcal{F}$ must be at least $\frac{1}{2}\zeta^{1-\gamma}|\nabla e(v_1\mathbf{k}(0,\omega_1)+v_2\mathbf{p}(0,\omega_2)+\mathbf{q})|\ge$ const $\delta\,\zeta^{1-\gamma}$. Suppose that $i=1$. Defi ne

$$\rho = e(v_1\mathbf{k}(0,\omega_1)+v_2\mathbf{p}(0,\omega_2)+\mathbf{q})$$

viewed as a function of $\alpha_1, \ldots, \alpha_{d-1}$ and $\beta_1, \ldots, \beta_{d-1}$. By (12), there must be a $1 \le j \le d-1$ such that

$$\begin{aligned} \left|\tfrac{\partial \rho}{\partial \alpha_j}\right| &= \left|\nabla e(v_1\mathbf{k}(0,\omega_1) + v_2\mathbf{p}(0,\omega_2) + \mathbf{q}) \cdot \tfrac{\partial \omega}{\partial \alpha_j}(\alpha_1, \ldots, \alpha_{d-1})\right| \\ &\ge \text{const}\, \delta\, \zeta^{1-\gamma} \begin{cases} \text{const} & \text{regular patch} \\ \text{const}\ \alpha_{d-1} & \text{singular patch} \end{cases} \end{aligned}$$

Make a final change of variables replacing α_j by ρ. The Jacobian for the composite change of variables from (ω_1, ω_2) to $(\alpha_1, \ldots, \alpha_{d-1}, \beta_1, \ldots, \beta_{d-1})$ and then to $\big((\alpha_i)_{\substack{1\le i\le d-1 \\ i\ne j}}, (\beta_i)_{1\le i\le d-1}, \rho\big)$ is bounded by

$$\text{const}\ \tfrac{1}{\delta}\zeta^{\gamma-1} \left\{ \begin{matrix} \text{const} & \text{regular patch} \\ \text{const}\ \alpha_{d-1}^{d-3} & \text{singular patch} \end{matrix} \right\} \le \ \text{const}\ \tfrac{1}{\delta}\zeta^{\gamma-1}$$

We thus have

$$T(\zeta) \le \ \text{const}\ \tfrac{1}{\delta}\zeta^{\gamma-1} \int_{-\zeta}^{\zeta} d\rho \le \ \text{const}\ \tfrac{1}{\delta}\zeta^{\gamma}$$

The optimal bound is when $\kappa(1-\gamma) = \gamma$, that is, $\gamma = \kappa/(1+\kappa)$. ■

3.3 The proof of Theorem 1.1

Proof of Theorem 1.1: Now that we have Proposition 3.6, the proof of Theorem 1.1 is almost identical to the corresponding proofs of [12]. The main change is that our current choice of localization operator simplifies the argument. Several proofs in this paper and its companion paper [11] are variants of the arguments of [12]. So we have provided, in Appendix A, a complete, self–contained proof that the value, $G(q)$, of each renormalized 1PI, two–legged graph is $C^{1-\varepsilon}$, using the simplest form of the argument in question. In particular, it does not use "volume improvement" bounds like Proposition 3.6. We here show how to use Proposition 3.6 to upgrade $C^{1-\varepsilon}$ to $C^{1+\varepsilon}$. This is a good time to read that Appendix, since we shall just explain the modifications to be made to it.

As in Appendix A, use (22) to introduce a scale expansion for each propagator and express $G(q)$ in terms of a renormalized tree expansion (24). We shall prove, by induction on the depth, D, of G^J, the bound

$$\sum_{J\in\mathcal{J}(j,t,R,G)} \sup_q \left|\partial_{q_0}^{s_0}\partial_{\mathbf{q}}^{s_1} G^J(q)\right| \le \text{const}_n |j|^{3n-2} M^{(1-s_0-s_1)j} \begin{cases} M^{\varepsilon j} & \text{if } s_0+s_1 \ge 1 \\ 1 & \text{if } s_0 = s_1 = 0 \end{cases} \tag{13}$$

for $s_0, s_1 \in \{0, 1, 2\}$. Here ε was specified in Proposition 3.6 and the other notation is as in Appendix A: n is the number of vertices in G and $\mathcal{J}(j, t, R, G)$ is the set of all assignments J of scales to the lines of G that have root scale j, that give forest t and that are compatible with the assignment R of renormalization labels to the two–legged forks of t. (This is explained in more detail just before (24).) If $s_0 + s_1 = 1$, the right hand side becomes $\mathrm{const}_n |j|^{3n-2} M^{\varepsilon j}$, which is summable over $j < 0$, implying that $G(q) = \sum_R \sum_{j<0} \sum_{J \in \mathcal{J}(j,t,R,G)} G^J(q)$ is C^1. To show that the first order derivatives of $G(q)$ are Hölder continuous of any degree strictly less than ε, just observe that if

$$\|f_j\|_\infty \le \mathrm{const}_n |j|^{3n-2} M^{\varepsilon j} \qquad \text{and} \qquad \|f'_j\|_\infty \le \mathrm{const}_n |j|^{3n-2} M^{\varepsilon j} M^{-j}$$

then

$$\begin{aligned} |f_j(x) - f_j(y)| &\le \min\{2\|f_j\|_\infty, \|f'_j\|_\infty |x-y|\} \\ &\le \mathrm{const}_n |j|^{3n-2} M^{\varepsilon j} \min\{2, M^{-j}|x-y|\} \\ &\le \mathrm{const}_n |j|^{3n-2} M^{\varepsilon j} M^{-\eta j} |x-y|^\eta \qquad \text{for any } 0 \le \eta \le 1 \end{aligned}$$

is summable over $j < 0$ for any $0 < \eta < \varepsilon$.

If $s_0 = s_1 = 0$, (13) is contained in Proposition A.1, so it suffices to consider $s_0 + s_1 \ge 1$. As in Appendix A, if $D > 0$, decompose the tree t into a pruned tree $\tilde{t}$ and insertion subtrees $\tau^1, \cdots, \tau^m$ by cutting the branches beneath all minimal $E_f = 2$ forks $f_1, \cdots, f_m$. In other words each of the forks $f_1, \cdots, f_m$ is an $E_f = 2$ fork having no $E_f = 2$ forks, except ϕ, below it in t. Each τ_i consists of the fork f_i and all of t that is above f_i. It has depth at most $D - 1$ so the corresponding subgraph G_{f_i} obeys (13). Think of each subgraph G_{f_i} as a generalized vertex in the graph $\tilde{G} = G/\{G_{f_1}, \cdots, G_{f_m}\}$. Thus $\tilde{G}$ now has two as well as four–legged vertices. These two–legged vertices have kernels of the form $T_i(k) = \sum_{j_{f_i} \le j_{\pi(f_i)}} \ell G_{f_i}(k)$ when f_i is a c–fork and of the form $T_i(k) = \sum_{j_{f_i} > j_{\pi(f_i)}} (\mathbb{1} - \ell) G_{f_i}(k)$ when f_i is an r–fork. At least one of the external lines[1] of G_{f_i} must be of scale precisely $j_{\pi(f_i)}$ so the momentum k passing through G_{f_i} lies in the support of $C_{j_{\pi(f_i)}}$. In the case of a c–fork $f = f_i$ we have, as in (27) and using the same notation, by the inductive hypothesis,

$$\sum_{j_f \le j_{\pi(f)}} \sum_{J_f \in \mathcal{J}(j_f, t_f, R_f, G_f)} \sup_k \left| \partial_{\mathbf{k}}^{s'_1} \ell G_f^{J_f}(k) \right| \le \sum_{j_f \le j_{\pi(f)}} \mathrm{const}_{n_f} |j_f|^{3n_f - 2} M^{j_f} M^{-s'_1(1-\varepsilon) j_f}$$

[1]Note that the root fork, $\emptyset$, of (24) does not carry an r, c label so that $\tilde{G}$ may not be simply a single two–legged c– or r–vertex. At least one external line of each G_{f_i} must be an internal line of $\tilde{G}$.

$$\leq \mathrm{const}_{n_f} |j_{\pi(f)}|^{3n_f-2} M^{j_{\pi(f)}} M^{-s_1'(1-\varepsilon) j_{\pi(f)}} \tag{14}$$

for $s_1' = 0, 1$. Note that the sum in the analog of (14) diverges when $s_1' = 2$, so it is essential that no more than one derivative act on any c–fork. As $\ell G_f^{J_f}(k)$ is independent of k_0, derivatives with respect to k_0 may not act on it. In the case of an r–fork $f = f_i$, we have, as in (29), using the mean value theorem in the case $s_0' = 0$,

$$\begin{aligned}
&\sum_{j_f > j_{\pi(f)}} \sum_{J_f \in \mathcal{J}(j_f, t_f, R_f, G_f)} \sup_k 1(C_{j_{\pi(f)}}(k) \neq 0) \left| \partial_{k_0}^{s_0'} \partial_{\mathbf{k}}^{s_1'} (\mathbb{1} - \ell) G_f^{J_f}(k) \right| \\
&\qquad \leq \sum_{j_f > j_{\pi(f)}} \sum_{J_f \in \mathcal{J}(j_f, t_f, R_f, G_f)} M^{(1-\min\{1, s_0'\}) j_{\pi(f)}} \sup_k \left| \partial_{k_0}^{\max\{1, s_0'\}} \partial_{\mathbf{k}}^{s_1'} G_f^{J_f}(k) \right| \\
&\qquad \leq \mathrm{const}_{n_f} M^{(1-\min\{1, s_0'\}) j_{\pi(f)}} \sum_{j_f > j_{\pi(f)}} |j_f|^{3n_f-2} M^{-(\max\{1, s_0'\} + s_1' - 1 - \varepsilon) j_f} \\
&\qquad \leq \mathrm{const}_{n_f} |j_{\pi(f)}|^{3n_f-2} M^{j_{\pi(f)}} M^{-s_0' j_{\pi(f)}} M^{-s_1' j_{\pi(f)}}
\end{aligned} \tag{15}$$

Denote by $\tilde{J}$ the restriction to $\tilde{G}$ of the scale assignment J. We bound $\tilde{G}^{\tilde{J}}$, which again is of the form (31), by a variant of the six step procedure followed in Appendix A. In fact the first five steps are almost identical.

1. Choose a spanning tree $\tilde{T}$ for $\tilde{G}$ with the property that $\tilde{T} \cap \tilde{G}_f^{\tilde{J}}$ is a connected tree for every $f \in t(\tilde{G}^{\tilde{J}})$.

2. Apply any q–derivatives. By the product rule each derivative may act on any line or vertex on the "external momentum path". It suffices to consider any one such action. Ensure, through a judicious use of integration by parts, that at most one derivative acts on any single c–fork. To do so, observe that a derivative with respect to the external momentum acting on a c–fork is, up to a sign, equal to the derivative with respect to any loop momentum that flows through the fork. So replace one external momentum derivative by a loop momentum derivative and integrate by parts to move the latter off of the c–fork.

3. Bound each two–legged renormalized subgraph (i.e. r–fork) by (15) and each two–legged counterterm (i.e. c–fork) by (14). Observe that when s_0' k_0–derivatives and s_1' $\mathbf{k}$–derivatives act on the vertex, the bound is no worse than $M^{-s_0' j_{\pi(f)}} M^{-s_1' j_{\pi(f)}}$ times the bound with no derivatives. (We shall not

need the factor $M^{s_1'\varepsilon j_{\pi(f)}}$ in (14). So we simply discard it.) As we have already observed, one of the external lines of the two–legged vertex must be of scale precisely $j_{\pi(f)}$. We write $M^{-s_0' j_{\pi(f)}} M^{-s_1' j_{\pi(f)}} = M^{-s_0' j_\ell} M^{-s_1' j_\ell}$, where ℓ is that line.

4. Bound all of the remaining vertex functions, (suitably differentiated) by their suprema in momentum space. We have already observed that if $s_0 = s_1 = 0$, the target bound (13) is contained in Proposition A.1, with $s_0 = s = 0$. In the event that $s_0 + s_1 \geq 1$, but all derivatives act on four–legged vertex functions, Proposition A.1, again with $s_0 = s = 0$ but with one or two four–legged vertex functions replaced by differentiated functions, again gives (13). So it suffi ces to consider the case that at least one derivative acts on a propagator or on a c– or r–fork.

5. Bound each propagator

$$|\partial_{k_0}^{s_0'} \partial_{\mathbf{k}}^{s_1'} C_{j_\ell}(k)| \leq \text{ const } M^{-(1+s_0'+s_1')j_\ell} \, 1(|ik_0 - e(\mathbf{k})| \leq M^{j_\ell}) \tag{16}$$

   Once again, when s_0' k_0–derivatives and s_1' $\mathbf{k}$–derivatives act on the propagator, the bound is no worse than $M^{-(s_0'+s')j_\ell}$ times the bound with no derivatives.

We now have $|\partial_{q_0}^{s_0} \partial_{\mathbf{q}}^{s_1} \tilde{G}^{\tilde{J}}(q)|$ bounded, uniformly in q by

$$\begin{aligned} \text{const}_n \prod_d M^{-j_{\ell_d}} \prod_{i=1}^{m} |j_{\pi(f_i)}|^{3n_{f_i}-2} M^{j_{\pi(f_i)}} \prod_{\ell \in \tilde{G}} M^{-j_\ell} \\ \int \prod_{\ell \in \tilde{G} \setminus \tilde{T}} đk_\ell \prod_{\ell \in \tilde{G}} 1(|ik_{\ell 0} - e(\mathbf{k}_\ell)| \leq M^{j_\ell}) \end{aligned} \tag{17}$$

Here d runs over the $s_0 + s_1 \geq 1$ derivatives in $\partial_{q_0}^{s_0} \partial_{\mathbf{q}}^{s_1}$ and ℓ_d refers to the specifi c line on which the derivative acted (or, in the case that the derivative acted on a c– or r–fork, the external line specifi ed in step 3).

For $\ell \in \tilde{T}$, the momentum k_ℓ is a signed sum of the loop momenta and external momentum flowing through ℓ. In Appendix A, we discarded the factors of the integrand $\prod_{\ell \in \tilde{G}} 1(|ik_{\ell 0} - e(\mathbf{k}_\ell)| \leq M^{j_\ell})$ with $\ell \in \tilde{T}$ at this point. Then the integrals over the loop momenta factorized and we bounded them by the volumes of their domains of integration, using Lemma 2.3. We now deviate from the argument of

Appendix A by exploiting the constraint that one factor $1(|ik_{\ell 0} - e(\mathbf{k}_\ell)| \le M^{j_\ell})$ with $\ell \in \tilde{T}$ imposes on the domain of integration.

We have reduced consideration to cases in which at least one derivative with respect to the external momentum acts either on a propagator in $\tilde{T}$ or on a two–legged c– or r–vertex in $\tilde{T}$, so that the associated line $\ell_d \in \tilde{T}$. Select any such ℓ_{d_0}. Recall that any line $\ell \in \tilde{G} \setminus \tilde{T}$ is associated with a loop Λ_ℓ that consists of ℓ and the linear subtree of $\tilde{T}$ joining the vertices at the ends of ℓ. By [11, Lemma 4.3], there exist two lines $\ell_1, \ell_2 \in \tilde{G} \setminus \tilde{T}$ such that $\ell_{d_0} \in \Lambda_{\ell_1} \cap \Lambda_{\ell_2}$. By [11, Lemma 4.4], $j_{\ell_1}, j_{\ell_2} \le j_{\ell_{d_0}}$. Now discard all of the factors $\prod_{\ell \in \tilde{G}} 1(|ik_{\ell 0} - e(\mathbf{k}_\ell)| \le M^{j_\ell})$ in the integrand of (17) with $\ell \in \tilde{T} \setminus \{\ell_{d_0}\}$. Choose the order of integration in (17) so that k_{ℓ_1} and k_{ℓ_2} are integrated first. By Proposition 3.6,

$$\int \prod_{\ell \in \{\ell_1, \ell_2\}} đk_\ell \prod_{\ell \in \{\ell_1, \ell_2, \ell_{d_0}\}} 1(|ik_{\ell 0} - e(\mathbf{k}_\ell)| \le M^{j_\ell}) \le \text{ const } M^{2j_{\ell_1}} M^{2j_{\ell_2}} M^{\varepsilon j_{d_0}} \tag{18}$$

Finally, integrate over the remaining loop momenta k_ℓ, $\ell \in \tilde{G} \setminus (\tilde{T} \cup \{\ell_1, \ell_2\})$ as in step 6 of $(A.1)$. The integral over each such k_ℓ is bounded by vol$\{\, k_{0\ell} \mid |k_{0\ell}| \le M^{j_\ell} \,\} \le 2M^{j_\ell}$ times the volume of $\{\, \mathbf{k}_\ell \mid |e(\mathbf{k}_\ell)| \le M^{j_\ell} \,\}$, which, by Lemma 2.3, is bounded by a constant times $|j_\ell| M^{j_\ell}$. We now have $|\partial_{q_0}^{s_0} \partial_{\mathbf{q}}^{s_1} \tilde{G}^{\tilde{J}}(q)|$ bounded, uniformly in q, by

$$\text{const}_n M^{\varepsilon j_{\ell_{d_0}}} \prod_d M^{-j_{\ell_d}} \prod_{i=1}^m |j_{\pi(f_i)}|^{3n_{f_i}-2} M^{j_{\pi(f_i)}} \prod_{\ell \in \tilde{G}} M^{-j_\ell} \prod_{\ell \in \tilde{G} \setminus \tilde{T}} |j_\ell| M^{2j_\ell}$$

For every derivative d, $j_{\ell_d} \ge j = j_\phi$, so that

$$M^{\varepsilon j_{\ell_{d_0}}} \prod_d M^{-j_{\ell_d}} = M^{-(1-\varepsilon) j_{\ell_{d_0}}} \prod_{d \ne d_0} M^{-j_{\ell_d}} \le M^{-s_0 j - s_1 j} M^{\varepsilon j}$$

Bounding each $|j_{\pi(f_i)}|^{3n_{f_i}-2} \le |j_{\pi(f_i)}|^{3n_{f_i}-1}$, we come to the conclusion that $|\partial_{q_0}^{s_0} \partial_{\mathbf{q}}^{s_1} \tilde{G}^{\tilde{J}}(q)|$ is bounded, uniformly in q, by

$$\text{const}_n M^{-s_0 j - s_1 j} M^{\varepsilon j} \prod_{i=1}^m |j_{\pi(f_i)}|^{3n_{f_i}-1} M^{j_{\pi(f_i)}} \prod_{\ell \in \tilde{G}} M^{-j_\ell} \prod_{\ell \in \tilde{G} \setminus \tilde{T}} |j_\ell| M^{2j_\ell} \tag{19}$$

This is exactly $M^{-s_0 j - s_1 j} M^{\varepsilon j}$ times the bound $(33)\big|_{s_0=s=0}$ of Appendix A. So $(36)\big|_{s_0=s=0}$ of Appendix A now gives (13). This completes the proof that the value of each graph contributing to the self–energy is $C^{1+\eta}$ in the external momentum, for every η strictly less than the ε of Proposition 3.6.

We may also apply this technique to connected four–legged graphs. There is no need for an induction argument because we already have all of the needed bounds on c– and r–forks. We just need to go through the $\tilde{G}$ argument once. When we do so, there are three changes:

- The overall power counting factor $M^{\frac{1}{2}j(4-E_\phi)}$ in (34), which took the value M^j for two–legged graphs, now takes the value 1 for four–legged graphs.

- We may only apply the overlapping loop bound (18) when we can find a line $\ell_3 \in \tilde{T}$ and two lines $\ell_1, \ell_2 \in \tilde{G} \setminus \tilde{T}$ with $\ell_3 \in \Lambda_{\ell_1} \cap \Lambda_{\ell_2}$. By [12, Lemma 2.34], this is the case if and only if $\tilde{G}$ is overlapping, as defined in [12, Definition 2.19]. By [12, Lemma 2.26], four–legged connected graphs fail to be overlapping if and only if they are dressed bubble chains, as defined in [12, Definition 2.24].

- To convert the $M^{\varepsilon j_{\ell_3}}$ from the overlapping loop integral (18) into the $M^{\varepsilon j}$ that we want in the final bound, we set f_3 to the highest fork with $\ell_3 \in \tilde{G}_{f_3}$ and write
$$M^{\varepsilon j_{\ell_3}} = M^{\varepsilon j_{f_3}} = M^{\varepsilon j} \prod_{\substack{f \in \tilde{t} \\ \phi < f \leq f_3}} M^{\varepsilon (j_f - j_{\pi(f)})}$$
The extra factors $M^{\varepsilon(j_f - j_{\pi(f)})}$, which are all at least one, are easily absorbed by (35) provided $E_f > 4$ for all forks f between the root ϕ and f_3. We may choose ℓ_3 so that this is the case precisely when G is *not* a generalized ladder. To see this, let $\hat{G}^J = G^J / \{G_f^J \mid E_f = 2, 4\}$ be the diagram G, but with both two– and four–legged subdiagrams G_f viewed as generalized vertices. Then we can find a suitable ℓ_3 if and only if $\hat{G}^J$ is overlapping which in turn is the case if and only if $\hat{G}^J$ is not a dressed bubble chain, which in turn is the case, for all labellings J, if and only if G is not a generalized ladder.

Thus when G is a connected four–legged graph, the right hand side of (13) is replaced by a constant times a power of j times

$$M^{(-s_0-s_1)j} \begin{cases} M^{\varepsilon j} & \text{if } G \text{ is not a generalized ladder} \\ 1 & \text{if } G \text{ is a generalized ladder} \end{cases}$$

for $s_0, s_1 \in \{0,1\}$. This implies that four–legged graphs, other than generalized ladders, are C^η functions of their external momenta for all η strictly smaller than ε.

For graphs G contributing to the higher correlation functions, we may once again repeat the same argument, but with $s_0 = s_1 = 0$ and without having to exploit overlapping loops, provided we use the L^1 norm, rather than the L^∞ norm, on the momentum space kernel of G. In [12], this norm was denoted $|\ \cdot\ |'$ and was defined in (1.46). See [12, (2.27) and Theorem 2.47] for the proof.

Denote by $K(e, \mathbf{q})$ the counterterm function for the dispersion relation $e(\mathbf{k})$ and by

$$C_j(e,k) = \tfrac{f(M^{-2j}|ik_0 - e(\mathbf{k})|^2)}{ik_0 - e(\mathbf{k})}$$

the scale j propagator for the dispersion relation $e(\mathbf{k})$. Observe that, for all $j_\ell < 0$ and $s_0' \in \{0,1\}$,

$$|\tfrac{\partial}{\partial t}\partial_{k_0}^{s_0'} C_{j_\ell}(e+th,k)|_{t=0}| \le \text{ const } \|h\|_\infty M^{-(2+s_0')j_\ell}\ 1(|ik_0 - e(\mathbf{k})| \le M^{j_\ell})$$

Thus the effect of a directional derivative with respect to the dispersion relation in direction h is to multiply (16) by $\|h\|_\infty M^{-j_\ell}$, which is $\|h\|_\infty$ times the effect of a ∂_{k_0} derivative. So the same argument that led to (13) gives

$$\sum_{J\in\mathcal{J}(j,t,R,G)} \sup_q |\tfrac{\partial}{\partial t}\partial_{q_0}^{s_0} G^J(e+th,q)|_{t=0}| \le \text{const}_n |j|^{3n-2} M^{-s_0 j} M^{\varepsilon j}\|h\|_\infty$$

for $s_0 \in \{0,1\}$. When $s_0 = 0$, this is summable over $j < 0$ so that

$$\sup_{\mathbf{q}} \Big|\tfrac{\partial}{\partial t}\Big[\sum_{r=1}^{R} \lambda^r K_r(e+th,\mathbf{q})|_{t=0}\Big]\Big| \le \text{const}_{\text{dKde}}\, |\lambda| \|h\|_\infty \tag{20}$$

The constant $\text{const}_{\text{dKde}} = \text{const}_{\text{dKde}}(e,v)$ depends on R and the various parameters in the hypotheses imposed by Theorem 1.1 on the dispersion relation e and two–body interaction v, like the C^3 norm of e, the eigenvalues of the Hessian of e at singular points, the C^2 norm of v and the constants Z_0, β_0 and κ of Hypothesis NN. Fix a two–body interaction v and a constant $A > 0$. Denote by $\mathcal{E}_A$ the set of dispersion relations such that $\text{const}_{\text{dKde}}(e,v) \le A$. If the dispersion relations e, e' and all interpolants $(1-t)e + te'$, $0 \le t \le 1$ are in $\mathcal{E}_A$, and if $|\lambda| < \frac{1}{A}$, then

$$e + \sum_{r=1}^{R} \lambda^r K_r(e) = e' + \sum_{r=1}^{R} \lambda^r K_r(e') \Longrightarrow e = e' \tag{21}$$

■

A Bounding General Diagrams - A Review

For the convenience of the reader, we here provide a review of the general diagram bounding technique of [12]. As a concrete example of the technique, we consider models in $d \geq 2$ for which the interaction v has C^1 Fourier transform and the dispersion relation e and its Fermi surface $\mathcal{F} = \{ \mathbf{k} \mid e(\mathbf{k}) = 0 \}$ obey

H1' $\{ \mathbf{k} \mid |e(\mathbf{k})| \leq 1 \}$ is compact.

H2' $e(\mathbf{k})$ is C^1.

H3' $e(\tilde{\mathbf{k}}) = 0$ and $\nabla e(\tilde{\mathbf{k}}) = \mathbf{0}$ simultaneously only for finitely many $\tilde{\mathbf{k}}$'s, called singular points.

H4' If $\tilde{\mathbf{k}}$ is a singular point then $[\frac{\partial^2}{\partial \mathbf{k}_i \partial \mathbf{k}_j} e(\tilde{\mathbf{k}})]_{1 \leq i,j \leq d}$ is nonsingular.

and we prove that, any graph contributing to the proper self–energy is C^s for any $s < 1$. Note that, in this appendix, we do *not* require the no–nesting condition of Hypothesis NN. The same methods apply to graphs with more than two legs as well.

Let G be any two–legged 1PI graph. We also use the symbol G to stand for the value of the graph G. Singularities of the Fermi surface have no influence on the ultraviolet regime, so we introduce a fixed ultraviolet cutoff by choosing a compactly supported C^∞ function $U(k)$ that is identically one on a neighbourhood of $\{0\} \times \mathcal{F}$ and use the propagator $C(k) = \frac{U(k)}{ik_0 - e(\mathbf{k})}$. If $M > 1$ and f is a suitable C_0^∞ function that is supported on $[M^{-4}, 1]$, we have the partition of unity [12, §2.1]

$$U(k) = \sum_{j<0} f(M^{-2j}|ik_0 - e(\mathbf{k})|^2)$$

and hence

$$C(k) = \sum_{j<0} C_j(k) \qquad \text{where} \qquad C_j(k) = \frac{f(M^{-2j}|ik_0 - e(\mathbf{k})|^2)}{ik_0 - e(\mathbf{k})} \tag{22}$$

Note that $f(M^{-2j}|ik_0 - e(\mathbf{k})|^2)$ and $C_j(k)$ vanish unless

$$M^{j-2} \leq |ik_0 - e(\mathbf{k})| \leq M^j$$

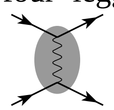
First, suppose that G is not renormalized. Expand each propagator of G using $C = \sum_{j<0} C_j$ to give

$$G = \sum_J G^J$$

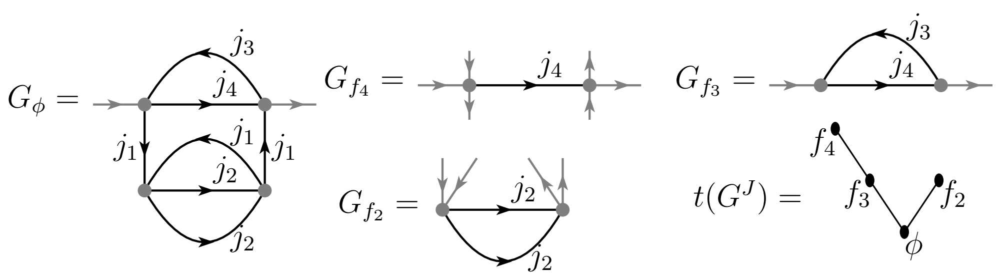
The sum runs over all possible labellings of the graph G, with each labeling consisting of an assignment $J = \{\, j_\ell < 0 \mid \ell \in G \,\}$ of scales to the lines of G. We now construct a natural hierarchy of subgraphs of G^J. This family of subgraphs will be a forest, meaning that if $G_f, G_{f'}$ are in the forest and intersect, either $G_f \subset G_{f'}$ or $G_{f'} \subset G_f$. First let, for each $j < 0$,

$$G^{(\geq j)} = \{\ \ell \in G^J \mid j_\ell \geq j\ \}$$

be the subgraph of G^J consisting of all lines of scale at least j. (Think of an interaction line as a generalised four–legged vertex



rather than a line.) There is no need for $G^{(\geq j)}$ to be connected. The forest $t(G^J)$ is the set of all connected subgraphs of G^J that are components of some $G^{(\geq j)}$. This forest is naturally partially ordered by containment. In order to make $t(G^J)$ look like a tree with its root at the bottom, we define, for $f, f' \in t(G^J)$, $f > f'$ if $G_f \subset G_{f'}$. We denote by $\pi(f)$ the highest fork of $t(G^J)$ below f and by ϕ the root element, i.e. the element with $G_\phi = G$. To each $G_f \in t(G)$ there is naturally associated the scale $j_f = \min\{\, j_\ell \mid \ell \in G_f \,\}$. In the example below $j_4 > j_3 > j_1$ and $j_2 > j_1$. External lines are in gray while internal lines are in black.



Reorganize the sum over J using

$$G = \sum_{j<0} \sum_{t \in \mathcal{F}(G)} \prod_{f \in t} \tfrac{1}{b_f!} \sum_{J \in \mathcal{J}(j,t,G)} G^J \tag{23}$$

where

$$\begin{aligned} \mathcal{F}(G) &= \text{the set of forests of subgraphs of } G \\ b_f &= \text{the number of upward branches at the fork } f \\ \mathcal{J}(j,t,G) &= \{\ \text{labellings } J \text{ of } G \mid t(G^J) = t,\ j_\phi = j\ \} \end{aligned}$$

A given labeling J of G is in $\mathcal{J}(j,t,G)$ if and only if

- for each $f \in t$, all lines of $G_f \setminus \cup_{\substack{f' \in t \\ f' > f}} G_{f'}$ have the same scale. Call the common scale j_f.
- if $f > f'$ then $j_f > j_{f'}$
- $j_\phi = j$

It is a standard result [12, (2.72)] that renormalization of the dispersion relation may be implemented by modifying (23) as follows.

- Each $\emptyset \neq f \in t$ for which G_f has two external lines is assigned a "renormalization label". This label can take the values r and c. The set of possible assignments of renormalization labels, i.e. the set of all maps from $\{\, f \in t \mid G_f$ has two external legs $\}$ to $\{r, c\}$, is denoted $\mathcal{R}(t)$.
- In the definition of the renormalized value of the graph G, the value of each subgraph G_f with renormalization label r is replaced by $(\mathbb{1} - \ell) G_f(k)$. Here ℓ is the localization operator, which we take[2] to be simply evaluation at $k_0 = 0$. For these r–forks, the constraint $j_f > j_{\pi(f)}$ still applies.
- In the definition of the renormalized value of the graph G, the value of each subgraph G_f with renormalization label c is replaced by $\ell G_f(k)$. For these c forks the constraint $j_f > j_{\pi(f)}$ is replaced by $j_f \leq j_{\pi(f)}$.

Given a graph G, a forest t of subgraphs of G and an assignment R of renormalization labels to the two–legged forks of t, we define $\mathcal{J}(j,t,R,G)$ to be the set of all assignments of scales to the lines of G obeying

- for each $f \in t$, all lines of $G_f \setminus \cup_{\substack{f' \in t \\ f' > f}} G_{f'}$ have the same scale. Call the common scale j_f.
- if G_f is not two–legged then $j_f > j_{\pi(f)}$

[2] The main property that the localization operator should have is that $\frac{(\mathbb{1}-\ell)G_f(k)}{ik_0 - e(\mathbf{k})}$ should be bounded for any (sufficiently smooth) G_f. Here is another possible localization operator for $d = 2$. In a neighbourhood of a regular point of the Fermi surface, define $\ell G_f(k) = G_f(k_0 = 0, P\mathbf{k})$ where $P\mathbf{k}$ is any reasonable projection of $\mathbf{k}$ onto the Fermi surface, as in [12, Section 2.2]. In a neighbourhood of a singular point, use a coordinate system in which $e(x,y) = xy$ and, in this coordinate system, define $\ell G_f(k_0, x, y) = G_f(0, x, 0) + G_f(0, 0, y) - G_f(0, 0, 0)$. Use a partition of unity to patch the different neighbourhoods together.

- if G_f is two–legged and $R_f = r$, then $j_f > j_{\pi(f)}$
- if G_f is two–legged and $R_f = c$ then $j_f \le j_{\pi(f)}$
- $j_\phi = j$

Then, the value of the graph G with all two–legged subdiagrams correctly renormalized is

$$G = \sum_{j<0} \sum_{t\in\mathcal{F}(G)} \prod_{f\in t} \tfrac{1}{b_f!} \sum_{R\in\mathcal{R}(t)} \sum_{J\in\mathcal{J}(j,t,R,G)} G^J \tag{24}$$

To derive bounds on G, when we are not interested in the dependence of those bounds on G and in particular on the order of perturbation theory, it suffices to derive bounds on $\sum_{j\le 0} \sum_{J\in\mathcal{J}(j,t,R,G)} G^J$ for each fixed t and R.

Proposition A.1 *Assume that the interaction has C^1 Fourier transform and the dispersion relation obeys* **H1'–H4'** *above. Let G be any two–legged 1PI graph of order n. Let t be a tree corresponding to a forest of subgraphs of G. Let R be an assignment of r, c labels to all forks $f > \phi$ of t for which G_f is two–legged. Let $\mathcal{J}(j,t,R,G)$ be the set of all assignments of scales to the lines of G that have root scale j and are consistent with t and R. Let $s \in (0,1)$. Then there is a constant* const , *depending on s but independent of j, such that*

$$\begin{aligned}
\sum_{J\in\mathcal{J}(j,t,R,G)} \sup_q |G^J(q)| &\le \mathrm{const}_n |j|^{3n-2} M^j \\
\sum_{J\in\mathcal{J}(j,t,R,G)} \sup_q |\partial_{q_0} G^J(q)| &\le \mathrm{const}_n |j|^{3n-2} \\
\sum_{J\in\mathcal{J}(j,t,R,G)} \sup_{q,\mathbf{p}} \tfrac{1}{|\mathbf{p}|^s} |G^J(q+\mathbf{p}) - G^J(q)| &\le \mathrm{const}_n |j|^{3n-2} M^{(1-s)j}
\end{aligned}$$

Remark A.2 *Note that here the root scale is not summed over and G_ϕ is not renormalized. But all internal scales are summed over and internal two–legged subgraphs that correspond to r and c forks are renormalized and localized respectively.*

Proof: The proof is by induction on the depth of the graph, which is defined by

$$D = \max\{\, n \mid \exists \text{ forks } f_1 > \cdots > f_n > \phi \text{ with } G_{f_1}, \cdots, G_{f_n} \text{ all two–legged} \,\}$$

The inductive hypothesis is that

$$\begin{aligned}\sum_{J\in\mathcal{J}(j,t,R,G)} \sup_q |\partial_{q_0}^{s_0} G^J(q)| &\le \mathrm{const}_n |j|^{3n-2} M^{(1-s_0)j}\\ \sum_{J\in\mathcal{J}(j,t,R,G)} \sup_{q,\mathbf{p}} \tfrac{1}{|\mathbf{p}|^s} |\partial_{q_0}^{s_0} G^J(q+\mathbf{p}) - \partial_{q_0}^{s_0} G^J(q)| &\le \mathrm{const}_n |j|^{3n-2} M^{(1-s-s_0)j}\end{aligned}$$

for $s_0 = 0, 1$ and all $s \in (0,1)$ (with the constant depending on s).

If $D > 0$, decompose the tree t into a pruned tree $\tilde{t}$ and insertion subtrees $\tau^1, \cdots, \tau^m$ by cutting the branches beneath all minimal $E_f = 2$ forks $f_1, \cdots, f_m$. In other words each of the forks $f_1, \cdots, f_m$ is an $E_f = 2$ fork having no $E_f = 2$ forks, except ϕ, below it in t. Each τ_i consists of the fork f_i and all of t that is above f_i. It has depth at most $D-1$ so the corresponding subgraph G_{f_i} obeys the conclusion of this Proposition. Think of each subgraph G_{f_i} as a generalized vertex in the graph $\tilde{G} = G/\{G_{f_1}, \cdots, G_{f_m}\}$. Thus $\tilde{G}$ now has two– as well as four–legged vertices. These two–legged vertices have kernels of the form

$$T_i(k) = \sum_{j_{f_i} \le j_{\pi(f_i)}} \ell G_{f_i}(k) \tag{25}$$

when f_i is a c–fork and of the form

$$T_i(k) = \sum_{j_{f_i} > j_{\pi(f_i)}} (\mathbb{1} - \ell) G_{f_i}(k) \tag{26}$$

when f_i is an r–fork. At least one of the external lines of G_{f_i} must be of scale precisely $j_{\pi(f_i)}$ so the momentum k passing through G_{f_i} lies in the support of $C_{j_{\pi(f_i)}}$. In the case of a c–fork $f = f_i$ we have, by the inductive hypothesis,

$$\begin{aligned}\sum_{j_f \le j_{\pi(f)}} \sum_{J_f \in \mathcal{J}(j_f,t_f,R_f,G_f)} \sup_k \left| \ell G_f^{J_f}(k) \right| &\le \sum_{j_f \le j_{\pi(f)}} \sum_{J_f \in \mathcal{J}(j_f,t_f,R_f,G_f)} \sup_k \left| G_f^{J_f}(k) \right|\\ &\le \sum_{j_f \le j_{\pi(f)}} \mathrm{const}_{n_f} |j_f|^{3n_f-2} M^{j_f}\\ &\le \mathrm{const}_{n_f} M^{j_{\pi(f)}} \sum_{i\ge 0} (|j_{\pi(f)}| + i)^{3n_f-2} M^{-i}\\ &\le \mathrm{const}_{n_f} |j_{\pi(f)}|^{3n_f-2} M^{j_{\pi(f)}} \sum_{i\ge 0} (i+1)^{3n_f-2} M^{-i}\\ &\le \mathrm{const}_{n_f} |j_{\pi(f)}|^{3n_f-2} M^{j_{\pi(f)}}\end{aligned} \tag{27}$$

Here n_f is the number of vertices of G_f and t_f and R_f are the restrictions of t and R respectively to forks $f' \geq f$. Hence J_f runs over all assignments of scales to the lines of G_f consistent with the original t and R and with the specified value of j_f. Similarly,

$$\sum_{j_f \leq j_{\pi(f)}} \sum_{J_f \in \mathcal{J}(j_f,t_f,R_f,G_f)} \sup_{k,\mathbf{p}} \tfrac{1}{|\mathbf{p}|^s} \Big| \ell G_f^{J_f}(k+\mathbf{p}) - \ell G_f^{J_f}(k) \Big| \\ \leq \mathrm{const}_{n_f} |j_{\pi(f)}|^{3n_f-2} M^{(1-s)j_{\pi(f)}} \tag{28}$$

Note that $\ell G_f^{J_f}(k)$ is independent of k_0 so that ∂_{k_0} may never act on it.

In the case of an r–fork $f = f_i$, we have

$$|(\mathbb{1}-\ell)G(k)| = |G(k_0,\mathbf{k}) - G(0,\mathbf{k})| \leq |k_0| \sup_k |\partial_{k_0} G(k)|$$

Hence, by the inductive hypothesis and, when $s_0 = 0$, the mean value theorem,

$$\begin{aligned} &\sum_{j_f > j_{\pi(f)}} \sum_{J_f \in \mathcal{J}(j_f,t_f,R_f,G_f)} \sup_k 1(C_{j_{\pi(f)}}(k) \neq 0) \Big| \partial_{k_0}^{s_0} (\mathbb{1}-\ell) G_f^{J_f}(k) \Big| \\ &\qquad \leq \sum_{j_f > j_{\pi(f)}} \sum_{J_f \in \mathcal{J}(j_f,t_f,R_f,G_f)} M^{(1-s_0)j_{\pi(f)}} \sup_k \Big| \partial_{k_0} G_f^{J_f}(k) \Big| \\ &\qquad \leq \mathrm{const}_{n_f} M^{(1-s_0)j_{\pi(f)}} \sum_{j_f > j_{\pi(f)}} |j_f|^{3n_f-2} \\ &\qquad \leq \mathrm{const}_{n_f} |j_{\pi(f)}|^{3n_f-1} M^{(1-s_0)j_{\pi(f)}} \end{aligned} \tag{29}$$

Similarly, for $|k_0| \leq M^{j_{\pi(f)}}$,

$$\begin{aligned} &\sum_{j_f > j_{\pi(f)}} \sum_{J_f \in \mathcal{J}(j_f,t_f,R_f,G_f)} \sup_{k,\mathbf{p}} \tfrac{1}{|\mathbf{p}|^s} \Big| \partial_{k_0}^{s_0} (\mathbb{1}-\ell) G_f^{J_f}(k+\mathbf{p}) - \partial_{k_0}^{s_0} (\mathbb{1}-\ell) G_f^{J_f}(k) \Big| \\ &\leq \sum_{j_f > j_{\pi(f)}} \sum_{J_f \in \mathcal{J}(j_f,t_f,R_f,G_f)} M^{(1-s_0)j_{\pi(f)}} \sup_{k,\mathbf{p}} \tfrac{1}{|\mathbf{p}|^s} \Big| \partial_{k_0} G_f^{J_f}(k+\mathbf{p}) - \partial_{k_0} G_f^{J_f}(k) \Big| \\ &\leq \mathrm{const}_{n_f} M^{(1-s_0)j_{\pi(f)}} \sum_{j_f > j_{\pi(f)}} |j_f|^{3n_f-2} M^{-j_f s} \\ &\leq \mathrm{const}_{n_f} |j_{\pi(f)}|^{3n_f-2} M^{(1-s_0-s)j_{\pi(f)}} \end{aligned} \tag{30}$$

We are now ready to bound $\tilde{G}^{\tilde{J}}$, where $\tilde{J}$ is the restriction of J to $\tilde{G}$. It is both convenient and standard to get rid of the conservation of momentum delta functions arising in the value of $\tilde{G}^{\tilde{J}}$ by integrating out some momenta. Then, instead of having one $(d+1)$–dimensional integration variable k for each line of the diagram, there is one for each momentum loop. Here is a convenient way to select these loops. Pick any spanning tree $\tilde{T}$ for $\tilde{G}$. A spanning tree is a subgraph of $\tilde{G}$ that is a tree and contains all the vertices of $\tilde{G}$. We associate to each line ℓ of $\tilde{G} \setminus \tilde{T}$ the "internal momentum loop" Λ_ℓ that consists of ℓ and the unique path in $\tilde{T}$ joining the ends of ℓ. The "external momentum path" is the unique path in $\tilde{T}$ joining the external legs. It carries the external momentum q. The loop Λ_ℓ carries momentum k_ℓ. The momentum $k_{\ell'}$ of each line $\ell' \in \tilde{T}$ is the signed sum of all loop and external momenta passing through ℓ'.

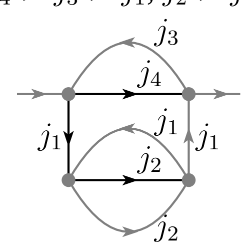
The form of the integral giving the value of $\tilde{G}^{\tilde{J}}(q)$ is then

$$\tilde{G}^{\tilde{J}}(q) = \int \prod_{\ell \in \tilde{G} \setminus \tilde{T}} \text{đ}k_\ell \prod_{\ell \in \tilde{G}} C_{j_\ell}(k_\ell) \prod_{\mathrm{v}} u_{\mathrm{v}} \qquad \text{where } \text{đ}k = \tfrac{d^{d+1}k}{(2\pi)^{d+1}} \tag{31}$$

Here $\tilde{T}$ is any spanning tree for $\tilde{G}$. The loops are labeled by the lines of $\tilde{G} \setminus \tilde{T}$. For each $\ell \in \tilde{T}$, the momentum k_ℓ is a signed sum of loop momenta and external momentum q. The product $\prod_{\mathrm{v}}$ runs over the vertices of $\tilde{G}$ and u_{v} is the vertex function for v. If v is one of the original interaction vertices then u_{v} is just v evaluated at the signed sum of loop and external momenta passing through v. If v is a two–legged vertex, then u_{v} is given either by (25) or by (26).

We are now ready to bound $\tilde{G}$ in six steps.

1. Choose a spanning tree $\tilde{T}$ for $\tilde{G}$ with the property that $\tilde{T} \cap \tilde{G}_f^{\tilde{J}}$ is a connected tree for every $f \in t(\tilde{G}^{\tilde{J}})$. $\tilde{T}$ can be built up inductively, starting with the smallest subgraphs $\tilde{G}_f$, because, by construction, every $\tilde{G}_f$ is connected and $t(\tilde{G}^J)$ is a forest. Such a spanning tree is illustrated below for the example given just before (23) with $j_4 > j_3 > j_1$, $j_2 > j_1$.



2. Apply any q–derivatives. By the product rule, or, in the case of a "discrete derivative", the "discrete product rule"

$$\begin{aligned} &f(k+\mathbf{q})g(k'+\mathbf{q}) - f(k)g(k') \\ &\qquad = [f(k+\mathbf{q}) - f(k)]g(k') + f(k+\mathbf{q})[g(k'+\mathbf{q}) - g(k')], \end{aligned}$$

each derivative may act on any line or vertex on the "external momentum path". It suffices to consider any one such action.

3. Bound each two–legged renormalized subgraph (i.e. r–fork) by (29,30) and each two–legged counterterm (i.e. c–fork) by (27,28). Observe that when s_0 k_0–derivatives and s $\mathbf{k}$–derivatives act on the vertex, the bound is no worse than $M^{-(s_0+s)j}$ times the bound with no derivatives, because we necessarily have $j \le j_{\pi(f)} < 0$.

4. Bound all remaining vertex functions, $u_{\rm v}$, (suitably differentiated) by their suprema in momentum space.

5. Bound each propagator

$$\begin{aligned} &|\partial_{k_0}^{s_0} C_{j_\ell}(k)| \le \text{ const } M^{-(1+s_0)j_\ell}\ 1(|ik_0 - e(\mathbf{k})| \le M^{j_\ell}) \\ &\tfrac{1}{|\mathbf{p}|^s} |\partial_{k_0}^{s_0} C_{j_\ell}(k+\mathbf{p}) - \partial_{k_0}^{s_0} C_{j_\ell}(k)| \le \text{ const } M^{-(1+s_0+s)j_\ell} \end{aligned} \tag{32}$$

Once again, when s_0 k_0–derivatives and s $\mathbf{k}$–derivatives act on the propagator, the bound is no worse than $M^{-(s_0+s)j}$ times the bound with no derivatives, because we necessarily have $j \le j_\ell < 0$. We now have $|\partial_{k_0}^{s_0} \tilde G^{\tilde J}(q)|$ and $\frac{1}{|\mathbf{p}|^s}\left|\partial_{k_0}^{s_0} \tilde G^{\tilde J}(q+\mathbf{p}) - \partial_{k_0}^{s_0} \tilde G^{\tilde J}(q)\right|$ bounded, uniformly in q and $\mathbf{p}$ by

$$\begin{aligned} \text{const}_n M^{-(s_0+s)j} \prod_{i=1}^{m} |j_{\pi(f_i)}|^{3n_{f_i}-1} M^{j_{\pi(f_i)}} \prod_{\ell\in\tilde G} M^{-j_\ell} \\ \int \prod_{\ell\in\tilde G\setminus\tilde T} đk_\ell \prod_{\ell\in\tilde G\setminus\tilde T} 1(|ik_{\ell 0} - e(\mathbf{k}_\ell)| \le M^{j_\ell}) \end{aligned}$$

with $s=0$ in the first case. We remark that for the bound on $|\partial_{k_0}^{s_0} \tilde G^{\tilde J}(q)|$ we may replace the $\prod_{\ell\in\tilde G\setminus\tilde T}$ in $\prod_{\ell\in\tilde G\setminus\tilde T} 1(|ik_{\ell 0} - e(\mathbf{k}_\ell)| \le M^{j_\ell})$ by $\prod_{\ell\in\tilde G}$. These extra integration constraints are not used in the current naive bound, but are used in other bounds that exploit "overlapping loops".

6. Integrate over the remaining loop momenta. Integration over k_ℓ with $\ell \in \tilde G \setminus \tilde T$ is bounded by vol$\{\ k_{0\ell} \mid |k_{0\ell}| \le M^{j_\ell}\ \} \le 2M^{j_\ell}$ times the volume of $\{\ \mathbf{k}_\ell \mid |e(\mathbf{k}_\ell)| \le M^{j_\ell}\ \}$, which, by Lemma 2.3, is bounded by a constant times $|j_\ell| M^{j_\ell}$.

The above six steps give that $|\partial_{k_0}^{s_0}\tilde G^{\tilde J}(q)|$ and $\frac{1}{|\mathbf{p}|^s}\left|\partial_{k_0}^{s_0}\tilde G^{\tilde J}(q+\mathbf{p})-\partial_{k_0}^{s_0}\tilde G^{\tilde J}(q)\right|$ are bounded, uniformly in q and $\mathbf{p}$ by

$$B^{\tilde J} = \mathrm{const}_n M^{-(s_0+s)j}\prod_{i=1}^{m}|j_{\pi(f_i)}|^{3n_{f_i}-1}M^{j_{\pi(f_i)}}\prod_{\ell\in\tilde G}M^{-j_\ell}\prod_{\ell\in\tilde G\setminus\tilde T}|j_\ell|M^{2j_\ell} \tag{33}$$

again with $s=0$ in the first case. Define the notation

$$\begin{aligned}
\tilde T_f &= \text{number of lines of } \tilde T\cap\tilde G_f\\
\tilde L_f &= \text{number of internal lines of } \tilde G_f\\
n_f &= \text{number of vertices of } G_f\\
E_f &= \text{number of external lines of } G_f\\
E_{\mathrm v} &= \text{number of lines hooked to vertex v}
\end{aligned}$$

Applying

$$M^{\alpha j_\ell} = M^{\alpha j_\phi}\prod_{\substack{f\in t\\ f>\phi\\ \ell\in G_f}} M^{\alpha(j_f-j_{\pi(f)})}$$

to each M^{-j_ℓ} and M^{2j_ℓ} and

$$\begin{aligned}
M^{j_{\pi(f_i)}} &= M^{j_\phi}\prod_{\substack{f\in\tilde t\\ \phi<f<f_i}} M^{j_f-j_{\pi(f)}}\\
&= M^{-\frac12(E_{f_i}-4)j_\phi}\prod_{\substack{f\in\tilde t\\ f>\phi\\ f_i\in\tilde G_f}} M^{-\frac12(E_{f_i}-4)(j_f-j_{\pi(f)})}
\end{aligned}$$

for each $1\le i\le m$ (thinking of f_i as a vertex of $\tilde G$) gives

$$\begin{aligned}
B^{\tilde J} \le\ & \mathrm{const}_n M^{-(s_0+s)j}|j|^{\tilde L_\phi-\tilde T_\phi+\sum(3n_{f_i}-1)}M^{j(\tilde L_\phi-2\tilde T_\phi-\sum_{\mathrm v\in\tilde G}\frac12(E_{\mathrm v}-4))}\\
&\prod_{\substack{f\in\tilde t\\ f>\phi}} M^{(j_f-j_{\pi(f)})(\tilde L_f-2\tilde T_f-\sum_{\mathrm v\in\tilde G_f}\frac12(E_{\mathrm v}-4))}
\end{aligned}$$

The sums $\sum_{v\in\tilde G}$ and $\sum_{v\in\tilde G_f}$ run over two– as well as four–legged generalized vertices. As

$$\tilde L_f = \tfrac12\Big(\sum_{\mathrm v\in\tilde G_f}E_{\mathrm v}-E_f\Big) \qquad \text{and} \qquad \tilde T_f=\sum_{\mathrm v\in\tilde G_f}1-1$$

$$\Longrightarrow \tilde{L}_f - 2\tilde{T}_f \;\; = \;\; \tfrac{1}{2}(4 - E_f + \sum_{\mathrm{v}\in\tilde{G}_f}(E_{\mathrm{v}} - 4))$$

and we have

$$\begin{aligned} B^{\tilde{J}} &\leq \mathrm{const}_n M^{-(s_0+s)j}|j|^{\tilde{L}_\phi - \tilde{T}_\phi + \sum(3n_{f_i}-1)} M^{\frac{1}{2}j(4-E_\phi)} \prod_{\substack{f\in\tilde{t}\\ f>\phi}} M^{\frac{1}{2}(j_f - j_{\pi(f)})(4-E_f)} \\ &= \mathrm{const}_n M^{-(s_0+s)j}|j|^{\tilde{L}_\phi - \tilde{T}_\phi + \sum(3n_{f_i}-1)} M^{j} \prod_{\substack{f\in\tilde{t}\\ f>\phi}} M^{\frac{1}{2}(j_f - j_{\pi(f)})(4-E_f)} \qquad (34) \end{aligned}$$

since $E_\phi = 2$. The scale sums are performed by repeatedly applying

$$\sum_{\substack{j_f\\ j_f > j_{\pi(f)}}} M^{\frac{1}{2}(j_f - j_{\pi(f)})(4-E_f)} \leq \begin{cases} |j| & \text{if } E_f = 4 \\ \frac{1}{M-1} & \text{if } E_f > 4 \end{cases} \qquad (35)$$

starting with the highest forks, and give at most $\tilde{L}_\phi - 1$ additional factors of $|j|$ since

$$\#\{f \in t(\tilde{G}^{\tilde{J}}),\ f \neq \phi\} \leq \tilde{L}_\phi - 1$$

Thus

$$\begin{aligned} \sum_{\tilde{J}\in\mathcal{J}(j,\tilde{t},\tilde{G})} B^{\tilde{J}} &\leq \mathrm{const}_n |j|^{2\tilde{L}_\phi - \tilde{T}_\phi - 1 + \sum(3n_{f_i}-1)} M^{(1-s_0-s)j} \\ &\leq \mathrm{const}_n |j|^{3n-2} M^{(1-s_0-s)j} \qquad (36) \end{aligned}$$

since, using $\tilde{n}_4$ to denote the number of four–legged vertices in $\tilde{G}$,

$$\begin{aligned} &2\tilde{L}_\phi - \tilde{T}_\phi - 1 + \sum_{i=1}^{m}(3n_{f_i} - 1) \\ &\qquad = 2\tfrac{1}{2}(4\tilde{n}_4 + 2m - 2) - (\tilde{n}_4 + m - 1) - 1 + 3\sum_{i=1}^{m} n_{f_i} - m \\ &\qquad = 3\tilde{n}_4 + 3\sum_{i=1}^{m} n_{f_i} - 2 \\ &\qquad = 3n - 2 \end{aligned}$$

This is the desired bound. ■

Corollary A.3 *Assume that the interaction has C^1 Fourier transform and the dispersion relation obeys* **H1'–H4'** *above. Let $G(q)$ be any graph contributing to the proper self–energy. Then, for every $0 < s < 1$,*

$$\begin{aligned} \sup_q |G(q)| &< \infty \\ \sup_{q,\mathbf{p}} \tfrac{1}{|\mathbf{p}|^s} |G(q+\mathbf{p}) - G(q)| &< \infty \end{aligned}$$

Proof: Both bounds are immediate from Proposition A.1. One merely has to sum over j, t and R. The bound on $\sup_q |G(q)|$ was also proven by these same methods in [21]. ■

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