Singular Fermi Surfaces II. The Two–Dimensional Case

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Abstract

We consider many–fermion systems with singular Fermi surfaces, which contain *Van Hove points* where the gradient of the band function $\mathbf{k} \mapsto e(\mathbf{k})$ vanishes. In a previous paper, we have treated the case of spatial dimension $d \geq 3$. In this paper, we focus on the more singular case $d = 2$ and establish properties of the fermionic self–energy to all orders in perturbation theory. We show that there is an asymmetry between the spatial and frequency derivatives of the self–energy. The derivative with respect to the Matsubara frequency diverges at the Van Hove points, but, surprisingly, the self–energy is C^1 in the spatial momentum to all orders in perturbation theory, provided the Fermi surface is curved away from the Van Hove points. In a prototypical example, the second spatial derivative behaves similarly to the first frequency derivative. We discuss the physical significance of these findings.

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Contents

1 Introduction

In this paper, we continue our analysis of (all–order) perturbative properties of the self–energy and the correlation functions in fermionic systems with a fixed non–nested singular Fermi surface. That is, the Fermi surface contains *Van Hove points*, where the gradient of the dispersion function vanishes, but satisfies a no–nesting condition away from these points. In a previous paper [1], we treated the case of spatial dimensions $d \geq 3$. Here we focus on the two–dimensional case, where the effects of the Van Hove points are strongest.

We have given a general introduction to the problem and some of the main questions in [1]. As discussed in [1], the no–nesting hypothesis is natural from a theoretical point of view, because it separates effects coming from the saddle points and nesting effects. Moreover, generically, nesting and Van Hove effects do not occur at the same Fermi level. In the following, we discuss those aspects of the problem that are specific to two dimensions. As already discussed in [1], the effects caused by saddle points of the dispersion function lying on the Fermi surface are believed to be strongest in two dimensions (we follow the usual jargon of calling the level set the Fermi "surface" even though it is a curve in $d = 2$). Certainly, the Van Hove singularities in the density of states of the noninteracting system are strongest in $d = 2$. As concerns many–body properties, we have shown in [1] that for $d \geq 3$, the overlapping loop estimates of [13] carry over essentially unchanged, which implies differentiability of the self–energy and hence a quasiparticle weight (Z–factor) close to 1 to all orders in renormalized perturbation theory.

In this paper, we show that for $d = 2$, there are more drastic changes. Namely, there is an asymmetry between the derivatives of the self–energy $\Sigma(q_0, \mathbf{q})$ with respect to the frequency variable q_0 and the spatial momentum $\mathbf{q}$. We prove that the spatial gradient $\nabla\Sigma$ is a bounded function to all orders in perturbation theory if the Fermi surface satisfies a no–nesting condition. By explicit calculation, we show that for a standard saddle point singularity, even the second–order contribution $\partial_0\Sigma_2(q_0, \mathbf{q}_s)$ diverges as $(\log |q_0|)^2$ at any Van Hove point $\mathbf{q}_s$ (if that point is on the Fermi surface).

This asymmetric behaviour is unlike the behaviour in all other cases that are under mathematical control: in one dimension, both $\partial_0\Sigma_2$ and $\partial_1\Sigma_2$ diverge like $\log |q_0|$ at the Fermi point. This is the first indication for vanishing of the Z–factor and the occurrence of anomalous decay exponents in this model. The point is, however, that once a suitable Z–factor is extracted, $Z\partial_1\Sigma_2$ remains of order 1 in one dimension, while for two–dimensional singular Fermi surfaces, the $\mathbf{p}$–

dependent function $Z(\mathbf{p})\nabla\Sigma(\mathbf{p})$ vanishes at the Van Hove points. In higher dimensions $d \geq 2$, and with a regular Fermi surface fulfilling a no-nesting condition very similar to that required here, Σ is continuously differentiable both in q_0 and in $\mathbf{q}$. Thus it is really the Van Hove points on the Fermi surface that are responsible for the asymmetry. In the last section of this paper, we point out some possible (but as yet unproven) consequences of this behaviour.

Our analysis is partly motivated by the two–dimensional Hubbard model, a lattice fermion model with a local interaction and a dispersion relation $\mathbf{k} \mapsto e(\mathbf{k})$ which, in suitable energy units, reads

$$e(\mathbf{k}) = -\cos k_1 - \cos k_2 + \theta(1 + \cos k_1 \cos k_2) - \mu. \tag{1}$$

The parameter μ is the chemical potential, used to adjust the particle density, and θ is a ratio of hopping parameters. As we shall explain now, the most interesting parameter range is $\mu \approx 0$ and

$$0 < \theta < 1.$$

The zeroes of the gradient of e are at $(0, 0)$, (π, π) and at $(\pi, 0)$, $(0, \pi)$. The first two are extrema, and the last two are the saddle points relevant for van Hove singularities (VHS). For $\mu = 0$, both saddle points are on the Fermi surface. For $\theta = 1$ the Fermi surface degenerates to the pair of lines $\{k_1 = 0\} \cup \{k_2 = 0\}$, so we assume that $\theta < 1$. For $\theta = 0$ and $\mu = 0$, the Fermi surface becomes the so–called Umklapp surface $U = \{k : k_1 \pm k_2 = \pm\pi\}$, which is nested since it has flat sides. This case has been studied in [2, 3, 4]. There, it was shown that for a local Hubbard interaction of strength λ, perturbation theory converges in the region of (β, λ) where $|\lambda|$ is small and $|\lambda|(\log\beta)^2 \ll 1$. We shall discuss this result further in Section 6. For $0 < \theta < 1$ the Fermi surface at $\mu = 0$ has nonzero curvature away from the Van Hove points $(\pi, 0)$ and $(0, \pi)$. Viewed from the point (π, π), it encloses a strictly convex region (as a subset of $\mathbb{R}^2$). There is ample evidence that in the Hubbard model, it is the parameter range $\theta > 0$ and electron density near to the van Hove density ($\mu \approx 0$) that is relevant for high–T_c superconductivity (see, e.g. [5, 6, 7, 8, 9]). In this parameter region, an important kinematic property is that the two saddle points at $(\pi, 0)$ and $(0, \pi)$ are connected by the vector $Q = (\pi, \pi)$, which has the property that $2Q = 0$ mod $2\pi\mathbb{Z}^2$. This modifies the leading order flow of the four–point function strongly (Umklapp scattering, [7, 8, 9]). The bounds we discuss here hold both in presence and absence of Umklapp scattering.

The interaction of the fermions is given by $\lambda\hat{v}$, where λ is the coupling constant and $\hat{v}$ is the Fourier transform of the two–body potential defining a density–density interaction. For the special case of the Hubbard model, two fermions interact only if they are at the same lattice point, so that $\hat{v}(\mathbf{k}) = 1$. Despite the simplicity of the Hamiltonian, little is known rigorously about the low–temperature phase diagram of the Hubbard model, even for small $|\lambda|$. In this paper, we do perturbation theory to all orders, i.e. we treat λ as a formal expansion parameter. For a discussion of the relation of perturbation theory to all orders to renormalization group flows obtained from truncations of the RG hierarchies, see the Introduction of [1].

Although our analysis is motivated by the Hubbard model, it applies to a much more general class of models. In this paper, we shall need only that the band function e has enough derivatives, as stated below, and a similar condition on the interaction. In fact, the interaction is allowed to be more general than just a density–density interaction: it may depend on frequencies, as well as the spin of the particles. See [13, 14, 15, 16] for details. As far as the singular points of e are concerned, we require that they are nondegenerate. The precise assumptions on e will be stated in detail below.

We add a few remarks to put these assumptions into perspective. No matter if we start with a lattice model or a periodic Schrödinger operator describing Bloch electrons in a crystal potential, the band function given by the Hamiltonian for the one–body problem is, under very mild conditions, a smooth, even analytic function. In such a class of functions, the occurrence of degenerate critical points is nongeneric, i.e. measure zero. In other words, if

$$e(\mathbf{k}_s + R\mathbf{k}) = -\varepsilon_1 k_1^2 + \varepsilon_2 k_2^2 + \dots$$

around a Van Hove point $\mathbf{k}_s$ (here R is a rotation that diagonalizes the Hessian at $\mathbf{k}_s$), getting even one of the two prefactors ε_i to vanish in a Taylor expansion requires a fine–tuning of the hopping parameters, in addition to the condition that the VH points are on S. Thus, in a one–body theory, an *extended VHS*, where the critical point becomes degenerate because, say, ε_1 vanishes, is nongeneric. On the other hand, experiments suggest [6] that ε_1 is very small in some materials, which seem to be modeled well by Hubbard–type band functions. On the theoretical side, in a renormalized expansion with counterterms, it is not the dispersion relation of the noninteracting system, but that of the interacting system, which appears in all fermionic covariances. It is thus an important theoretical question to decide what effects the interaction has on the dispersion relation and in particular whether an extended VHS can be caused by the interaction. We shall discuss this question further in Section 6.

2 Main Results

In this section we state our hypotheses on the dispersion function and the Fermi surface, and then state our main result.

2.1 Hypotheses on the dispersion relation

We make the following hypotheses on the dispersion relation e and its Fermi surface $\mathcal{F} = \{\ \mathbf{k} \mid e(\mathbf{k}) = 0\ \}$ in $d = 2$.

H1 $\{\ \mathbf{k} \mid |e(\mathbf{k})| \le 1\ \}$ is compact.

H2 $e(\mathbf{k})$ is C^r with $r \ge 7$.

H3 $e(\tilde{\mathbf{k}}) = 0$ and $\nabla e(\tilde{\mathbf{k}}) = \mathbf{0}$ simultaneously only for finitely many $\tilde{\mathbf{k}}$'s, called *Van Hove points* or *singular points*.

H4 If $\tilde{\mathbf{k}}$ is a singular point then $[\frac{\partial^2}{\partial \mathbf{k}_i \partial \mathbf{k}_j} e(\tilde{\mathbf{k}})]_{1 \le i,j \le d}$ is nonsingular and has one positive eigenvalue and one negative eigenvalue.

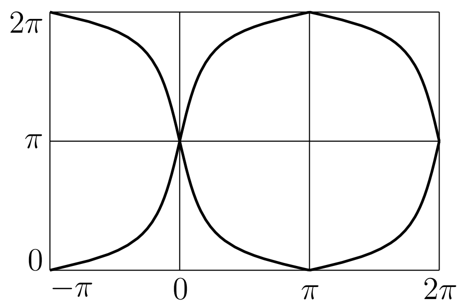
H5 There is at worst polynomial flatness. This means the following. Let $\tilde{\mathbf{k}} \in \mathcal{F}$. Suppose that $k_2 - \tilde{k}_2 = f(k_1 - \tilde{k}_1)$ is a C^{r-2} curve contained in $\mathcal{F}$ in a neighbourhood of $\tilde{\mathbf{k}}$. (If $\tilde{\mathbf{k}}$ is a singular point, there can be two such curves.) Then some derivative of $f(x)$ at $x = 0$ of order at least two and at most $r-2$ does not vanish. Similarly if the roles of the first and second coordinates are exchanged.

H6 There is at worst polynomial nesting. This means the following. Let $\tilde{\mathbf{k}} \in \mathcal{F}$ and $\tilde{\mathbf{p}} \in \mathcal{F}$ with $\tilde{\mathbf{k}} \ne \tilde{\mathbf{p}}$. Suppose that $k_2 - \tilde{k}_2 = f(k_1 - \tilde{k}_1)$ is a C^{r-2} curve contained in $\mathcal{F}$ in a neighbourhood of $\tilde{\mathbf{k}}$ and $k_2 - \tilde{p}_2 = g(k_1 - \tilde{p}_1)$ is a C^{r-2} curve contained in $\mathcal{F}$ in a neighbourhood of $\tilde{\mathbf{p}}$. Then some derivative of $f(x) - g(x)$ at $x = 0$ of order at most $r - 2$ does not vanish. Similarly if the roles of the first and second coordinates are exchanged. If $e(\mathbf{k})$ is not even, we further assume a similar nonvanishing when f gives a curve in $\mathcal{F}$ in a neighbourhood of any $\tilde{\mathbf{k}} \in \mathcal{F}$ and g gives a curve in $-\mathcal{F}$ in a neighbourhood of any $\tilde{\mathbf{p}} \in -\mathcal{F}$.

We denote by n_0 the largest nonflatness or nonnesting order plus one, and assume that

$$r \ge 2n_0 + 1$$

The Fermi surface for the Hubbard model with $0 < \theta < 1$ and $\mu = 0$, when viewed from (π, π), encloses a convex region. See the fi gure below. It has nonzero curvature except at the singular points. If one writes the equation of (one branch of) the Fermi surface near the singular point $(0, \pi)$ in the form $k_2 - \pi = f(k_1)$, then $f^{(3)}(0) \neq 0$. So this Fermi surface satisfi es the nonflatness and no–nesting conditions with $n_0 = 4$.



2.2 Main theorem

In the following, we state our main results about the fermionic self–energy. A discussion will be given at the end of the paper, in Section 6.

Theorem 2.1 *Let $\mathcal{B} = \mathbb{R}^2/2\pi\mathbb{Z}^2$ and $e \in C^7(\mathcal{B}, \mathbb{R})$. Assume that the Fermi surface $S = \{\mathbf{k} \in \mathcal{B} : e(\mathbf{k}) = 0\}$ contains points where $\nabla e(\mathbf{k}) = 0$, and that the Hessian of e at these points is nonsingular. Moreover, assume that away from these points, the Fermi surface can have at most finite–order tangencies with its (possibly reflected) translates and is at most polynomially flat. [These hypotheses have been spelled out in detail in* **H1–H6** *above.] As well, the interaction v is assumed to be short–range, so that its Fourier transform $\hat{v}$ is C^2.*

Then there is a counterterm function $K \in C^1(\mathcal{B}, \mathbb{R})$, given as a formal power series $K = \sum_{r \geq 1} K_r \lambda^r$ in the coupling constant λ, such that the renormalized expansion for all Green functions, at temperature zero, is finite to all orders in λ.

1. *The self–energy is given as a formal power series $\Sigma = \sum_{r \geq 1} \Sigma_r \lambda^r$, where for all $r \in \mathbb{N}$ and all $\omega \in \mathbb{R}$, the function $\mathbf{k} \mapsto \Sigma_r(\omega, \mathbf{k}) \in C^1(\mathcal{B}, \mathbb{C})$. Specifically, we have*

$$
\begin{aligned}
\|\Sigma_r\|_\infty &\leq \text{const} \\
\|\nabla \Sigma_r\|_\infty &\leq \text{const}
\end{aligned}
$$

with the constants depending on r. Moreover, the function $\omega \mapsto \Sigma_r(\omega, \mathbf{k})$ is C^1 in ω for all $k \in \mathcal{B} \setminus \overline{V}$, where V denotes the integer lattice generated by all van Hove points, and the bar means the closure in $\mathcal{B}$.

2. *For e given by the normal form $e(\mathbf{k}) = k_1 k_2$, which has a van Hove point at $\mathbf{k} = \mathbf{0}$, the second order contribution Σ_2 to the self–energy obeys*

$$\begin{aligned} \textit{Im}\ \partial_\omega \Sigma_2(\omega, 0) &= -a_1\ (\log|\omega|)^2 + O(|\log|\omega||) \\ \text{Re}\ \partial_{k_1}\partial_{k_2}\Sigma_2(\omega, 0) &= \quad a_2\ (\log|\omega|)^2 + O(|\log|\omega||) \end{aligned}$$

$\partial_{k_1}^2 \Sigma_2(\omega, 0)$ and $\partial_{k_2}^2 \Sigma_2(\omega, 0)$ grow at most linearly in $\log|\omega|$. The explicit values of $a_1 > 0$ and $a_2 > 0$ are given in Lemmas 5.1 and 5.2, below.

Theorem 2.1 is a statement about the zero temperature limit of Σ. That is, $\Sigma_r(\omega, \mathbf{k})$ and its derivatives are computed at a positive temperature $T = \beta^{-1}$, where they are C^2 in ω and $\mathbf{q}$, and then the limit $\beta \to \infty$ is taken. (Because only one–particle–irreducible graphs contribute to Σ_r, it is indeed a regular function of ω for all $\omega \in \mathbb{R}$ at any inverse temperature $\beta < \infty$.)

The bounds to all orders stated in item 1 of Theorem 2.1 generalize to low positive temperatures in an obvious way: the length–of–overlap estimates and the singularity analysis done below only use the spatial geometry of the Fermi surface for e, which is unaffected by the temperature. The other changes are merely to replace some derivatives with respect to frequency by finite differences, which only leads to trivial changes.

Our explicit computation of the asymptotics in the model case of item 2 of Theorem 2.1 uses that several contributions to these derivatives vanish in the limit $\beta \to \infty$, and that certain cancellations occur in the remaining terms. For this reason, the result stated in item 2 is a result at zero temperature. (In particular, the coefficients in the $O(\log|\omega|)$ terms are just numbers.) However, we do not expect any significant change in the asymptotics at low temperature and small ω to occur. That is, we expect the low–temperature asymptotics to contain only terms whose supremum over $|\omega| \geq \pi/\beta$ is at most of order $(\log\beta)^2$, and the square of the logarithm to be present.

2.3 Heuristic explanation of the asymmetry

We refer to the different behaviour of $\nabla\Sigma$ (which is bounded to all orders) and $\partial_\omega\Sigma$ (which is $\log^2$–divergent in second order) as the asymmetry in the derivatives of Σ. In the case of a regular Fermi surface, no–nesting implies that Σ_r is in $C^{1+\delta}$ with a Hölder exponent δ that depends on the no–nesting assumption. A similar bound was shown in [1] for Fermi surfaces with singularities in $d \geq 3$ dimensions. In [1], we formulated a slight generalization of the no-nesting hypothesis of [13], and again proved a volume improvement estimate, which implies the above–mentioned Hölder continuity of the first derivatives.

In the more special case of a regular Fermi surface with strictly positive curvature, we have given, in [14, 15], bounds on certain second derivatives of the self–energy with respect to momentum. We briefly review that discussion for the second–order contribution, to motivate why there is a difference between the spatial and the frequency derivatives.

For simplicity, we assume a local interaction, and consider the infrared part of the two–loop contribution

$$I(q_0, \mathbf{q}) = \langle C(\omega_1, e(\mathbf{p}_1))C(\omega_2, e(\mathbf{p}_2))C(\omega, \tilde{e})\rangle$$

where $\omega = q_0 + v_1\omega_1 + v_2\omega_2$, $v_i = \pm 1$ and

$$\tilde{e} = e(\mathbf{q} + v_1\mathbf{p}_1 + v_2\mathbf{p}_2)$$

The angular brackets denote integration of $\mathbf{p}_1$ and $\mathbf{p}_2$ over the two-dimensional Brillouin zone and Matsubara summation of ω_1 and ω_2, over the set $\frac{\pi}{\beta}(2\mathbb{Z}+1)$. By infrared part we mean that the fermion propagators are of the form

$$C(\omega, E) = \frac{U(\omega^2 + E^2)}{\mathrm{i}\omega - E}$$

where U is a suitable cutoff function that is supported in a small, fixed neighbourhood of zero.

The third denominator depends on the external momentum $(q_0, \mathbf{q})$ and derivatives with respect to the external momentum increase the power of that denominator, which may lead to bad behaviour as $\beta \to \infty$. The main idea why some derivatives behave better than expected by simple counting of powers (see [14]) is that in dimension two and higher, there are, in principle, enough integrations to make a change of variables so that $\tilde{e}$, $e(\mathbf{p}_1)$ and $e(\mathbf{p}_2)$ all become integration variables. This puts all dependence on the external variable $\mathbf{q}$ into the Jacobian J of this change of variables. If J were C^k with uniform bounds, $I(q_0, \mathbf{q})$ would be C^k in the spatial momentum $\mathbf{q}$, and (by integration by parts) also in q_0. However, J always has singularities, and the leading contributions to the derivatives of $I(q_0, \mathbf{q})$ come from the vicinity of these singularities. It was proven in [14, 15] that if the Fermi surface is regular and has strictly positive curvature, these singularities of the Jacobian are harmless, provided derivatives are taken tangential to the Fermi surface.

To explain the change of variables, we first show it for the case without van Hove singularities and then discuss the changes required when van Hove singularities are present.

In a neighborhood of the Fermi surface, we introduce coordinates ρ and θ, so that $\mathbf{p} = P(\rho, \theta)$. The coordinates are chosen such that $\rho = e(\mathbf{p})$ and that $P(\rho, \theta + \pi)$ is the antipode of $P(\rho, \theta)$. (We are assuming that the Fermi surface is strictly convex – see [14].) Doing this for $\mathbf{p}_1$ and $\mathbf{p}_2$, with corresponding Jacobian $J = \det P'$, we have

$$I(q_0, \mathbf{q}) = \left[\int d\theta_1 \; d\theta_2 \; J(\rho_1, \theta_1) J(\rho_2, \theta_2) \; C(\omega, \tilde{e}) \right]_{1,2}$$

where $[F]_{1,2}$ now denotes multiplying F by $C(\omega_1, \rho_1) C(\omega_2, \rho_2)$ and integrating over ρ_1 and ρ_2 and summing over the frequencies. To remove the $\mathbf{q}$–dependence from $C(\omega, \tilde{e})$, one now wants to change variables from θ_1 or θ_2 to $\tilde{e}$. This works except near points where

$$\frac{\partial \tilde{e}}{\partial \theta_1} = \frac{\partial \tilde{e}}{\partial \theta_2} = 0.$$

These equations determine the singularities of the Jacobian. The detailed analysis of their solutions is in [14]. Essentially, if one requires that the momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ are on the FS, i.e. $\rho_1 = \rho_2 = 0$, that $\mathbf{q} = P(0, \theta_{\mathbf{q}})$ is on the FS *and* that the sum $\mathbf{q} + v_1 P(0, \theta_1) + v_2 P(0, \theta_2)$ is on the FS, then the only solutions are $\theta_1 \in \{\theta_{\mathbf{q}}, \theta_{\mathbf{q}} + \pi\}$ and $\theta_2 \in \{\theta_{\mathbf{q}}, \theta_{\mathbf{q}} + \pi\}$. (The general case of momenta near to the Fermi surface is then treated by a deformation argument which requires that $\nabla e \neq 0$ and that the curvature be nonzero.) A detailed analysis of the singularity in J, in which strict convexity enters again, then implies that the self–energy is regular.

The conditions needed for the above argument fail at the Van Hove points. But, introducing a partition of unity on the Fermi surface, they still hold away from the singular points. So the only contributions that may fail to have derivatives come from $\mathbf{q} + v_1 \mathbf{p}_1 + v_2 \mathbf{p}_2$ in a small neighbourhood of the singular points. When a derivative with respect to q_0 is taken, the integrand contains a factor of $-\mathrm{i}(\mathrm{i}\omega - \tilde{e})^{-2}$. When a derivative with respect to $\mathbf{q}$ is taken, the integrand contains a factor $\nabla e(\mathbf{q} + v_1 \mathbf{p}_1 + v_2 \mathbf{p}_2)(\mathrm{i}\omega - \tilde{e})^{-2}$. Because we are in a small neighbourhood of the singular point, the numerator, $\nabla e(\mathbf{q} + v_1 \mathbf{p}_1 + v_2 \mathbf{p}_2)$, in the latter expression is small, and vanishes at the singular point. This suggests that the first derivative with respect to $\mathbf{q}$ may be better behaved than the first derivative with respect to q_0, as is indeed the case – see item 1 of Theorem 2.1. A second derivative with respect to $\mathbf{q}$ may act on the numerator, $\nabla e(\mathbf{q} + v_1\mathbf{p}_1 + v_2\mathbf{p}_2)$, and eliminate its zero. This suggests that the second derivative with respect to $\mathbf{q}$ behaves like the first derivative with respect to q_0, as is indeed the case – see item 2 of Theorem 2.1.

The above heuristic discussion is only to provide a motivation as to why the asymmetry in the derivatives of Σ_2 occurs. The proof does not make use of the idea of the change of variables to $\tilde{e}$, but rather of length–of–overlap estimates, which partially replace the overlapping loop estimates, away from the singular points. This allows us to show the convergence of the first $\mathbf{q}$–derivative under conditions **H1–H6**, which are significantly weaker than strict convexity, and it also allows us to treat the situation with Umklapp scattering, which had to be excluded in second order in [14], and which is the reason for the restriction on the density in [14].

3 Fermi Surface

In this section we prove bounds on the size of the overlap of the Fermi surface with translates of a tubular neighbourhood of the Fermi surface. These bounds make precise the geometrical idea that for non–nested surfaces (here: curves), the non–flatness condition **H5** strongly restricts such lengths of overlap.

3.1 Normal form for $e(\mathbf{k})$ near a singular point

Lemma 3.1 *Let $d = 2$ and assume* **H2–H5** *with $r \geq n_0 + 1$. Assume that $\tilde{\mathbf{k}} = \mathbf{0}$ is a singular point of e. Then there are*

- *integers $2 \leq \nu_1, \nu_2 < n_0$,*
- *a constant, nonsingular matrix A and*
- *$C^{r-2-\max\{\nu_1,\nu_2\}}$ functions $a(\mathbf{k})$, $b(\mathbf{k})$ and $c(\mathbf{k})$ that are bounded and bounded away from zero*

such that in a neighbourhood of the origin

$$e(A\mathbf{k}) = a(\mathbf{k})(k_1 - k_2^{\nu_1} b(\mathbf{k}))(k_2 - k_1^{\nu_2} c(\mathbf{k})). \tag{2}$$

Proof: Let λ_1 and λ_2 be the eigenvalues of $[\frac{\partial^2}{\partial \mathbf{k}_i \partial \mathbf{k}_j} e(\mathbf{0})]_{1\le i,j\le 2}$ and set $\tilde\lambda = \sqrt{|\lambda_2/\lambda_1|}$. By the Morse lemma [17, Lemma 1.1 of Chapter 6], there is a C^{r-2} diffeomorphism $\mathbf{x}(\mathbf{k})$ with $\mathbf{x}(\mathbf{0}) = \mathbf{0}$ such that

$$\begin{aligned} e(\mathbf{k}) &= \lambda_1 x_1(\mathbf{k})^2 + \lambda_2 x_2(\mathbf{k})^2 \\ &= \lambda_1 (\, x_1(\mathbf{k})^2 - \tilde\lambda^2 x_2(\mathbf{k})^2 \,) \\ &= \lambda_1 (\, x_1(\mathbf{k}) - \tilde\lambda x_2(\mathbf{k}) \,)\,(\, x_1(\mathbf{k}) + \tilde\lambda x_2(\mathbf{k}) \,) \\ &= \lambda_1 (\, a_1 k_1 + a_2 k_2 - \tilde x_1(\mathbf{k}) \,)\,(\, b_1 k_1 + b_2 k_2 - \tilde x_2(\mathbf{k}) \,) \end{aligned}$$

with $\tilde x_1(\mathbf{k})$ and $\tilde x_2(\mathbf{k})$ vanishing to at least order two at $\mathbf{k} = 0$. Here $a_1 k_1 + a_2 k_2$ and $b_1 k_1 + b_2 k_2$ are the degree one parts of the Taylor expansions of $x_1(\mathbf{k}) - \tilde\lambda x_2(\mathbf{k})$ and $x_1(\mathbf{k}) + \tilde\lambda x_2(\mathbf{k})$ respectively. Since the Jacobian $\det D\mathbf{x}(\mathbf{0})$ of the diffeomorphism at the origin is nonzero and $\det \begin{bmatrix} 1 & -\tilde\lambda \\ 1 & \tilde\lambda \end{bmatrix} = 2\tilde\lambda \neq 0$, we have

$$\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = \det \left\{ \begin{bmatrix} 1 & -\tilde\lambda \\ 1 & \tilde\lambda \end{bmatrix} D\mathbf{x}(\mathbf{0}) \right\} \neq 0$$

Setting

$$A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}^{-1}$$

we have

$$e(A\mathbf{k}) = \lambda_1 (k_1 - \tilde x_1(A\mathbf{k}))(k_2 - \tilde x_2(A\mathbf{k}))$$

Write

$$k_1 - \tilde x_1(A\mathbf{k}) = k_1 - k_1 f_1(\mathbf{k}) - k_2^{\nu_1} g_1(k_2)$$

with

$$\begin{aligned} k_1 f_1(\mathbf{k}) &= \tilde x_1(A\mathbf{k}) - \tilde x_1(A\mathbf{k})|_{k_1=0} \\ k_2^{\nu_1} g_1(k_2) &= \tilde x_1(A\mathbf{k})|_{k_1=0} \end{aligned}$$

Since $\tilde x_1(A\mathbf{k})$ vanishes to order at least two at $\mathbf{k} = 0$,

$$f_1(\mathbf{k}) = \int_0^1 [\tfrac{\partial}{\partial k_1} \tilde x_1(A\mathbf{k})]_{\mathbf{k}=(tk_1,k_2)}\, dt$$

is C^{r-3} and vanishes to order at least one at $\mathbf{k} = 0$ and, in particular, $|f_1(\mathbf{k})| \le \frac{1}{2}$ for all $\mathbf{k}$ in a neighbourhood of the origin. We choose ν_1 to be the power of the first nonvanishing term in the Taylor expansion of $\tilde{x}_1(A\mathbf{k})|_{k_1=0}$. Since this function must vanish to order at least two in k_2, we have that $\nu_1 \geq 2$. By the nonflatness condition, applied to the curve implicitly determined by $k_1 = \tilde{x}_1(A\mathbf{k})$, $\nu_1 \leq r-2$. So $g_1(k_2)$ is $C^{r-2-\nu_1}$ and is bounded and bounded away from zero in a neighbourhood of $k_2 = 0$.

In a similar fashion, write

$$k_2 - \tilde{x}_2(A\mathbf{k}) = k_2 - k_2 f_2(\mathbf{k}) - k_1^{\nu_2} g_2(k_1)$$

with

$$\begin{aligned} k_2 f_2(\mathbf{k}) &= \tilde{x}_2(A\mathbf{k}) - \tilde{x}_2(A\mathbf{k})|_{k_2=0} \\ k_1^{\nu_2} g_2(k_1) &= \tilde{x}_2(A\mathbf{k})|_{k_2=0} \end{aligned}$$

Then we have the desired decomposition (2) with

$$\begin{aligned} a(\mathbf{k}) &= \lambda_1(1-f_1(\mathbf{k}))(1-f_2(\mathbf{k})) \\ b(\mathbf{k}) &= \frac{g_1(k_2)}{1-f_1(\mathbf{k})}\,, \qquad c(\mathbf{k}) = \frac{g_2(k_1)}{1-f_2(\mathbf{k})}. \end{aligned}$$

■

We remark that it is possible to impose weaker regularity hypotheses by exploiting that $k_2^{\nu_1} b(\mathbf{k})$, resp. $k_1^{\nu_2} c(\mathbf{k})$, is a C^{r-3} function whose k_2, resp. k_1, derivatives of order strictly less than ν_1, resp. ν_2, vanish at $k_2 = 0$, resp. $k_1 = 0$.

3.2 Length of overlap estimates

It follows from the normal form derived in Lemma 3.1 that under the hypotheses **H2–H5** the curvature of the Fermi surface may vanish as one approaches the singular points. Thus, even if the Fermi surface is curved away from these points, there is no uniform lower bound on the curvature. Curvature effects are very important in the analysis of regularity estimates, and in a situation without uniform bounds these curvature effects improve power counting only at scales lower than a scale set by the rate at which the curvature vanishes. Thus it becomes natural to define, at a given scale, scale–dependent neighbourhoods of the singular points, outside of which curvature improvements hold. The estimates for the length of overlaps that we prove in this section allow us to make this idea precise. They hold under much more general conditions than a nonvanishing curvature, namely the nonnesting and nonflatness assumptions **H5** and **H6** suffice. We first discuss the special case corresponding to the normal form in the vicinity of a singular point, and then deal with the general case.

3.2.1 Length of overlap – special case

Lemma 3.2 *Let* $\nu_1 \geq 2$ *and* $\nu_2 \geq 2$ *be integers and*

$$e(x,y) = (x - y^{\nu_1} b(x,y))(y - x^{\nu_2} c(x,y))$$

with b *and* c *bounded and bounded away from zero and with* $b, c \in C^{\nu_2+1}$*. Let* $u(x)$ *obey*

$$u(x) = x^{\nu_2} c(x, u(x))$$

for all x *in a neighbourhood of* 0*. That is,* $y = u(x)$ *lies on the Fermi curve* $e(x,y) = 0$*. There are constants* C *and* $D > 0$ *such that for all* $\varepsilon > 0$ *and* $0 < \delta \leq |(X,Y)| \leq D$

$$\text{Vol}\,\{\, x \in \mathbb{R} \mid |x| \leq D,\ |e(X+x, Y+u(x))| \leq \varepsilon \,\} \leq C(\tfrac{\varepsilon}{\delta})^{1/\nu_2}$$

Proof: Write

$$e(X+x, Y+u(x)) = F(x,X,Y)G(x,X,Y) \tag{3}$$

with

$$\begin{aligned}
F(x,X,Y) &= X + x - (Y+u(x))^{\nu_1} b(X+x, Y+u(x)) \\
G(x,X,Y) &= Y + u(x) - (X+x)^{\nu_2} c(X+x, Y+u(x)) \\
&= Y - \{(X+x)^{\nu_2} - x^{\nu_2}\} c(X+x, Y+u(x)) \\
&\quad - x^{\nu_2}\{c(X+x, Y+u(x)) - c(x, u(x))\}
\end{aligned}$$

Observe that, for all allowed x, X and Y,

$$|F(x,X,Y)| \leq \tfrac{1}{100} \qquad |\tfrac{\partial}{\partial x} F(x,X,Y)| \geq \tfrac{99}{100}$$

since x, X, Y and $u(x)$ all have to be $O(D)$ small. For our analysis of $G(x,X,Y)$ we consider two separate cases.

Case 1: $|Y| \geq \kappa |X|$ with κ a constant to be chosen shortly. Since $c(X+x, Y+u(x)) - c(x,u(x))$ vanishes to first order in (X,Y), for all x

$$\begin{aligned}
|x^{\nu_2}\{c(X+x, Y+u(x)) - c(x,u(x))\}| &\leq \tfrac{1}{100}[|X| + |Y|] \\
|\tfrac{\partial}{\partial x}[x^{\nu_2}\{c(X+x, Y+u(x)) - c(x,u(x))\}]| &\leq \tfrac{1}{100}[|X| + |Y|]
\end{aligned}$$

Since $(X+x)^{\nu_2} - x^{\nu_2}$ vanishes to first order in X, for all x

$$\begin{aligned}
|\{(X+x)^{\nu_2} - x^{\nu_2}\} c(X+x, Y+u(x))| &\leq \tilde{\kappa}|X| \\
|\tfrac{\partial}{\partial x}[\{(X+x)^{\nu_2} - x^{\nu_2}\} c(X+x, Y+u(x))]| &\leq \tilde{\kappa}|X|
\end{aligned}$$

We choose $\kappa = \max\{2, 200\tilde{\kappa}\}$. Then

$$|G(x,X,Y)| \geq \tfrac{98}{100}|Y| \qquad |\tfrac{\partial}{\partial x}G(x,X,Y)| \leq \tfrac{2}{100}|Y|$$

Thus, by (3) and the product rule,

$$|\tfrac{\partial}{\partial x}e(X+x, Y+u(x))| \;\;\geq\;\; (\tfrac{99}{100}\tfrac{98}{100} - \tfrac{1}{100}\tfrac{2}{100})|Y| \geq \tfrac{1}{2}|Y|$$

and, by Lemma A.1,

$$\text{Vol}\,\{\; x \in \mathbb{R} \mid |x| \leq D,\; |e(X+x, Y+u(x))| \leq \varepsilon \;\} \leq 4\tfrac{\varepsilon}{|Y|/2} \leq 16\tfrac{\varepsilon}{\delta}.$$

Case 2: $|Y| \leq \kappa|X|$. In this case we bound the ν_2^{th} x–derivative away from zero. We claim that the dominant term comes from one derivative acting on F and ν_2-1 derivatives acting on G. Observe that for $|X|,\ |Y|,\ |x| \leq D$ with D sufficiently small

$$\left|\tfrac{d^m}{dx^m}u(x)\right| \;\;\leq\;\; O(D) \qquad \text{for } 0 \leq m < \nu_2$$

since $u(x) = x^{\nu_2}c(x, u(x))$ and consequently

$$\begin{aligned} |F(x,X,Y)| \;&\leq\; O(D) \\ |\tfrac{\partial}{\partial x}F(x,X,Y)| \;&\geq\; 1 - O(D) \\ |\tfrac{\partial^m}{\partial x^m}F(x,X,Y)| \;&\leq\; O(D) \qquad \text{for } 1 < m \leq \nu_2 \end{aligned}$$

Furthermore, since $c(X+x, Y+u(x)) - c(x, u(x))$ vanishes to first order in (X,Y), for all x,

$$|\tfrac{\partial^m}{\partial x^m}[x^{\nu_2}\{c(X+x, Y+u(x)) - c(x,u(x))\}]| \leq O(D)[|X| + |Y|]$$

for $0 \leq m < \nu_2$ and

$$|\tfrac{\partial^m}{\partial x^m}[x^{\nu_2}\{c(X+x, Y+u(x)) - c(x,u(x))\}]| \leq O(1)[|X| + |Y|]$$

for $m = \nu_2$. Since $(X+x)^{\nu_2} - \nu_2 X x^{\nu_2-1} - x^{\nu_2}$ vanishes to second order in X, for all x,

$$\begin{aligned} |\tfrac{\partial^m}{\partial x^m}[\{(X+x)^{\nu_2} - \nu_2 X x^{\nu_2-1} - x^{\nu_2}\}c(X+x, Y+u(x))]| \;&\leq\; O(|X|^2) \\ &\leq\; O(D)|X| \end{aligned}$$

for all $0 \le m \le \nu_2$. Finally

$$\begin{array}{rcll} |\frac{\partial^m}{\partial x^m}[\nu_2 X x^{\nu_2-1} c(X+x, Y+u(x))]| & \le & O(D)|X| & \text{for } 0 \le m < \nu_2 - 1 \\ |\frac{\partial^m}{\partial x^m}[\nu_2 X x^{\nu_2-1} c(X+x, Y+u(x))]| & \ge & \kappa'|X| - O(D)|X| & \text{for } m = \nu_2 - 1 \\ |\frac{\partial^m}{\partial x^m}[\nu_2 X x^{\nu_2-1} c(X+x, Y+u(x))]| & \le & O(1)|X| & \text{for } m = \nu_2 \end{array}$$

with $\kappa' = \nu_2! \inf |c(x,y)| > 0$. Consequently,

$$\begin{aligned} |\tfrac{\partial^{\nu_2}}{\partial x^{\nu_2}}\ & e(X+x, Y+u(x))| \\ &= |\nu_2 \tfrac{\partial F}{\partial x} \tfrac{\partial^{\nu_2-1} G}{\partial x^{\nu_2-1}} + \sum_{\substack{m=0 \\ m \ne 1}}^{\nu_2} \tbinom{\nu_2}{m} \tfrac{\partial^m F}{\partial x^m} \tfrac{\partial^{\nu_2-m} G}{\partial x^{\nu_2-m}}| \\ &\ge (1-O(D))(\kappa' - O(D))|X| - O(D)[|X| + |Y|] \\ &\ge (1-O(D))(\kappa' - O(D))|X| - O(D)(1+\kappa)|X| \\ &\ge \tfrac{\kappa'}{2}|X| \end{aligned}$$

if D is small enough. Hence, by Lemma A.1,

$$\begin{aligned} \text{Vol}\ \{\ x \in \mathbb{R}\ |\ |x| \le D,\ |e(X+x, Y+u(x))| \le \varepsilon\ \} \\ \le 2^{\nu_2+1} \left(\tfrac{\varepsilon}{\kappa'|X|/2}\right)^{1/\nu_2} \le 2^{\nu_2+1} \left(\tfrac{2\sqrt{1+\kappa^2}}{\kappa'} \tfrac{\varepsilon}{\delta}\right)^{1/\nu_2} \end{aligned}$$

■

3.2.2 Length of overlap – general case

Proposition 3.3 *Assume* **H1–H6** *with* $r \ge 2n_0 + 1$. *There is a constant* $D > 0$ *such that for all* $0 < \delta < 1$ *and each sign* $\pm$ *the measure of the set of* $\mathbf{p} \in \mathbb{R}^2$ *such that*

$$\ell\Big(\{\ \mathbf{k} \in \mathcal{F}\ |\ |e(\mathbf{p} \pm \mathbf{k})| \le M^j\ \}\Big) \ge \left(\tfrac{M^j}{\delta}\right)^{1/n_0} \qquad \textit{for some } j < 0$$

is at most $D\delta^2$. *Here* ℓ *is the Euclidean measure (length) on* $\mathcal{F}$. *Recall that* n_0 *is the largest nonflatness or nonnesting order plus one.*

Lemma 3.4 *Let* $r \ge 2n_0 + 1$. *For each* $\tilde{\mathbf{p}} \in \mathbb{R}^2$ *and* $\tilde{\mathbf{k}} \in \mathcal{F}$, *there are constants* $d, D' > 0$ *(possibly depending on* $\tilde{\mathbf{p}}$ *and* $\tilde{\mathbf{k}}$*) such that for each sign* $\pm$, *all* $j < 0$, *and all* $\mathbf{p} \in \mathbb{R}^2$ *obeying* $|\mathbf{p} - \tilde{\mathbf{p}}| \le d$

$$\ell\Big(\{\ \mathbf{k} \in \mathcal{F}\ |\ |\mathbf{k} - \tilde{\mathbf{k}}| \le d,\ |e(\mathbf{p} \pm \mathbf{k})| \le M^j\ \}\Big) \le D' \left(\tfrac{M^j}{|\mathbf{p} - \tilde{\mathbf{p}}|}\right)^{1/n_0}$$

Proof of Proposition 3.3, assuming Lemma 3.4. For each $\tilde{\mathbf{p}} \in \mathbb{R}^2$ and $\tilde{\mathbf{k}} \in \mathcal{F}$, let $d_{\tilde{\mathbf{p}},\tilde{\mathbf{k}}}$, $D'_{\tilde{\mathbf{p}},\tilde{\mathbf{k}}}$ be the constants of the Lemma and set

$$\mathcal{O}_{\tilde{\mathbf{p}},\tilde{\mathbf{k}}} = \{ \ (\mathbf{p}, \mathbf{k}) \in \mathbb{R}^2 \times \mathcal{F} \mid |\mathbf{p} - \tilde{\mathbf{p}}| < d_{\tilde{\mathbf{p}},\tilde{\mathbf{k}}}, \ |\mathbf{k} - \tilde{\mathbf{k}}| < d_{\tilde{\mathbf{p}},\tilde{\mathbf{k}}} \ \}$$

Since $\mathcal{F} = \{ \ \mathbf{k} \mid e(\mathbf{k}) = 0 \ \}$ and $\{ \ \mathbf{k} \mid |e(\mathbf{k})| \le 1 \ \}$ are compact, there is an $R > 0$ such that if $|\mathbf{p}| > R$, then $\{ \ \mathbf{k} \in \mathcal{F} \mid |e(\mathbf{p} \pm \mathbf{k})| \le M^j \ \}$ is empty for all $j < 0$. Since $\{ \ \mathbf{p} \in \mathbb{R}^2 \mid |\mathbf{p}| \le R \ \} \times \mathcal{F}$ is compact, there are $(\tilde{\mathbf{p}}_1, \tilde{\mathbf{k}}_1)$, $\dots$, $(\tilde{\mathbf{p}}_N, \tilde{\mathbf{k}}_N)$ such that

$$\{ \ \mathbf{p} \in \mathbb{R}^2 \mid |\mathbf{p}| \le R \ \} \times \mathcal{F} \subset \bigcup_{i=1}^{N} \mathcal{O}_{\tilde{\mathbf{p}}_i,\tilde{\mathbf{k}}_i}$$

Fix any $0 < \delta < 1$ and set, for each $1 \le i \le N$, $\delta_i = (N D'_{\tilde{\mathbf{p}}_i,\tilde{\mathbf{k}}_i})^{n_0} \delta$. If $|\mathbf{p} - \tilde{\mathbf{p}}_i| > \delta_i$ for all $1 \le i \le N$, then for all $j < 0$

$$\begin{aligned}
&\ell(\{ \ \mathbf{k} \in \mathcal{F} \mid |e(\mathbf{p} \pm \mathbf{k})| \le M^j \ \}) \\
&\le \ \ell\Big(\bigcup_{\substack{1 \le i \le n \\ |\mathbf{p} - \tilde{\mathbf{p}}_i| \le d_{\tilde{\mathbf{p}}_i,\tilde{\mathbf{k}}_i}}} \{ \ \mathbf{k} \in \mathcal{F} \mid |\mathbf{k} - \tilde{\mathbf{k}}_{\tilde{\mathbf{p}}_i,\tilde{\mathbf{k}}_i}| \le d_{\tilde{\mathbf{p}}_i,\tilde{\mathbf{k}}_i}, \ |e(\mathbf{p} \pm \mathbf{k})| \le M^j \ \} \Big) \\
&\le \ \sum_{i=1}^{N} D'_{\tilde{\mathbf{p}}_i,\tilde{\mathbf{k}}_i} \left(\tfrac{M^j}{|\mathbf{p} - \tilde{\mathbf{p}}_i|} \right)^{1/n_0} < \sum_{i=1}^{N} D'_{\tilde{\mathbf{p}}_i,\tilde{\mathbf{k}}_i} \left(\tfrac{M^j}{\delta_i} \right)^{1/n_0} \\
&\le \ \sum_{i=1}^{N} \tfrac{1}{N} \left(\tfrac{M^j}{\delta} \right)^{1/n_0} = \left(\tfrac{M^j}{\delta} \right)^{1/n_0}
\end{aligned}$$

Consequently the measure of the set of $\mathbf{p} \in \mathbb{R}^2$ for which

$$\ell\Big(\{ \ \mathbf{k} \in \mathcal{F} \mid |e(\mathbf{p} \pm \mathbf{k})| \le M^j \ \} \Big) \ge (\tfrac{M^j}{\delta})^{1/n_0} \qquad \text{for some } j < 0$$

is at most

$$\sum_{i=1}^{N} \pi \delta_i^2 \le D \delta^2 \qquad \text{where} \quad D = \sum_{i=1}^{N} \pi (N D'_{\tilde{\mathbf{p}}_i,\tilde{\mathbf{k}}_i})^{2n_0}.$$

■

Proof of Lemma 3.4. We give the proof for $\mathbf{p} + \mathbf{k}$. The other case is similar. In the event that $\tilde{\mathbf{p}} + \tilde{\mathbf{k}} \notin \mathcal{F}$, there is a $d > 0$ and an integer $j_0 < 0$ such that

$\{\ \mathbf{k} \in \mathcal{F} \mid |\mathbf{k} - \tilde{\mathbf{k}}| \le d,\ |e(\mathbf{p}+\mathbf{k})| \le M^j\ \}$ is empty for all $|\mathbf{p} - \tilde{\mathbf{p}}| \le d$ and $j < j_0$. So we may assume that $\tilde{\mathbf{p}} + \tilde{\mathbf{k}} \in \mathcal{F}$.

Case 1: $\tilde{\mathbf{p}} + \tilde{\mathbf{k}}$ is not a singular point. By a rotation and translation of the $\mathbf{k}$ plane, we may assume that $\tilde{\mathbf{p}} + \tilde{\mathbf{k}} = \mathbf{0}$ and that the tangent line to $\mathcal{F}$ at $\tilde{\mathbf{p}} + \tilde{\mathbf{k}}$ is $k_2 = 0$. Then, as in Section 3.1, there are

- ○ $\nu \in \mathbb{N}$ with $2 \le \nu < n_0$ and
- ○ $C^{r-\nu}$ functions $a(\mathbf{q})$ and $b(\mathbf{q})$ that are bounded and bounded away from zero

such that

$$e(\mathbf{q}) = a(\mathbf{q})(q_2 - q_1^\nu b(\mathbf{q}))$$

in a neighbourhood of $\mathbf{0}$. (Choose $q_2 a(\mathbf{q}) = e(\mathbf{q}) - e(\mathbf{q})|_{q_2=0}$ and $q_1^\nu a(\mathbf{q}) b(\mathbf{q}) = e(\mathbf{q})|_{k_q=0}$.) If the tangent line to $\mathcal{F}$ at $\tilde{\mathbf{k}}$ (when $\tilde{\mathbf{k}}$ is a singular point the tangent line of one branch) is not parallel to $k_1 = 0$ and if d is small enough, we can write the equation of $\mathcal{F}$ for $\mathbf{k}$ within a distance d of $\tilde{\mathbf{k}}$ (when $\tilde{\mathbf{k}}$ is a singular point, the equation of the branch under consideration) as $k_2 - \tilde{k}_2 = (k_1 - \tilde{k}_1)^{\nu'} c(k_1 - \tilde{k}_1)$ for some $1 \le \nu' < n_0$ and some C^{r-n_0-1} function c that is bounded and bounded away from zero (when $\tilde{\mathbf{k}}$ is a singular point, by Lemma 3.1). Then, for $\mathbf{k} \in \mathcal{F}$, we have, writing $\mathbf{p} - \tilde{\mathbf{p}} = (X, Y)$ and $k_1 - \tilde{k}_1 = x$,

$$\begin{aligned} e(\mathbf{p}+\mathbf{k}) &= e(\mathbf{p} - \tilde{\mathbf{p}} + \mathbf{k} - \tilde{\mathbf{k}}) = e((X,Y) + (x, x^{\nu'} c(x))) \\ &= A(x,X,Y)(Y + x^{\nu'} c(x) - (X+x)^\nu B(x,X,Y)) \end{aligned}$$

where

$$\begin{aligned} A(x,X,Y) &= a(X+x, Y + x^{\nu'} c(x)) \\ B(x,X,Y) &= b(X+x, Y + x^{\nu'} c(x)) \end{aligned}$$

and $c(x)$ are bounded and bounded away from zero.

Observe that $y = x^{\nu'} c(x)$ is the equation of a fragment of $\mathcal{F}$ translated so as to move $\tilde{\mathbf{k}}$ to $\mathbf{0}$ and $y = x^\nu b(x,y)$ is the equation of a fragment of $\mathcal{F}$ translated so as to move $\tilde{\mathbf{p}} + \tilde{\mathbf{k}}$ to $\mathbf{0}$. If $\tilde{\mathbf{p}} = \mathbf{0}$ these two fragments may be identical. That is $x^{\nu'} c(x) \equiv x^\nu b(x, x^{\nu'} c(x))$. (Of course, in this case $\nu = \nu'$.) If $\tilde{\mathbf{p}} \ne \mathbf{0}$, the nonnesting condition says that there is an $n \in \mathbb{N}$ such that if $y = x^\nu b(x,y)$ is rewritten in the form $y = x^\nu C(x)$, then the $n^{\rm th}$ derivative of $x^\nu C(x) - x^{\nu'} c(x)$ must not vanish at $x = 0$. Let $n < n_0$ be the smallest such natural number. Since derivatives of $x^\nu C(x)$ at $x=0$ of order strictly lower than n agree with the corresponding derivatives of $x^{\nu'}c(x)$, the $n^{\rm th}$ derivatives at $x=0$ of $x^\nu b(x, x^\nu C(x))$ and $x^\nu b(x, x^{\nu'}c(x))$ coincide. Since $x^\nu C(x) \equiv x^\nu b(x, x^\nu C(x))$, the $n^{\rm th}$ derivative of $x^{\nu'}c(x) - x^\nu b(x, x^{\nu'}c(x)) = x^{\nu'}c(x) - x^\nu B(x,0,0)$ must not vanish at $x=0$.

- If $\nu' < \nu$,

  $$\frac{d^{\nu'}}{dx^{\nu'}}(Y + x^{\nu'}c(x) - (X+x)^\nu B(x,X,Y)) = \nu'!\, c(x) + O(d) = \nu'!\, c(0) + O(d)$$

  is uniformly bounded away from zero, if d is small enough.

- If $\nu' > \nu$,

  $$\frac{d^\nu}{dx^\nu}(Y + x^{\nu'}c(x) - (X+x)^\nu B(x,X,Y)) = -\nu!\, B(0,0,0) + O(d)$$

  is uniformly bounded away from zero, if d is small enough.

- If $\nu' = \nu$ and $x^{\nu'}c(x) \not\equiv x^\nu B(x,0,0)$, then, as the function $x^\nu B(x,0,0) - (X+x)^\nu B(x,X,Y)$ vanishes for all x if $X = Y = 0$,

  $$\begin{aligned} & \frac{d^n}{dx^n}\Big[Y + x^{\nu'}c(x) - (X+x)^\nu B(x,X,Y)\Big] \\ &= \frac{d^n}{dx^n}\Big[Y + x^{\nu'}c(x) - x^\nu B(x,0,0) \\ &\qquad + [x^\nu B(x,0,0) - (X+x)^\nu B(x,X,Y)]\Big] \\ &= \frac{d^n}{dx^n}(x^{\nu'}c(x) - x^\nu B(x,0,0)) + O(|X|+|Y|) \\ &= \frac{d^n}{dx^n}(x^{\nu'}c(x) - x^\nu B(x,0,0))|_{x=0} + O(d) + O(|X|+|Y|) \end{aligned}$$

  is uniformly bounded away from zero, if d is small enough.

- If $\nu' = \nu$ and $x^{\nu'}c(x) \equiv x^\nu B(x,0,0)$ and $|Y| \le |X|$

  $$\begin{aligned} Y &+ x^{\nu'}c(x) - (X+x)^\nu B(x,X,Y) \\ &= Y - \nu X x^{\nu-1} B(x,X,Y) \\ &\quad - \{(X+x)^\nu - \nu X x^{\nu-1} - x^\nu\} B(x,X,Y) \\ &\quad - x^\nu \{B(x,X,Y) - B(x,0,0)\} \end{aligned}$$

  so that

  $$\begin{aligned} & \frac{d^{\nu-1}}{dx^{\nu-1}}(Y + x^{\nu'}c(x) - (X+x)^\nu B(x,X,Y)) \\ &= -\nu!\, X[B(0,0,0) + O(d)] + O(d)O(|X|+|Y|) \end{aligned}$$

  is bounded away from zero by $\frac{1}{2}\nu!\, |XB(0,0,0)|$, if d is small enough.

In all of the above cases, by Lemma A.1,

$$\ell\Big(\{\ \mathbf{k}\in\mathcal{F}\mid |\mathbf{k}-\tilde{\mathbf{k}}|\le d,\ |e(\mathbf{p}+\mathbf{k})|\le M^j\ \}\Big)\le\sqrt{1+c_1^2}\,2^{\nu_0+1}(\tfrac{c_0M^j}{\rho})^{1/\nu_0}$$

where ν_0 is one of ν', ν, n or $\nu-1$, the constant c_0 is the inverse of the infimum of $a(\mathbf{k})$, the constant c_1 is the maximum slope of $\mathcal{F}$ within a distance d of $\tilde{\mathbf{k}}$ and ρ is either a constant or a constant times X with X at least a constant times $|\mathbf{p}-\tilde{\mathbf{p}}|$. There are two remaining possibilities. One is that the tangent line to $\mathcal{F}$ at $\tilde{\mathbf{k}}$ (when $\tilde{\mathbf{k}}$ is a singular point the tangent line of one branch) is parallel to $k_1=0$. This case is easy to handle because the two fragments of $\mathcal{F}$ are almost perpendicular, so that

$$\ell\Big(\{\ \mathbf{k}\in\mathcal{F}\mid |\mathbf{k}-\tilde{\mathbf{k}}|\le d,\ |e(\mathbf{p}+\mathbf{k})|\le M^j\ \}\Big)\le\ \text{const}\ M^j$$

The final possibility is

- If $\nu'=\nu$ and $x^{\nu'}c(x)\equiv x^\nu B(x,0,0)$ and $|Y|\ge|X|$

  $$\begin{aligned}Y\ &+\ x^{\nu'}c(x)-(X+x)^\nu B(x,X,Y)\\ &=\ Y-\{(X+x)^\nu-x^\nu\}B(x,X,Y)-x^\nu\{B(x,X,Y)-B(x,0,0)\}\end{aligned}$$

  so that

  $$|Y+x^{\nu'}c(x)-(X+x)^\nu B(x,X,Y)|\ge|Y|-O(d)O(|X|+|Y|)\ge\tfrac{1}{2}|Y|$$

  if d is small enough. As a result $\{\ \mathbf{k}\in\mathcal{F}\mid |\mathbf{k}-\tilde{\mathbf{k}}|\le d,\ |e(\mathbf{p}+\mathbf{k})|\le M^j\ \}$ is empty if $|Y|$ is larger than some constant times M^j. On the other hand, if $|Y|$ is smaller than a constant times M^j, then $\frac{M^j}{|\mathbf{p}-\tilde{\mathbf{p}}|}$ is larger than some constant.

Case 2: $\tilde{\mathbf{p}}+\tilde{\mathbf{k}}$ is a singular point. By Lemma 3.1,

$$e(\tilde{\mathbf{p}}+\tilde{\mathbf{k}}+\mathbb{M}\mathbf{q})=a(\mathbf{q})(q_1-q_2^{\nu_1}b(\mathbf{q}))(q_2-q_1^{\nu_2}c(\mathbf{q}))$$

where $2\le\nu_1,\nu_2<n_0$ are integers, $\mathbb{M}$ is a constant, nonsingular matrix and $a(\mathbf{k})$, $b(\mathbf{k})$ and $c(\mathbf{k})$ are $C^{r-2-\max\{\nu_1,\nu_2\}}$ functions that are bounded and bounded away from zero.

Suppose that the tangent line to $\mathbb{M}^{-1}\mathcal{F}$ at $\tilde{\mathbf{k}}$ (when $\tilde{\mathbf{k}}$ is a singular point, the tangent line of one branch) makes an angle of at most 45° with the x–axis. Otherwise exchange the roles of the q_1 and q_2 coordinates. If d is small enough, we can write the equation of $\mathcal{F}$ for $\mathbf{k}$ within a distance d of $\tilde{\mathbf{k}}$ (when $\tilde{\mathbf{k}}$ is a singular point, the equation of the branch under consideration) as

$$(\mathbb{M}^{-1}(\mathbf{k}-\tilde{\mathbf{k}}))_2 = (\mathbb{M}^{-1}(\mathbf{k}-\tilde{\mathbf{k}}))_1^{\nu'} v((\mathbb{M}^{-1}(\mathbf{k}-\tilde{\mathbf{k}}))_1)$$

for some $1 \le \nu' < n_0$ and some C^{r-n_0-1} function v that is bounded and bounded away from zero. Then, writing $\mathbb{M}^{-1}(\mathbf{p}-\tilde{\mathbf{p}}) = (X, Y)$ and $(\mathbb{M}^{-1}(\mathbf{k}-\tilde{\mathbf{k}}))_1 = x$ and assuming that $\mathbf{k} \in \mathcal{F}$,

$$\begin{aligned} e(\mathbf{p}+\mathbf{k}) &= e(\tilde{\mathbf{p}}+\tilde{\mathbf{k}}+\mathbb{M}\mathbb{M}^{-1}(\mathbf{p}-\tilde{\mathbf{p}}+\mathbf{k}-\tilde{\mathbf{k}})) \\ &= e(\tilde{\mathbf{p}}+\tilde{\mathbf{k}}+\mathbb{M}(X+x, Y+x^{\nu'}v(x))) \\ &= A(x,X,Y)F(x,X,Y)G(x,X,Y) \end{aligned}$$

where

$$\begin{aligned} A(x,X,Y) &= a(X+x, Y+x^{\nu'}v(x)) \\ F(x,X,Y) &= X+x-(Y+x^{\nu'}v(x))^{\nu_1}B(x,X,Y) \\ G(x,X,Y) &= Y+x^{\nu'}v(x)-(X+x)^{\nu_2}C(x,X,Y) \\ B(x,X,Y) &= b(X+x, Y+x^{\nu'}v(x)) \\ C(x,X,Y) &= c(X+x, Y+x^{\nu'}v(x)) \end{aligned}$$

The functions $A(x,X,Y)$, $B(x,X,Y)$, $C(x,X,Y)$ and $v(x)$ are all C^{r-n_0-1} and bounded and bounded away from zero.

As in case 1, $y = x^{\nu'}v(x)$ is the equation of a fragment of $\mathbb{M}^{-1}\mathcal{F}$ translated so as to move $\tilde{\mathbf{k}}$ to $\mathbf{0}$ and $y = x^{\nu_2}c(x,y)$ is the equation of a fragment of $\mathbb{M}^{-1}\mathcal{F}$ translated so as to move $\tilde{\mathbf{p}}+\tilde{\mathbf{k}}$ to $\mathbf{0}$. If $\tilde{\mathbf{p}} = \mathbf{0}$, these two fragments may be identical in which case $x^{\nu'}v(x) \equiv x^{\nu_2}c(x, x^{\nu'}v(x))$ and $\nu_2 = \nu'$. This case has already been dealt with in Lemma 3.2. Otherwise, the nonnesting condition says that there is an $n \in \mathbb{N}$ such that if $y = x^{\nu_2}c(x,y)$ is rewritten in the form $y = x^{\nu_2}V(x)$, then the n^{th} derivative of $x^{\nu_2}V(x) - x^{\nu'}v(x)$ must not vanish at $x = 0$. Let $n < n_0$ be the smallest such natural number. Since $x^{\nu_2}V(x) \equiv x^{\nu_2}c(x, x^{\nu_2}V(x))$ and since derivatives at 0 of $x^{\nu_2}V(x)$ of order lower than n agree with the corresponding derivatives of $x^{\nu'}v(x)$, the n^{th} derivative of $x^{\nu'}v(x) - x^{\nu_2}c(x, x^{\nu'}v(x)) = x^{\nu'}v(x) - x^{\nu_2}C(x,0,0)$ must not vanish at $x = 0$. So the remaining cases are:

- If $\nu' < \nu_2$, then

  $$\tfrac{d^{\nu'}}{dx^{\nu'}}G(x,X,Y) = \nu'!\, v(0) + O(d)$$

  Since

  $$F(x,X,Y) = O(d) \qquad \tfrac{d}{dx}F(x,X,Y) = 1 + O(d)$$

and applying zero to $\nu' - 1$ x–derivatives to $G(x,X,Y)$ gives $O(d)$, we have

$$\tfrac{d^{\nu'+1}}{dx^{\nu'+1}} F(x,X,Y)G(x,X,Y) = (\nu'+1)!\, v(0) + O(d)$$

uniformly bounded away from zero, if d is small enough.

- If $\nu' > \nu_2$,

$$\tfrac{d^{\nu_2}}{dx^{\nu_2}} G(x,X,Y) = -\nu_2!\, C(0,0,0) + O(d)$$

  Again

$$F(x,X,Y) = O(d) \qquad \tfrac{d}{dx} F(x,X,Y) = 1 + O(d)$$

  and applying zero to $\nu_2 - 1$ x–derivatives to $G(x,X,Y)$ gives $O(d)$, so that

$$\tfrac{d^{\nu_2+1}}{dx^{\nu_2+1}} F(x,X,Y)G(x,X,Y) = -(\nu_2+1)!\, C(0,0,0) + O(d)$$

  is uniformly bounded away from zero, if d is small enough.

- If $\nu' = \nu_2$ and $x^{\nu'} v(x) \not\equiv x^{\nu_2} C(x,0,0)$

$$\begin{aligned}\tfrac{d^n}{dx^n} G(x,X,Y) &= \tfrac{d^n}{dx^n} \Big[Y + x^{\nu'} v(x) - x^{\nu_2} C(x,0,0) \\ &\qquad\qquad + [x^{\nu_2} C(x,0,0) - (X+x)^{\nu_2} C(x,X,Y)] \Big] \\ &= \tfrac{d^n}{dx^n} (x^{\nu'} v(x) - x^{\nu_2} C(x,0,0))\big|_{x=0} + O(d) + O(|X|+|Y|)\end{aligned}$$

  and applying strictly fewer than n derivatives gives $O(d)$. As in the last two cases

$$\tfrac{d^{n+1}}{dx^{n+1}} F(x,X,Y)G(x,X,Y) = n \tfrac{d^n}{dx^n} (x^{\nu'} v(x) - x^{\nu_2} C(x,0,0))\big|_{x=0} + O(d)$$

  is uniformly bounded away from zero, if d is small enough.

The lemma now follows by Lemma A.1, as in Case 1. ■

4 Regularity

4.1 The gradient of the self–energy

4.1.1 The second order contribution

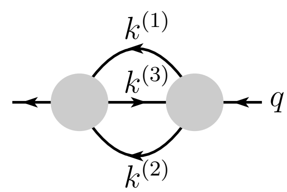
Let

$$C(k) = \frac{U(k)}{ik_0 - e(\mathbf{k})}$$

where the ultraviolet cutoff $U(k)$ is a smooth compactly supported function that is identically one for all k with $|ik_0 - e(\mathbf{k})|$ sufficiently small. We consider the value

$$F(q) = \int d^3k^{(1)} d^3k^{(2)} d^3k^{(3)}\ \delta(k^{(1)} + k^{(2)} - k^{(3)} - q)\ C(k^{(1)})C(k^{(2)})C(k^{(3)}) \\ V(k^{(1)}, k^{(2)}, k^{(3)}, q)$$

of the diagram



The function V is a second order polynomial in the interaction function $\hat{v}$. For details, as well as the generalization to frequency–dependent interactions, see [14]. For the purposes of the present discussion, all we need is a simple regularity assumption on V.

Lemma 4.1 *Assume* **H1–H6**. *If* $V(k^{(1)}, k^{(2)}, k^{(3)}, q)$ *is* C^1*, then* $F(q)$ *is* C^1 *in the spatial coordinates* $\mathbf{q}$*.*

Proof: Introduce our standard partition of unity of a neighbourhood of the Fermi surface [13, §2.1]

$$U(k) = \sum_{j<0} f(M^{-2j}|ik_0 - e(\mathbf{k})|^2)$$

where $f(M^{-2j}|ik_0 - e(\mathbf{k})|^2)$ vanishes unless $M^{j-2} \le |ik_0 - e(\mathbf{k})| \le M^j$. We have

$$C(k) = \sum_{j<0} C_j(k) \qquad \text{where} \qquad C_j(k) = \frac{f(M^{-2j}|ik_0 - e(\mathbf{k})|^2)}{ik_0 - e(\mathbf{k})}$$

and

$$F(q) = \sum_{j_1,j_2,j_3<0} \int d^3k^{(1)}d^3k^{(2)}d^3k^{(3)}\ \delta(k^{(1)}+k^{(2)}-k^{(3)}-q) \\ C_{j_1}(k^{(1)})C_{j_2}(k^{(2)})C_{j_3}(k^{(3)})\,V$$

Route the external momentum q through the line with smallest $|j_i|$ (i.e. use the delta function to evaluate the integral over the $k^{(i)}$ corresponding to the smallest $|j_i|$) and apply $\nabla_{\mathbf{q}}$. Rename the remaining integration variables $k^{(i)}$ to k and p. Permute the indices so that $j_1 \le j_2 \le j_3$. If the $\nabla_{\mathbf{q}}$ acts on V, the estimate is easy. For each fixed j_1, j_2 and j_3,

- ∘ the volume of the domain of integration is bounded by a constant times $|j_1|M^{2j_1}|j_2|M^{2j_2}$, by Lemma 2.3 of [1]. (The k_0 and p_0 components contribute $M^{j_1}M^{j_2}$ to this bound.)
- ∘ and the integrand is bounded by const $M^{-j_1}M^{-j_2}M^{-j_3}$.

so that

$$\begin{aligned}\sum_{j_1\le j_2\le j_3<0} \int d^3kd^3p\ |C_{j_1}(k)C_{j_2}(p)C_{j_3}(\pm k\pm p\pm q)\ \nabla_{\mathbf{q}}V| \\ \le \text{ const} \sum_{j_1\le j_2\le j_3<0} |j_1|M^{j_1}|j_2|M^{j_2}M^{-j_3} \le \text{ const} \sum_{j_1\le j_2<0}|j_1|M^{j_1}|j_2| \\ \le \text{ const} \sum_{j_1<0}|j_1|^3M^{j_1}\end{aligned}$$

is uniformly bounded. So assume that the $\nabla_{\mathbf{q}}$ acts on $C_{j_3}(\pm k \pm p \pm q)$. The terms of interest are now of the form

$$\int d^3kd^3p\ V\ C_{j_1}(k)C_{j_2}(p)\nabla_{\mathbf{q}}C_{j_3}(\pm k\pm p\pm q)$$

with $j_1 \le j_2 \le j_3 < 0$ and

$$\nabla_{\mathbf{q}}C_{j_3}(\pm k\pm p\pm q) = \pm\Big[\tfrac{\nabla e(\tilde{\mathbf{k}})}{[i\tilde{k}_0-e(\tilde{\mathbf{k}})]^2} f\big(M^{-2j_3}|i\tilde{k}_0-e(\tilde{\mathbf{k}})|^2\big) \\ +\tfrac{2M^{-2j_3}e(\tilde{\mathbf{k}})\nabla e(\tilde{\mathbf{k}})}{i\tilde{k}_0-e(\tilde{\mathbf{k}})} f'\big(M^{-2j_3}|i\tilde{k}_0-e(\tilde{\mathbf{k}})|^2\big)\Big]_{\tilde{k}=\pm k\pm p\pm q} \tag{4}$$

Observe that $|\nabla_{\mathbf{q}}C_{j_3}(\pm k\pm p\pm q)| \le$ const M^{-2j_3} since $|i\tilde{k}_0 - e(\tilde{\mathbf{k}})| \ge$ const M^{j_3} and $|e(\tilde{\mathbf{k}})| \le$ const′ M^{j_3} on the support of $f(M^{-2j_3}|i\tilde{k}_0 - e(\tilde{\mathbf{k}})|^2)$. Choose three small constants $\eta, \tilde{\eta}, \varepsilon > 0$ such that $0 < \varepsilon \le \frac{1}{2n_0}$, $0 < \eta < \frac{2n_0-1}{2n_0+2}\varepsilon$ and $\tilde{\eta} \le \frac{2\eta+\varepsilon}{2n_0+2\eta+\varepsilon}$. Here n_0 is the integer in Proposition 3.3. For example, if $n_0 = 3$, we can choose $\varepsilon = \frac{1}{6}$, $\eta = \frac{1}{10} < \frac{5}{48}$ and $\tilde{\eta} = \frac{1}{20}$.

Reduction 1: For any $\tilde{\eta} > 0$, it suffices to consider $j_1 \le j_2 \le j_3 \le (1-\tilde{\eta})j_1$. For the remaining terms, we simply bound

$$\begin{aligned}\Big|\int d^3k d^3p\, V\, C_{j_1}(k) C_{j_2}(p) \nabla_{\mathbf{q}} C_{j_3}(\pm k \pm p \pm q)\Big| \\ \le \text{ const } \int d^3k d^3p\, f\big(\tfrac{|ik_0 - e(\mathbf{k})|^2}{M^{2j_1}}\big) M^{-j_1}\, f\big(\tfrac{|ip_0 - e(\mathbf{p})|^2}{M^{2j_2}}\big) M^{-j_2}\, M^{-2j_3} \\ \le \text{ const } |j_1| M^{j_1}\, |j_2| M^{j_2}\, M^{-2j_3}\end{aligned}$$

and

$$\begin{aligned}\sum_{\substack{j_1 \le j_2 \le j_3 < 0 \\ j_3 \ge (1-\tilde{\eta})j_1}} |j_1| M^{j_1}\, |j_2| M^{j_2}\, M^{-2j_3} \\ \le \text{ const } \sum_{j_1 \le j_2 \le j_3 < 0} |j_1| M^{j_1}\, |j_2| M^{j_2}\, M^{-j_3 - (1-\tilde{\eta})j_1} \\ \le \text{ const } \sum_{j_1 \le j_2 < 0} |j_1|\, |j_2|\, M^{j_1}\, M^{-(1-\tilde{\eta})j_1} \\ = \text{ const } \sum_{j_1 < 0} |j_1|^3 M^{\tilde{\eta} j_1} < \infty\end{aligned}$$

Reduction 2: For any $\eta > 0$, it suffices to consider $(\mathbf{k}, \mathbf{p})$ with $|\pm\mathbf{k} \pm \mathbf{p} \pm \mathbf{q} - \tilde{\mathbf{q}}| \ge M^{\eta j_3}$ for all singular points $\tilde{\mathbf{q}}$. Let $\Xi_{j_3}(\mathbf{k})$ be the characteristic function of the set

$$\{\ \mathbf{k} \in \mathbb{R}^2 \mid |\mathbf{k} - \tilde{\mathbf{q}}| \ge M^{\eta j_3} \text{ for all singular points } \tilde{\mathbf{q}}\ \}$$

If $|\pm\mathbf{k} \pm \mathbf{p} \pm \mathbf{q} - \tilde{\mathbf{q}}| \le M^{\eta j_3}$ for some singular point $\tilde{\mathbf{q}}$, then $|\nabla e(\pm\mathbf{k} \pm \mathbf{p} \pm \mathbf{q})| \le$ const $M^{\eta j_3}$ and we may bound

$$\begin{aligned}\Big|\int d^3k d^3p\, V\, C_{j_1}(k) C_{j_2}(p) \nabla_{\mathbf{q}} C_{j_3}(\pm k \pm p \pm q)\ (1 - \Xi_{j_3}(\pm\mathbf{k} \pm \mathbf{p} \pm \mathbf{q}))\Big| \\ \le \text{ const } \int d^3k d^3p\, f\big(\tfrac{|ik_0 - e(\mathbf{k})|^2}{M^{2j_1}}\big) M^{-j_1}\, f\big(\tfrac{|ip_0 - e(\mathbf{p})|^2}{M^{2j_2}}\big) M^{-j_2}\, M^{-(2-\eta)j_3} \\ \le \text{ const } |j_1| M^{j_1}\, |j_2| M^{j_2}\, M^{-(2-\eta)j_3}\end{aligned}$$

and

$$\sum_{j_1\le j_2\le j_3<0} |j_1|M^{j_1}\ |j_2|M^{j_2}\ M^{-(2-\eta)j_3}$$
$$\le \text{ const } \sum_{j_1\le j_2<0} |j_1|M^{j_1}\ |j_2|M^{j_2}\ M^{-(2-\eta)j_2}$$
$$\le \text{ const } \sum_{j_1\le j_2<0} |j_1|^2 M^{j_1}\ M^{-j_2+\eta j_2}$$
$$\le \text{ const } \sum_{j_1<0} |j_1|^2 M^{j_1}\ M^{-j_1+\eta j_1}$$
$$= \text{ const } \sum_{j_1<0} |j_1|^2 M^{\eta j_1} < \infty$$

Current status: It remains to bound

$$\sum_{\substack{j_1\le j_2\le j_3<0\\ j_3\le(1-\tilde\eta)j_1}} \Big|\int d^3k d^3p\ V\ C_{j_1}(k)C_{j_2}(p)\nabla_{\mathbf q}C_{j_3}(\pm k\pm p\pm q)\ \Xi_{j_3}(\pm\mathbf k\pm\mathbf p\pm\mathbf q)\Big|$$
$$\le \text{ const } \sum_{\substack{j_1\le j_2\le j_3<0\\ j_3\le(1-\tilde\eta)j_1}} \int d^3kd^3p\ f\big(\tfrac{|ik_0-e(\mathbf k)|^2}{M^{2j_1}}\big)M^{-j_1}\ f\big(\tfrac{|ip_0-e(\mathbf p)|^2}{M^{2j_2}}\big)M^{-j_2}$$
$$M^{-2j_3}\ \Xi_{j_3}(\pm\mathbf k\pm\mathbf p\pm\mathbf q)\chi_{j_3}(\pm\mathbf k\pm\mathbf p\pm\mathbf q)$$
$$\le \text{ const } \sum_{\substack{j_1\le j_2\le j_3<0\\ j_3\le(1-\tilde\eta)j_1}} M^{-2j_3}\int d^2\mathbf k d^2\mathbf p\ \chi_{j_1}(\mathbf k)\ \chi_{j_2}(\mathbf p)\ (\chi_{j_3}\Xi_{j_3})\ (\pm\mathbf k\pm\mathbf p\pm\mathbf q)$$
$$\le \text{ const } \sum_{\substack{j_1,j_2,j_3<0\\ \frac{j_3}{1-\tilde\eta}\le j_1,j_2\le j_3}} M^{-2j_3}\int d^2\mathbf k d^2\mathbf p\ \chi_{j_1}(\mathbf k)\ \chi_{j_2}(\mathbf p)\ (\chi_{j_3}\Xi_{j_3})\ (\pm\mathbf k\pm\mathbf p\pm\mathbf q)$$

where $\chi_j(\mathbf k)$ is the characteristic function of the set of $\mathbf k$'s with $|e(\mathbf k)|\le M^j$.

Make a change of variables with $\pm\mathbf k\pm\mathbf p\pm\mathbf q$ becoming the new $\mathbf k$ integration variable. This gives

$$\text{const} \sum_{\substack{j_1,j_2,j_3<0\\ \frac{j_3}{1-\tilde\eta}\le j_1,j_2\le j_3}} M^{-2j_3}\int d^2\mathbf k d^2\mathbf p\ \chi_{j_1}(\pm\mathbf k\pm\mathbf p\pm\mathbf q)\ \chi_{j_2}(\mathbf p)\ \chi_{j_3}(\mathbf k)\ \Xi_{j_3}(\mathbf k)$$

Reduction 3: It suffices to show that, for each fixed $\tilde{\mathbf{k}}$, $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{q}}$ in $\mathbb{R}^2$, there are (possibly $\tilde{\mathbf{k}}$, $\tilde{\mathbf{p}}$, $\tilde{\mathbf{q}}$ dependent, but j_i independent) constants c and C such that

$$\int_{|\mathbf{k}-\tilde{\mathbf{k}}|\le c} d^2\mathbf{k} \int_{|\mathbf{p}-\tilde{\mathbf{p}}|\le c} d^2\mathbf{p}\, \chi_{j_1}(\pm\mathbf{k}\pm\mathbf{p}\pm\mathbf{q})\, \chi_{j_2}(\mathbf{p})\, \chi_{j_3}(\mathbf{k})\, \Xi_{j_3}(\mathbf{k}) \\ \le C|j_2| M^{j_2} M^{(1-\eta)j_3} M^{\varepsilon j_3 - \eta' j_3} \qquad (5)$$

for all $\mathbf{q}$ obeying $|\mathbf{q}-\tilde{\mathbf{q}}| \le c$. The constant η' will be chosen later and will obey $\varepsilon - \eta - \eta' > 0$. Since

$$\{\ (\mathbf{k},\mathbf{p},\mathbf{q}) \in \mathbb{R}^6 \mid |e(\mathbf{k})| \le 1, |e(\mathbf{p})| \le 1,\ |e(\pm\mathbf{k}\pm\mathbf{p}\pm\mathbf{q})| \le 1\ \}$$

is compact, once this is proven we will have the bound

$$\begin{aligned} &\text{const} \sum_{\substack{j_1,j_2,j_3<0 \\ \frac{j_3}{1-\tilde{\eta}}\le j_1,j_2\le j_3}} M^{-2j_3} \int d^2\mathbf{k} d^2\mathbf{p}\, \chi_{j_1}(\pm\mathbf{k}\pm\mathbf{p}\pm\mathbf{q})\, \chi_{j_2}(\mathbf{p})\, \chi_{j_3}(\mathbf{k})\, \Xi_{j_3}(\mathbf{k}) \\ &\le \text{const} \sum_{\substack{j_1,j_2,j_3<0 \\ \frac{j_3}{1-\tilde{\eta}}\le j_1,j_2\le j_3}} M^{-2j_3} |j_2| M^{j_2} M^{(1-\eta)j_3} M^{\varepsilon j_3 - \eta' j_3} \\ &\le \text{const} \sum_{j_3<0} \left(\tfrac{|j_3|}{1-\tilde{\eta}}\right)^3 M^{(\varepsilon-\eta-\eta')j_3} \\ &\le \text{const} \end{aligned}$$

since $\varepsilon > \eta + \eta'$. Furthermore, if $\tilde{\mathbf{k}}$ or $\tilde{\mathbf{p}}$ or $\pm\tilde{\mathbf{k}}\pm\tilde{\mathbf{p}}\pm\tilde{\mathbf{q}}$ does not lie on $\mathcal{F}$, we can choose c sufficiently small that the integral of (5) vanishes whenever $|\mathbf{q}-\tilde{\mathbf{q}}| \le c$ and $|j_1|, |j_2|, |j_3|$ are large enough. So it suffices to require that $\tilde{\mathbf{k}}$, $\tilde{\mathbf{p}}$ and $\pm\tilde{\mathbf{k}}\pm\tilde{\mathbf{p}}\pm\tilde{\mathbf{q}}$ all lie on $\mathcal{F}$.

Reduction 4: If $\tilde{\mathbf{k}}$ is not a singular point, make a change of variables to $\rho = e(\mathbf{k})$ and an "angular" variable θ. So the condition $\chi_{j_3}(\mathbf{k}) \ne 0$ forces $|\rho| \le$ const M^{j_3}. If $\tilde{\mathbf{k}}$ is a singular point, the condition $\Xi_{j_3}(\mathbf{k}) \ne 0$ forces $|\mathbf{k}-\tilde{\mathbf{k}}| \ge$ const $M^{\eta j_3}$ and this, in conjunction with the condition that $\chi_{j_3}(\mathbf{k}) \ne 0$, forces $\mathbf{k}$ to lie fairly near one of the two branches of $\mathcal{F}$ at $\tilde{\mathbf{k}}$ at least a distance const $M^{\eta j_3}$ from $\tilde{\mathbf{k}}$. Using Lemma 3.1, we can make a change of variables such that $e(\mathbf{k}(\rho,\theta)) = \rho\theta$ and either $|\theta| \ge$ const $M^{\eta j_3}$ or $|\rho| \ge$ const $M^{\eta j_3}$. Possibly exchanging the roles of ρ and θ, we may, without loss of generality assume the former. Then the condition $\chi_{j_3}(\mathbf{k}) \ne 0$ forces $|\rho| \le$ const $M^{(1-\eta)j_3}$. Thus, regardless of whether $\tilde{\mathbf{k}}$ is singular or not,

$$\int_{|\mathbf{k}-\tilde{\mathbf{k}}|\le c} d^2\mathbf{k} \int_{|\mathbf{p}-\tilde{\mathbf{p}}|\le c} d^2\mathbf{p}\, \chi_{j_1}(\pm\mathbf{k}\pm\mathbf{p}\pm\mathbf{q})\, \chi_{j_2}(\mathbf{p})\, \chi_{j_3}(\mathbf{k})\, \Xi_{j_3}(\mathbf{k})$$
$$\le \text{ const} \int_{\substack{|\theta|\le 1 \\ |\rho|\le \text{const } M^{(1-\eta)j_3}}} d\rho d\theta \int_{|\mathbf{p}-\tilde{\mathbf{p}}|\le c} d^2\mathbf{p}\, \chi_{j_1}(\pm\mathbf{k}(\rho,\theta)\pm\mathbf{p}\pm\mathbf{q})\, \chi_{j_2}(\mathbf{p})$$
$$\le \text{ const} \int_{\substack{|\theta|\le 1 \\ |\rho|\le \text{const } M^{(1-\eta)j_3}}} d\rho d\theta \int_{|\mathbf{p}-\tilde{\mathbf{p}}|\le c} d^2\mathbf{p}\, \chi_{j'}(\pm\mathbf{k}(0,\theta)\pm\mathbf{p}\pm\mathbf{q})\, \chi_{j_2}(\mathbf{p})$$
$$\le \text{ const } M^{(1-\eta)j_3} \int_{|\theta|\le 1} d\theta \int_{|\mathbf{p}-\tilde{\mathbf{p}}|\le c} d^2\mathbf{p}\, \chi_{j'}(\pm\mathbf{k}(0,\theta)\pm\mathbf{p}\pm\mathbf{q})\, \chi_{j_2}(\mathbf{p})$$

where $M^{j'} = M^{j_1} + \text{ const } M^{(1-\eta)j_3} \le \text{ const } M^{(1-\eta)j_3}$. Thus it suffices to prove that

$$\int_{|\mathbf{p}-\tilde{\mathbf{p}}|\le c} d^2\mathbf{p} \int_{|\theta|\le 1} d\theta\, \chi_{j'}(\pm\mathbf{k}(0,\theta)\pm\mathbf{p}\pm\mathbf{q})\, \chi_{j_2}(\mathbf{p}) \le C|j_2|M^{j_2}M^{\varepsilon j_3-\eta' j_3}$$

for all $\mathbf{q}$ obeying $|\mathbf{q}-\tilde{\mathbf{q}}|\le c$.

The Final Step: We apply Proposition 3.3 with $j = j'$ and $\delta = M^{(1+\varepsilon)j_3/2}$. Denote by $\tilde{\chi}(\mathbf{p})$ the characteristic function of the set of $\mathbf{p}$'s with

$$\mu\Big(\{\ -1\le\theta\le 1 \mid |e(\ \pm\mathbf{p}\pm\mathbf{q}\pm\mathbf{k}(0,\theta))|\le M^{j'}\ \}\Big) \ge c_1\big(\tfrac{M^{j'}}{M^{(1+\varepsilon)j_3/2}}\big)^{1/n_0}$$

where c_1 is the supremum of $\frac{d\theta}{ds}$ (s is arc length). If $\pm\mathbf{p}\pm\mathbf{q}$ is not in the set of measure $D\delta^2$ specified in Proposition 3.3, then

$$\ell\Big(\{\ \mathbf{k}\in\mathcal{F} \mid |e(\ \pm\mathbf{p}\pm\mathbf{q}\pm\mathbf{k}))|\le M^{j'}\ \}\Big) < \big(\tfrac{M^{j'}}{\delta}\big)^{1/n_0}$$

and hence

$$\mu\Big(\{\ -1\le\theta\le 1 \mid |e(\ \pm\mathbf{p}\pm\mathbf{q}\pm\mathbf{k}(0,\theta))|\le M^{j'}\ \}\Big)$$
$$= \int_{-1}^{1} d\theta\, \chi\Big(|e(\ \pm\mathbf{p}\pm\mathbf{q}\pm\mathbf{k}(0,\theta))|\le M^{j'}\Big)$$
$$= \int ds \tfrac{d\theta}{ds}\, \chi\Big(|e(\ \pm\mathbf{p}\pm\mathbf{q}\pm\mathbf{k})|\le M^{j'}\Big)$$
$$\le c_1\ell\Big(\{\ \mathbf{k}\in\mathcal{F} \mid |e(\ \pm\mathbf{p}\pm\mathbf{q}\pm\mathbf{k}))|\le M^{j'}\ \}\Big)$$
$$< c_1\big(\tfrac{M^{j'}}{M^{(1+\varepsilon)j_3/2}}\big)^{1/n_0}$$

so that $\tilde{\chi}(\mathbf{p}) = 0$. Thus $\tilde{\chi}(\mathbf{p})$ vanishes except on a set of measure $DM^{(1+\varepsilon)j_3}$ and

$$\begin{aligned}
\int d^2\mathbf{p} \int_{|\theta|\le 1} & d\theta\, \chi_{j'}(\pm\mathbf{k}(0,\theta) \pm \mathbf{p} \pm \mathbf{q})\, \chi_{j_2}(\mathbf{p}) \\
&\le \int d^2\mathbf{p}\, \tilde{\chi}(\mathbf{p})\chi_{j_2}(\mathbf{p}) \int_{|\theta|\le 1} d\theta\, \chi_{j'}(\pm\mathbf{k}(0,\theta) \pm \mathbf{p} \pm \mathbf{q}) \\
&\qquad + \int d^2\mathbf{p}\, (1-\tilde{\chi}(\mathbf{p}))\chi_{j_2}(\mathbf{p}) \int_{|\theta|\le 1} d\theta\, \chi_{j'}(\pm\mathbf{k}(0,\theta) \pm \mathbf{p} \pm \mathbf{q}) \\
&\le 2\int d^2\mathbf{p}\, \tilde{\chi}(\mathbf{p}) + \text{ const } \int d^2\mathbf{p}\, \chi_{j_2}(\mathbf{p}) (\tfrac{M^{j'}}{M^{(1+\varepsilon)j_3/2}})^{1/n_0} \\
&\le \text{ const } M^{(1+\varepsilon)j_3} + \text{ const } |j_2| M^{j_2} (\tfrac{M^{j'}}{M^{(1+\varepsilon)j_3/2}})^{1/n_0} \\
&\le \text{ const } M^{j_3} M^{\varepsilon j_3} + \text{ const } |j_2| M^{j_2} (M^{(1-\eta)j_3 - \frac{1+\varepsilon}{2} j_3})^{1/n_0} \\
&\le \text{ const } M^{j_2} M^{(\varepsilon - \frac{\tilde{\eta}}{1-\tilde{\eta}})j_3} + \text{ const } |j_2| M^{j_2} (M^{(\frac{1}{2} - \eta - \frac{\varepsilon}{2})j_3})^{1/n_0} \\
&\qquad\qquad \text{since } j_3 \le (1-\tilde{\eta})j_2 \le j_2 - \tfrac{\tilde{\eta}}{1-\tilde{\eta}} j_3 \\
&\le \text{ const } |j_2| M^{j_2} M^{\varepsilon j_3 - \eta' j_3}
\end{aligned}$$

provided $\varepsilon \le \frac{1}{2n_0}$, $\eta' \ge \frac{\tilde{\eta}}{1-\tilde{\eta}}$ and $\eta' \ge \frac{\eta}{n_0} + \frac{\varepsilon}{2n_0}$.

We choose $\eta' = \frac{\eta}{n_0} + \frac{\varepsilon}{2n_0}$. Then the remaining conditions

$$\varepsilon > \eta + \eta' \iff \varepsilon > \eta + \tfrac{\eta}{n_0} + \tfrac{\varepsilon}{2n_0} \iff \tfrac{2n_0-1}{2n_0}\varepsilon > \tfrac{n_0+1}{n_0}\eta \iff \eta < \tfrac{2n_0-1}{2n_0+2}\varepsilon \quad (6)$$

and

$$\eta' \ge \tfrac{\tilde{\eta}}{1-\tilde{\eta}} \iff \tilde{\eta} \le \tfrac{\eta'}{1+\eta'} = \tfrac{2\eta+\varepsilon}{2n_0+2\eta+\varepsilon}$$

are satisfied because of the conditions we imposed when we chose $\eta, \tilde{\eta}$ and ε just before Reduction 1.

We have now verified that $\nabla_{\mathbf{q}} F(q)$ is a uniformly convergent sum (over j_1, j_2, and j_3) of the continuous functions $\nabla_{\mathbf{q}} \int d^3k d^3p\, V\, C_{j_1}(k) C_{j_2}(p) C_{j_3}(\pm k \pm p \pm q)$. Hence $F(q)$ is C^1 in $\mathbf{q}$. ■

4.1.2 The general diagram

The argument of the last section applies equally well to general diagrams.

Lemma 4.2 *Let $G(q)$ be the value of any two–legged 1PI graph with external momentum q. Then $G(q)$ is C^1 with respect to the spatial components $\mathbf{q}$.*

Proof: We shall simply merge the argument of the last section with the general bounding argument of [1, Appendix A]. This is a good time to read that Appendix, since we shall just explain the modifications to be made to it. In addition to the small constants $\eta, \eta', \tilde{\eta}, \varepsilon > 0$ of Lemma 4.1, we choose a small constant $\bar{\varepsilon} > 0$ and require[1] that

$$0 < \varepsilon \le \tfrac{1}{2n_0},\ 0 < \eta < \tfrac{2n_0-1}{2n_0+2}\varepsilon,\ \eta' = \tfrac{\eta}{n_0} + \tfrac{\varepsilon}{2n_0} \text{ and } \tfrac{\tilde{\eta}}{1-\tilde{\eta}} < \min\{\eta', \varepsilon - \eta - \eta'\}$$

and

$$\bar{\varepsilon} \le \min\{\eta\,,\ \tilde{\eta}\,,\ (1+\varepsilon-\eta-\eta')(1-\tilde{\eta})-1\}$$

with n_0 being the integer in Proposition 3.3. All of these conditions may be satisfied by

- ◦ choosing $0 < \varepsilon \le \frac{1}{2n_0}$ and then
- ◦ choosing $0 < \eta < \frac{2n_0-1}{2n_0+2}\varepsilon$ (by (6), this ensures that $\varepsilon - \eta - \eta' > 0$) and then
- ◦ choosing $0 < \tilde{\eta} < 1$ so that $\frac{\tilde{\eta}}{1-\tilde{\eta}} < \min\{\eta', \varepsilon - \eta - \eta'\}$ (this ensures that the expression $(1+\varepsilon-\eta-\eta')(1-\tilde{\eta}) - 1 > 0$) and then
- ◦ choosing $\bar{\varepsilon} > 0$ so that $\bar{\varepsilon} \le \min\{\eta\,,\ \tilde{\eta}\,,\ (1+\varepsilon-\eta-\eta')(1-\tilde{\eta})-1\}$

As in [1, Appendix A], use [1, (22)] to introduce a scale expansion for each propagator and express $G(q)$ in terms of a renormalized tree expansion [1, (24)]. We shall prove, by induction on the depth, D, of G^J, the bound

$$\sum_{J \in \mathcal{J}(j,t,R,G)} \sup_q |\partial_{q_0}^{s_0} \partial_{\mathbf{q}}^{s_1} G^J(q)| \le \text{const}_n |j|^{3n-2} M^j M^{-s_0 j} M^{-s_1(1-\bar{\varepsilon})j} \tag{7}$$

for $s_0, s_1 \in \{0, 1\}$. The notation is as in [1, Appendix A]: n is the number of vertices in G and $\mathcal{J}(j,t,R,G)$ is the set of all assignments J of scales to the lines of G that have root scale j, that give forest t and that are compatible with the assignment R of renormalization labels to the two–legged forks of t. (This is explained in more detail just before [1, (24)].) If $s_0 = 0$ and $s_1 = 1$, the right hand side becomes $\text{const}_n |j|^{3n-2} M^{\bar{\varepsilon} j}$, which is summable over $j < 0$, implying that $G(q)$ is C^1 with respect to the spatial components $\mathbf{q}$. If $s_1 = 0$, (7) is contained in [1, Proposition A.1], so it suffices to consider $s_1 = 1$.

[1]The first three conditions as well as the condition that $\frac{\tilde{\eta}}{1-\tilde{\eta}} \le \eta'$ were already present in Lemma 4.1. The other conditions are new.

As in [1, Appendix A], if $D > 0$, decompose the tree t into a pruned tree $\tilde{t}$ and insertion subtrees $\tau^1, \cdots, \tau^m$ by cutting the branches beneath all minimal $E_f = 2$ forks $f_1, \cdots, f_m$. In other words each of the forks $f_1, \cdots, f_m$ is an $E_f = 2$ fork having no $E_f = 2$ forks, except ϕ, below it in t. Each τ_i consists of the fork f_i and all of t that is above f_i. It has depth at most $D - 1$ so the corresponding subgraph G_{f_i} obeys (7). Think of each subgraph G_{f_i} as a generalized vertex in the graph $\tilde{G} = G/\{G_{f_1}, \cdots, G_{f_m}\}$. Thus $\tilde{G}$ now has two as well as four–legged vertices. These two–legged vertices have kernels of the form $T_i(k) = \sum_{j_{f_i} \le j_{\pi(f_i)}} \ell G_{f_i}(k)$ when f_i is a c–fork and of the form $T_i(k) = \sum_{j_{f_i} > j_{\pi(f_i)}} (\mathbb{1} - \ell) G_{f_i}(k)$ when f_i is an r–fork. At least one of the external lines of G_{f_i} must be of scale precisely $j_{\pi(f_i)}$ so the momentum k passing through G_{f_i} lies in the support of $C_{j_{\pi(f_i)}}$. In the case of a c–fork $f = f_i$ we have, as in [1, (27)] and using the same notation, by the inductive hypothesis,

$$\begin{aligned}\sum_{j_f \le j_{\pi(f)}} \sum_{J_f \in \mathcal{J}(j_f, t_f, R_f, G_f)} \sup_k \left| \partial_{\mathbf{k}}^{s_1} \ell G_f^{J_f}(k) \right| &\le \sum_{j_f \le j_{\pi(f)}} \mathrm{const}_{n_f} |j_f|^{3n_f - 2} M^{j_f} M^{-s_1(1-\bar{\varepsilon})j_f} \\ &\le \mathrm{const}_{n_f} |j_{\pi(f)}|^{3n_f-2} M^{j_{\pi(f)}} M^{-s_1(1-\bar{\varepsilon})j_{\pi(f)}} \qquad (8)\end{aligned}$$

for $s_1 = 0, 1$. As $\ell G_f^{J_f}(k)$ is independent of k_0 derivatives with respect to k_0 may not act on it. In the case of an r–fork $f = f_i$, we have, as in [1, (29)],

$$\begin{aligned}\sum_{j_f > j_{\pi(f)}} \sum_{J_f \in \mathcal{J}(j_f, t_f, R_f, G_f)} &\sup_k \mathbb{1}(C_{j_{\pi(f)}}(k) \ne 0) \left| \partial_{k_0}^{s_0} \partial_{\mathbf{k}}^{s_1} (\mathbb{1} - \ell) G_f^{J_f}(k) \right| \\ &\le \sum_{j_f > j_{\pi(f)}} \sum_{J_f \in \mathcal{J}(j_f, t_f, R_f, G_f)} M^{(1-s_0)j_{\pi(f)}} \sup_k \left| \partial_{k_0} \partial_{\mathbf{k}}^{s_1} G_f^{J_f}(k) \right| \\ &\le \mathrm{const}_{n_f} M^{(1-s_0)j_{\pi(f)}} \sum_{j_f > j_{\pi(f)}} |j_f|^{3n_f-2} M^{-s_1(1-\bar{\varepsilon})j_f} \\ &\le \mathrm{const}_{n_f} |j_{\pi(f)}|^{3n_f-1} M^{j_{\pi(f)}} M^{-s_0 j_{\pi(f)}} M^{-s_1(1-\bar{\varepsilon})j_{\pi(f)}} \qquad (9)\end{aligned}$$

Denote by $\tilde{J}$ the restriction to $\tilde{G}$ of the scale assignment J. We bound $\tilde{G}^{\tilde{J}}$, which again is of the form [1, (31)], by a variant of the six step procedure followed in [1, Appendix A]. In fact the first four steps are identical.

1. Choose a spanning tree $\tilde{T}$ for $\tilde{G}$ with the property that $\tilde{T} \cap \tilde{G}_f^{\tilde{J}}$ is a connected tree for every $f \in t(\tilde{G}^{\tilde{J}})$.

2. Apply any q–derivatives. By the product rule each derivative may act on any line or vertex on the "external momentum path". It suffices to consider any one such action.

3. Bound each two–legged renormalized subgraph (i.e. r–fork) by (9) and each two–legged counterterm (i.e. c–fork) by (8). Observe that when s_0' k_0–derivatives and s_1' $\mathbf{k}$–derivatives act on the vertex, the bound is no worse than $M^{-s_0' j} M^{-s_1'(1-\bar{\varepsilon})j}$ times the bound with no derivatives, because we necessarily have $j \le j_{\pi(f)} < 0$.

4. Bound all remaining vertex functions, $u_{\rm v}$, (suitably differentiated) by their suprema in momentum space.

We have already observed that if $s_1 = 0$, the bound (7) is contained in [1, Proposition A.1], with $s = 0$. In the event that $s_1 = 1$, but the spatial gradient acts on a vertex, [1, Proposition A.1], again with $s = 0$ but with either one v replaced by its gradient or with an extra factor of $M^{-(1-\bar{\varepsilon})j}$ coming from Step 3, again gives (7).

So it suffices to consider the case that $s_1 = 1$ and the spatial gradient acts on a propagator of the "external momentum path". It is in this case that we apply the arguments of Lemma 4.2. The heart of those arguments was the observation that, when the gradient acted on a line ℓ_3 of scale j_3, the line ℓ_3 also lay on distinct momentum loops, Λ_{ℓ_1} and Λ_{ℓ_2}, generated by lines, ℓ_1 and ℓ_2 of scales j_{ℓ_1} and j_{ℓ_2} with $j_{\ell_1}, j_{\ell_2} \le j_{\ell_3}$. This is still the case and is proven in Lemmas 4.3 and 4.4 below. So we may now apply the procedure of Lemma 4.1.

Reduction 1: It suffices to consider $j \le j_{\ell_3} \le (1-\tilde{\eta})j$. For the remaining terms, we simply bound the differentiated propagator, as in the argument following (4), by

$$\begin{aligned} |\partial_{q_0}^{s_0'} \partial_{\mathbf{q}} C_{j_{\ell_3}}(k_{\ell_3}(q))| &\le \text{const } M^{-2j_{\ell_3}} M^{-s_0' j_{\ell_3}} \\ &\le \text{const } M^{-j_{\ell_3}} M^{-s_0' j_{\ell_3}} M^{-(1-\tilde{\eta})j} \\ &\le \text{const } M^{-j_{\ell_3}} M^{-s_0' j_{\ell_3}} M^{-(1-\bar{\varepsilon})j} \end{aligned}$$

So for the terms with $j_{\ell_3} > (1-\tilde{\eta})j$, the effect of the spatial gradient is to degrade the $s_1 = 0$ bound by at most a factor of const $M^{-(1-\bar{\varepsilon})j}$ and we may apply the rest of [1, Proposition A.1], starting with step 5, without further modification.

Reduction 2: It suffices to consider loop momenta $(k_\ell)_{\ell\in\tilde G\setminus\tilde T}$, in the domain of integration of [1, (31)], for which the momentum $\mathbf{k}_{\ell_3}$ flowing through ℓ_3 (which is a linear combination of $\mathbf{q}$ and various loop momenta) remains a distance at least $M^{\eta j_{\ell_3}}$ from all singular points. If $\mathbf{k}_{\ell_3}$ is at most a distance $M^{\eta j_{\ell_3}}$ from some singular point then $|\partial_{\mathbf{k}} e(\mathbf{k}_{\ell_3})| \le$ const $M^{\eta j_{\ell_3}}$ and we may use the ∇e in the numerator of (4) to improve the bound on the ℓ_3 propagator to

$$\begin{aligned} |\partial_{q0}^{s_0'}\partial_{\mathbf{q}} C_{j_{\ell_3}}(k_{\ell_3}(q))| &\le \text{const } M^{-2j_{\ell_3}} M^{-s_0' j_{\ell_3}} M^{\eta j_{\ell_3}} \\ &\le \text{const } M^{-j_{\ell_3}} M^{-s_0' j_{\ell_3}} M^{-(1-\eta)j} \\ &\le \text{const } M^{-j_{\ell_3}} M^{-s_0' j_{\ell_3}} M^{-(1-\bar\varepsilon)j} \end{aligned}$$

Once again, in this case, the effect of the spatial gradient is to degrade the $s_1 = 0$ bound by at most a factor of const $M^{-(1-\bar\varepsilon)j}$ and we may apply the rest of the proof [1, Proposition A.1], starting with step 5, without further modification.

Reduction 3: Now apply step 5, that is, bound every propagator. The extra $\partial_{\mathbf{q}}$ acting on $C_{j_{\ell_3}}$ gives a factor of $M^{-s_1 j_{\ell_3}} \le M^{-s_1 j}$ worse than the bound achieved in step 5 of [1, Proposition A.1]. Prepare for the application of step 6, the integration over loop momenta, by ordering the integrals in such a way that the two integrals executed first (that is the two innermost integrals) are those over k_{ℓ_1} and k_{ℓ_2}. The momentum flowing through ℓ_3 is of the form $k_{\ell_3} = \pm k_{\ell_1} \pm k_{\ell_2} + q'$, where q' is some linear combination of the external momentum q and possibly various other loop momenta. Make a change of variables from $\mathbf{k}_{\ell_1}$ and $\mathbf{k}_{\ell_2}$ to $\mathbf{k} = \mathbf{k}_{\ell_3} = \pm\mathbf{k}_{\ell_1} \pm \mathbf{k}_{\ell_2} + \mathbf{q}'$ and $\mathbf{p} = \mathbf{k}_{\ell_2}$. It now suffices to show that, for each fixed $\tilde{\mathbf{k}}$, $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{q}}'$ in $\mathbb{R}^2$, there are (possibly $\tilde{\mathbf{k}}$, $\tilde{\mathbf{p}}$, $\tilde{\mathbf{q}}'$ dependent, but j_{ℓ_i} independent) constants c and C such that

$$\int_{|\mathbf{k}-\tilde{\mathbf{k}}|\le c} d^2\mathbf{k} \int_{|\mathbf{p}-\tilde{\mathbf{p}}|\le c} d^2\mathbf{p}\, \chi_{j_{\ell_1}}(\pm\mathbf{k}\pm\mathbf{p}\pm\mathbf{q}')\, \chi_{j_{\ell_2}}(\mathbf{p})\, \chi_{j_{\ell_3}}(\mathbf{k})\, \Xi_{j_{\ell_3}}(\mathbf{k}) \le C|j_{\ell_2}| M^{j_{\ell_2}} M^{(1-\eta)j_{\ell_3}} M^{\varepsilon j_{\ell_3} - \eta' j_{\ell_3}} \tag{10}$$

for all $\mathbf{q}'$ obeying $|\mathbf{q}' - \tilde{\mathbf{q}}'| \le c$. Recall that $\chi_j(\mathbf{k})$ is the characteristic function of the set of $\mathbf{k}$'s with $|e(\mathbf{k})| \le M^j$ and $\Xi_j(\mathbf{k})$ is the characteristic function of the set

$$\{\ \mathbf{k} \in \mathbb{R}^2 \mid |\mathbf{k} - \tilde{\mathbf{q}}| \ge M^{\eta j} \text{ for all singular points } \tilde{\mathbf{q}}\ \}$$

Once proven, the bound (10) (together with the usual compactness argument) replaces the bound

$$\int d^2\mathbf{k}_{\ell_1} \int d^2\mathbf{k}_{\ell_2}\ \chi_{j_{\ell_1}}(\mathbf{k}_{\ell_1})\ \chi_{j_{\ell_2}}(\mathbf{k}_{\ell_2}) \le \text{ const } |j_{\ell_1}| M^{j_{\ell_1}}\ |j_{\ell_2}| M^{j_{\ell_2}}$$

used in step 6 of [1, Proposition A.1]. Since $j \le j_{\ell_1}, j_{\ell_2} \le j_{\ell_3} \le (1-\tilde\eta)j$, the bound (10) constitutes an improvement by a factor of

$$\begin{aligned} \text{const}\ M^{j_{\ell_2}} M^{(1-\eta)j_{\ell_3}} M^{\varepsilon j_{\ell_3} - \eta' j_{\ell_3}} M^{-j_{\ell_1}}\ M^{-j_{\ell_2}} &\le \text{const}\ M^{(1-\eta-\eta'+\varepsilon)j_{\ell_3}} M^{-j_{\ell_1}} \\ &\le \text{const}\ M^{[(1-\eta-\eta'+\varepsilon)(1-\tilde\eta)-1]j} \\ &\le \text{const}\ M^{\bar\varepsilon j} \end{aligned}$$

So once we proven the bound (10), we may continue with the rest of the proof of [1, Proposition A.1], without further modifi cation, and show that the inductive hypothesis (7) is indeed preserved.

In fact the bound (10) has already been proven in Reduction 4 and The Final Step of the proof of Lemma 4.1. ■

Lemma 4.3 *Let G be any two–legged 1PI graph with each vertex having an even number of legs. Let T be a spanning tree for G. Assume that the two external legs of G are hooked to two distinct vertices and that ℓ_3 is a line of G that is in the linear subtree of T joining the external legs. Recall that any line ℓ not in T is associated with a loop Λ_ℓ that consists of ℓ and the linear subtree of T joining the vertices at the ends of ℓ. There exist two lines ℓ_1 and ℓ_2, not in T such that $\ell_3 \in \Lambda_{\ell_1} \cap \Lambda_{\ell_2}$.*

Proof: Since T is a tree, $T \setminus \{\ell_3\}$ necessarily contains exactly two connected components T_1 and T_2 (though one could consist of just a single vertex). On the other hand, since G is 1PI, $G \setminus \{\ell_3\}$ must be connected. So there must be a path in $G \setminus T$ that connects the two components of $T \setminus \{\ell_3\}$. Since T is a spanning tree, every line of $G \setminus T$ joins two vertices of T, so we may alway choose the connecting path to consist of a single line. Let ℓ_1 be any such line. Then $\ell_3 \in \Lambda_{\ell_1}$. If $G \setminus \{\ell_1, \ell_3\}$ is still connected, then there must be a second line $\ell_2 \ne \ell_1$ of $G \setminus T$ that connects the two components of $T \setminus \{\ell_3\}$. Again $\ell_2 \in \Lambda_{\ell_1}$. If $G \setminus \{\ell_1, \ell_3\}$ is not connected, it consists of two connected components G_1 and G_2 with G_1 containing T_1 and G_2 containing T_2. Each of T_1 and T_2 must contain exactly one external vertex of G. So each of G_1 and G_2 must have exactly one external leg that is also an external leg of G. As ℓ_1 and ℓ_3 are the remaining external legs of both G_1 and G_2, each has three external legs, which is impossible. ■

Lemma 4.4 *Let G and T be as in Lemma 4.3. Let J be an assignment of scales to the lines of G such that $T \cap G^J_f$ is a connected tree for every fork $f \in t(G^J)$.*

(See step 1 of [1, Proposition A.1].) Let $\ell_3 \in T$. Let $\ell \in G \setminus T$ connect the two components of $T \setminus \{\ell_3\}$, then $j_\ell \le j_{\ell_3}$.

Proof: Let G' be the connected component containing ℓ of the subgraph of G consisting of lines having scales $j \ge j_\ell$. By hypothesis, $G' \cap T$ is a spanning tree for G'. So the linear subtree of T joining the vertices of ℓ is completely contained in G'. But ℓ_3 is a member of that line's subtree and so has scale $j_{\ell_3} \ge j_\ell$. ■

4.2 The frequency derivative of the self–energy

We show to all orders that singularities in this derivative can only occur at the closure of the lattice generated by the Van Hove points. That the singularities really occur is shown by explicit calculations in model cases in the following sections.

Lemma 4.5 *Let $G(q)$ be the value of any two–legged 1PI graph with external momentum q. Let B be the closure of the set of momenta of the form $(0, \mathbf{q})$ with $\mathbf{q} = \sum_{i=1}^{n} (-1)^{s_i} \tilde{\mathbf{q}}_i$ where $n \in \mathbb{N}$ and, for each $1 \le i \le n$, $s_i \in \{0, 1\}$ and $\tilde{\mathbf{q}}_i$ is a singular point. Then $G(q)$ is C^1 with respect to q_0 on $\mathbb{R}^3 \setminus B$.*

Proof: The proof is similar to that of Lemma 4.2. Introduce scales in the standard way and denote the root scale j. Choose a spanning tree T in the standard way. View c– and r–forks as vertices. The external momentum is always routed through the spanning tree so the derivative may only act on vertices and on lines of the spanning tree. The cases in which the derivative acts on an interaction vertex or c–fork are trivial. If the derivative acts on an r–fork, the effect on the bound [1, (29)], namely a factor of $M^{-j_{\pi(f)}}$, is the same as the effect on the bound [1, (32)] when the derivative acts on a propagator attached to the r–fork. So suppose that the derivative acts on a line ℓ_3 of the spanning tree that has scale j_3. We know from Lemma 4.3 in the last section that there exist two different lines ℓ_1 and ℓ_2, not in T such that ℓ_3 lies on the loops associated to ℓ_1 and ℓ_2. We also know, from Lemma 4.4, that the scales j' of all loop momenta running through ℓ_3, including the scales j_1 and j_2 of the two lines chosen, obey $j' \le j_3$.

Reduction 1: It suffices to consider $j \le j_1, j_2 \le j_3 \le (1-\tilde{\eta})j$. For the remaining terms, we simply bound

$$\left|\frac{d}{dq_0} C_{j_3}(\pm q \pm \text{internal momenta})\right| \le \text{ const } M^{-2j_3} \le \text{ const } M^{-j_3} M^{-(1-\tilde{\eta})j}$$

After one sums over all scales except the root scale j, one ends up with

$$\text{const } |j|^n M^j M^{-(1-\tilde{\eta})j} = \text{ const } |j|^n M^{\tilde{\eta} j}$$

which is still summable.

Reduction 2: Denote by k, p and $\pm k \pm p + q'$ the momenta flowing in the lines ℓ_1, ℓ_2 and ℓ_3, respectively. Here q' is plus or minus the external momentum q possibly plus or minus some some other loop momenta. In this reduction we prove that it suffices to consider $(\mathbf{k}, \mathbf{p})$ with $|\mathbf{k} - \tilde{\mathbf{k}}| \le M^{\eta j}$ and $|\mathbf{p} - \tilde{\mathbf{p}}| \le M^{\eta j}$ and $| \pm \mathbf{k} \pm \mathbf{p} \pm \mathbf{q}' - \tilde{\mathbf{q}}| \le M^{\eta j}$ for some singular points $\tilde{\mathbf{k}}$, $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{q}}$.

Suppose that at least one of $\mathbf{k}$, $\mathbf{p}$, $\pm\mathbf{k} \pm \mathbf{p} \pm \mathbf{q}'$ is farther than $M^{\eta j}$ from all singular points. We can make a change of variables (just for the purposes of computing the volume of the domain of integration) such that $\mathbf{k}$ is at least a distance $M^{\eta j}$ from all singular points. After the change of variables, the indices j_1, j_2 and j_3 are no longer ordered, but all are still between j and $(1 - \tilde{\eta})j$. Let $\Xi_j(\mathbf{k})$ be the characteristic function of the set

$$\{ \ \mathbf{k} \in \mathbb{R}^2 \mid |\mathbf{k} - \tilde{\mathbf{k}}| \ge M^{\eta j} \text{ for all singular points } \tilde{\mathbf{k}} \ \}$$

We claim that

$$\begin{aligned} \text{Vol} \{ \ (\mathbf{k}, \mathbf{p}) \ &\in \ \mathbb{R}^4 \mid \chi_{j_1}(\mathbf{k}) \Xi_j(\mathbf{k}) \chi_{j_2}(\mathbf{p}) \chi_{j_3}(\pm\mathbf{k} \pm \mathbf{p} \pm \mathbf{q}') \ne 0 \ \} \\ &\le \ C|j_2| M^{j_1 - \eta j} M^{j_2} M^{\varepsilon j - \eta j} \end{aligned}$$

This would constitute a volume improvement of $M^{(\varepsilon - 2\eta)j}$ and would provide summability if $2\eta < \varepsilon$. By the usual compactness arguments, it suffices to show that, for each fixed $\tilde{\mathbf{k}}$, $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{q}}$ in $\mathbb{R}^2$, there are (possibly $\tilde{\mathbf{k}}$, $\tilde{\mathbf{p}}$, $\tilde{\mathbf{q}}$ dependent, but j, j_i independent) constants c and C such that

$$\int_{|\mathbf{k} - \tilde{\mathbf{k}}| \le c} d^2\mathbf{k} \int_{|\mathbf{p} - \tilde{\mathbf{p}}| \le c} d^2\mathbf{p} \, \chi_{j_1}(\mathbf{k}) \Xi_j(\mathbf{k}) \chi_{j_2}(\mathbf{p}) \chi_{j_3}(\pm\mathbf{k} \pm \mathbf{p} \pm \mathbf{q}') \le C|j_2| M^{j_1 - \eta j} M^{j_2} M^{\varepsilon j - \eta j}$$

for all $\mathbf{q}'$ obeying $|\mathbf{q}' - \tilde{\mathbf{q}}| \le c$. Furthermore, if $\tilde{\mathbf{k}}$ or $\tilde{\mathbf{p}}$ or $\pm\tilde{\mathbf{k}} \pm \tilde{\mathbf{p}} \pm \tilde{\mathbf{q}}'$ does not lie on $\mathcal{F}$, we can choose c sufficiently small that the integral vanishes whenever $|\mathbf{q}' - \tilde{\mathbf{q}}| \le c$ and $|j_1|, |j_2|, |j_3|$ are large enough. So it suffices to require that $\tilde{\mathbf{k}}$, $\tilde{\mathbf{p}}$ and $\pm\tilde{\mathbf{k}} \pm \tilde{\mathbf{p}} \pm \tilde{\mathbf{q}}'$ all lie on $\mathcal{F}$.

If $\tilde{\mathbf{k}}$ is not a singular point, make a change of variable to $\rho = e(\mathbf{k})$ and an "angular" variable θ. If $\tilde{\mathbf{k}}$ is a singular point, the condition $\Xi_j(\mathbf{k}) \ne 0$ forces

$|\mathbf{k}-\tilde{\mathbf{k}}| \geq M^{\eta j}$. We can make a change of variables such that $e(\mathbf{k}(\rho,\theta)) = \rho\theta$ and either $|\theta| \geq$ const $M^{\eta j}$ or $|\rho| \geq$ const $M^{\eta j}$. Possibly exchanging the roles of ρ and θ, we may, without loss of generality assume the former. In all of these cases, the condition $\chi_{j_1}(\mathbf{k}) \neq 0$ forces $|\rho| \leq$ const $M^{j_1-\eta j}$. Thus

$$\begin{aligned}
&\int_{|\mathbf{k}-\tilde{\mathbf{k}}|\leq c} d^2\mathbf{k} \int_{|\mathbf{p}-\tilde{\mathbf{p}}|\leq c} d^2\mathbf{p}\ \chi_{j_1}(\mathbf{k})\Xi_j(\mathbf{k})\chi_{j_2}(\mathbf{p})\chi_{j_3}(\pm\mathbf{k}\pm\mathbf{p}\pm\mathbf{q}') \\
&\qquad \leq \text{ const } \int_{\substack{|\theta|\leq 1 \\ |\rho|\leq \text{const} M^{j_1-\eta j}}} d\rho d\theta \int_{|\mathbf{p}-\tilde{\mathbf{p}}|\leq c} d^2\mathbf{p}\ \chi_{j_2}(\mathbf{p})\chi_{j_3}(\pm\mathbf{k}(\rho,\theta)\pm\mathbf{p}\pm\mathbf{q}') \\
&\qquad \leq \text{ const } \int_{\substack{|\theta|\leq 1 \\ |\rho|\leq \text{const} M^{j_1-\eta j}}} d\rho d\theta \int_{|\mathbf{p}-\tilde{\mathbf{p}}|\leq c} d^2\mathbf{p}\ \chi_{j'}(\pm\mathbf{k}(0,\theta)\pm\mathbf{p}\pm\mathbf{q}')\ \chi_{j_2}(\mathbf{p}) \\
&\qquad \leq \text{ const } M^{j_1-\eta j} \int_{|\theta|\leq 1} d\theta \int_{|\mathbf{p}-\tilde{\mathbf{p}}|\leq c} d^2\mathbf{p}\ \chi_{j'}(\pm\mathbf{k}(0,\theta)\pm\mathbf{p}\pm\mathbf{q}')\ \chi_{j_2}(\mathbf{p})
\end{aligned}$$

where $M^{j'} = M^{j_3} +$ const $M^{j_1-\eta j} \leq$ const $M^{(1-\eta-\tilde{\eta})j}$. Thus it suffices to prove that

$$\int_{|\mathbf{p}-\tilde{\mathbf{p}}|\leq c} d^2\mathbf{p} \int_{|\theta|\leq 1} d\theta\ \chi_{j'}(\pm\mathbf{k}(0,\theta)\pm\mathbf{p}\pm\mathbf{q}')\ \chi_{j_2}(\mathbf{p}) \leq C|j_2|M^{j_2}M^{\varepsilon j-\eta j}$$

for all $\mathbf{q}'$ obeying $|\mathbf{q}'-\tilde{\mathbf{q}}| \leq c$.

We again apply Proposition 3.3 with $j = j'$ and $\delta = M^{(1+\varepsilon)j_2/2}$. If we denote by $\tilde{\chi}(\mathbf{p})$ the characteristic function of the set of $\mathbf{p}$'s with

$$\mu\Big(\{\ -1\leq\theta\leq 1 \mid |e(\ \pm\mathbf{p}\pm\mathbf{q}'\pm\mathbf{k}(0,\theta))| \leq M^{j'}\ \}\Big) \geq c_1\big(\tfrac{M^{j'}}{M^{(1+\varepsilon)j_2/2}}\big)^{1/n_0}$$

where c_1 is the supremum of $\frac{d\theta}{ds}$ (s is arc length), then $\tilde{\chi}(\mathbf{p})$ vanishes except on a set of measure $DM^{(1+\varepsilon)j_2}$ and

$$\begin{aligned}
&\int d^2\mathbf{p} \int_{|\theta|\leq 1} d\theta\ \chi_{j'}(\pm\mathbf{p}\pm\mathbf{q}'\pm\mathbf{k}(0,\theta))\ \chi_{j_2}(\mathbf{p}) \\
&\qquad \leq \int d^2\mathbf{p}\ \tilde{\chi}(\mathbf{p})\chi_{j_2}(\mathbf{p}) \int_{|\theta|\leq 1} d\theta\ \chi_{j'}(\pm\mathbf{p}\pm\mathbf{q}'\pm\mathbf{k}(0,\theta)) \\
&\qquad\qquad + \int d^2\mathbf{p}\ (1-\tilde{\chi}(\mathbf{p}))\chi_{j_2}(\mathbf{p}) \int_{|\theta|\leq 1} d\theta\ \chi_{j'}(\pm\mathbf{p}\pm\mathbf{q}'\pm\mathbf{k}(0,\theta)) \\
&\qquad \leq 2\int d^2\mathbf{p}\ \tilde{\chi}(\mathbf{p}) + \text{ const } \int d^2\mathbf{p}\ \chi_{j_2}(\mathbf{p})\big(\tfrac{M^{j'}}{M^{(1+\varepsilon)j_2/2}}\big)^{1/n_0}
\end{aligned}$$

$$
\begin{aligned}
&\leq \text{ const } M^{(1+\varepsilon)j_2} + \text{ const } |j_2| M^{j_2} \big(\tfrac{M^{j'}}{M^{(1+\varepsilon)j_2/2}}\big)^{1/n_0} \\
&\leq \text{ const } M^{j_2} M^{\varepsilon(1-\tilde\eta)j} + \text{ const } |j_2| M^{j_2} \big(M^{(1-\eta-\tilde\eta)j - \frac{1+\varepsilon}{2}j}\big)^{1/n_0} \\
&\leq \text{ const } M^{j_2} M^{\varepsilon(1-\tilde\eta)j} + \text{ const } |j_2| M^{j_2} M^{\frac{1}{n_0}(\frac{1}{3}-\eta)j} \\
&\leq \text{ const } |j_2| M^{j_2} M^{\varepsilon j - \eta j}
\end{aligned}
$$

provided $\tilde\eta \leq \eta \leq \frac{1}{12}$ and $\varepsilon \leq \min\{\frac{1}{6}, \frac{1}{3n_0}\}$.

End stage of proof The momentum flowing through the differentiated line is of the form $\pm q \pm k_1 \pm \ldots \pm k_n$ where q is the external momentum and each of the k_i's is a loop momentum of a line not in the spanning tree whose scale is no closer to zero than the scale of the differentiated line. If $q \notin B$, then either q_0 is nonzero or the infimum of $|\pm\mathbf{q} \pm \tilde{\mathbf{k}}_1 \pm \ldots \pm \tilde{\mathbf{k}}_n - \tilde{\mathbf{k}}_{n+1}|$, with the $\tilde{\mathbf{k}}_i$'s running over all singular points, is nonzero. So there exists a j_0 such that when the root scale obeys $j \leq j_0$, either the zero component of one of $k_1, \ldots, k_n$, $\pm q \pm k_1 \pm \ldots \pm k_n$ has magnitude larger than const M^j, in which case the corresponding covariance vanishes, or the distance of one of $\mathbf{k}_1, \ldots, \mathbf{k}_n$, $\pm\mathbf{q} \pm \mathbf{k}_1 \pm \ldots \pm \mathbf{k}_n$ to the nearest singular point is at least $M^{\eta j}$, in which case we can apply the argument of reduction 2. (If it is one of $\mathbf{k}_1, \ldots, \mathbf{k}_n$ whose distance to the nearest singular point is at least $M^{\eta j}$, we may choose as the ℓ_1 of Lemma 4.3 the line initiating that loop momentum.) ■

5 Singularities

In this section, we do a two–loop calculation for a typical case to show that the frequency derivative of the self–energy is indeed divergent in typical situations, and to calculate the second spatial derivative at the singular points.

By the Morse lemma, there are coordinates (x, y) such that in a neighbourhood of the Van Hove singularity, the dispersion relation becomes

$$e(\mathbf{k}) = \tilde e(x, y) = x\, y.$$

Here we consider the case of a Van Hove singularity at $k = 0$ with $e(\mathbf{k}) = \frac{k_1 k_2}{(2\pi)^2}$. (In particular the Van Hove singularity is on the Fermi surface for $\mu = 0$.) The nonlinearities induced by the changes of variables are absent in this example, and moreover, the curvature is zero on the Fermi surface. We rescale to $x = \frac{k_1}{2\pi}$, $y = \frac{k_2}{2\pi}$ and, for definiteness, take the integration region for each variable to be $[-1, 1]$.

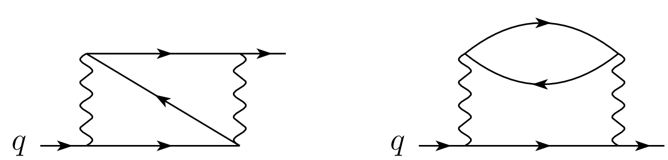
For this case, we determine the asymptotics of derivatives of the two–loop contribution to the self–energy as a function of q_0 for small q_0. We find that again, the gradient of the self–energy is bounded (in fact, the correction is zero in that case) but that the q_0–derivative is indeed divergent. Power counting by standard scales suggests that this derivative diverges at zero temperature like $|\log q_0|^3$. However, there is a cancellation of the leading singularity which is not seen when taking absolute values, so that the true behaviour is only $(\log q_0)^2$. We then also calculate the asymptotics of the second spatial derivative and find that it is of the same order as the first frequency derivative. Finally, we do the calculation for the one–loop contributions to the four–point function, to compare the coefficients of different divergences in perturbation theory.

The physical significance of these results will be discussed in Section 6.

5.1 Preparations

We restrict to a local potential. Since we have only considered short–range interactions, the potential is smooth in momentum space. For differentiability questions, a momentum dependence could only make a difference if the potential vanished at the singular points or other special points, so the restriction to a local potential, which is constant in momentum space, is not a loss of generality.

There are two graphs contributing, one of vertex correction type and the other of vacuum polarization type (the graphs with insertion of first–order self–energy graphs have been eliminated by renormalization through a shift in μ).



The latter gets a (-1) from the fermion loop and a 2 from the spin sum. Thus the total contribution is

$$\Sigma_2(q_0, q) = -\frac{1}{\beta^2} \sum_{\omega_1,\omega_2} \frac{1}{(2\pi)^4} \int d^2k_1 d^2k_2 d^2k_3 \; \delta(q - k_1 + k_2 - k_3) \\ C(\omega_1, e(k_1)) \; C(\omega_2, e(k_2)) \; C(q_0 - \omega_1 + \omega_2, e(k_3))$$

with

$$C(\omega, E) = \frac{1}{\mathrm{i}\omega - E}.$$

Call $E_i = e(k_i)$ and

$$\langle F \rangle_q = \int \mathrm{d}\rho_q(\mathbf{k})\ F(\mathbf{k})$$

where $\mathbf{k} = (k_1, k_2, k_3)$ and $\mathrm{d}\rho_q(\mathbf{k}) = \frac{1}{(2\pi)^4} \int d^2k_1 d^2k_2 d^2k_3\ \delta(q - k_1 + k_2 - k_3)$. The frequency summation gives

$$\Sigma_2(q_0, q) = -\left\langle \frac{(f_\beta(E_1) + b_\beta(E_2 - E_3))\ (f_\beta(E_2) - f_\beta(E_3))}{\mathrm{i}q_0 + (E_2 - E_3 - E_1)} \right\rangle_q \tag{11}$$

where $f_\beta(E) = (1 + \mathrm{e}^{\beta E})^{-1}$ is the Fermi function and $b_\beta(E) = (\mathrm{e}^{\beta E} - 1)^{-1}$ is the Bose function. Since

$$b_\beta(E_2 - E_3)\ [f_\beta(E_2) - f_\beta(E_3)] = f_\beta(E_2)[f_\beta(E_3) - 1]$$

the numerator is bounded in magnitude by 2. The denominator is bounded below in magnitude by $|q_0|$, so the integrand is C^∞ in q_0, for all $q_0 \neq 0$ and all $\beta \geq 0$. Because the integral is over a compact region of momenta, the same holds for the integral. The limit $\beta \to \infty$ exists and has the same property. The structure of the denominator may suggest that for it to almost vanish requires only the combination $E_2 - E_3 - E_1$ to get small, but a closer look reveals that each E_i has to be small: at $T = 0$, the Fermi functions become step functions, $f_\beta(E) \to \Theta(-E)$ and $b_\beta(E) \to -\Theta(-E)$, and then the factors in the numerator imply that all summands in $E_2 - E_3 - E_1$ really have the same sign, i.e. $|E_2 - E_3 - E_1| = |E_2| + |E_3| + |E_1|$, so all $|E_i|$ must be small for the energy difference to be small. At finite β, when the E_i's are "of the wrong sign" the exponential suppression provided by the numerator compensates for the $|q_0| \geq \frac{\pi}{\beta}$ in the denominator.

Because

$$\mathrm{i}q_0 - e(q) - \lambda^2 \Sigma_2(q_0, q) = \mathrm{i}q_0(1 + \mathrm{i}\lambda^2 \partial_0 \Sigma_2(0, q)) - (e(q) + \lambda^2 \Sigma_2(0, q)) + \dots \tag{12}$$

and because $\partial_0 \Sigma_2(0, q)$ is purely imaginary, we are interested in

$$Z_2(q) = \left(1 - \lambda^2\ \mathrm{Im}\ \partial_0 \Sigma_2(0, q)\right)^{-1}\ . \tag{13}$$

The value $q_0 = 0$ is not an allowed fermionic Matsubara frequency at $T > 0$. We shall keep $q_0 \neq 0$ in the calculations. In discussions about temperature dependence, we shall replace $|q_0|$ by π/β.

The second order contribution to $\mathrm{Im}\ \partial_0 \Sigma$ is

$$\mathrm{Im}\ \partial_0 \Sigma_2(q_0, q) = \Big\langle \Phi_{q_0}(E_2 - E_3 - E_1) \\ [f_\beta(E_1) + b_\beta(E_2 - E_3)]\ [f_\beta(E_2) - f_\beta(E_3)] \Big\rangle_q \tag{14}$$

with

$$\Phi_{q_0}(\varepsilon) = \text{ Re } \frac{1}{(\mathrm{i}q_0+\varepsilon)^2} = \frac{\varepsilon^2 - q_0^2}{(\varepsilon^2+q_0^2)^2}. \tag{15}$$

5.2 q_0–derivative

The above expression (14) for Im $\partial_0 \Sigma_2(q_0, q)$ shows that it is an even function of q_0 and that q_0 serves as a regulator so that even at zero temperature, a singularity can develop only in the limit $q_0 \to 0$. We therefore calculate it as a function of q_0 at zero temperature. In the following, we take $q_0 > 0$.

As $\beta \to \infty$, the Fermi function $f_\beta(E) \to \Theta(-E)$ and for $E \neq 0$, the Bose function $b_\beta(E) \to -\Theta(-E)$. In this limit, the integrand vanishes except when $E_2 E_3 < 0$ and $E_1(E_2 - E_3) < 0$. This reduces to the two cases

$$E_1 > 0, E_2 < 0, E_3 > 0 \text{ and } E_1 < 0, E_2 > 0, E_3 < 0. \tag{16}$$

In both cases, the combination of f_β,b_β's in the numerator is -1. Thus

$$\text{Im } \partial_0 \Sigma_2(q_0, q) = -\Big\langle \Phi_{q_0}(E_2 - E_3 - E_1) \Big[\mathbb{1}(E_1 > 0 \wedge E_2 < 0 \wedge E_3 > 0) + \mathbb{1}(E_1 < 0 \wedge E_2 > 0 \wedge E_3 < 0) \Big] \Big\rangle_q$$

Recall that we are considering the case of a Van Hove point at $k = (x, y) = 0$ for the dispersion relation $e(k) = xy$ with $x = k_1/2\pi$ and $y = k_2/2\pi$, so that $\frac{\mathrm{d}^2 k}{(2\pi)^2} = \mathrm{d}x \ \mathrm{d}y$. Moreover we set $q = 0$, and use the delta function to fix E_1 in terms of E_2 and E_3, so that

$$E_2 = xy \qquad E_3 = x'y' \qquad E_1 = (x - x')(y - y')$$

and

$$\varepsilon = E_2 - E_3 - E_1 = xy' - 2x'y' + x'y.$$

Recall also that, for definiteness, we are taking the integration region for each variable to be $[-1, 1]$. The sign conditions on the E_i impose conditions on the variables $x, \ldots$, which are listed in Appendix B, and which we use to transform the integration region to $[0, 1]^4$.

At $T = 0$, only $n \in \mathcal{M} = \{1, 2, 3, 4, 9, 10, 11, 12\}$ from the table in Appendix B contribute. Thus

$$\text{Im } \partial_0 \Sigma_2(q_0, 0) = -\int_{[0,1]^4} \mathrm{dxdydx'dy'} \sum_{n \in \mathcal{M}} \mathbb{1}(\rho_n) \ \Phi_{q_0}(\varepsilon_n(\mathrm{x}, \mathrm{y}, \mathrm{x'}, \mathrm{y'}))$$

with ε_n and ρ_n given in the table in Appendix B. By (31), (32) and (33), and because Φ_{q_0} is even in ε, all eight terms give the same contribution. Hence

$$\text{Im } \partial_0 \Sigma_2(q_0, 0) = -2I(q_0) \tag{17}$$

where

$$I(q_0) = 4 \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}y' \int_0^1 \mathrm{d}x' \int_0^{x'} \mathrm{d}x \; \Phi_{q_0} \left((2x' - x)y' + yx' \right). \tag{18}$$

Lemma 5.1 *Let I be defined as in (18) and $0 < q_0 < \frac{1}{2}$. Then*

$$\textit{Im } \partial_0 \Sigma_2(q_0, 0) = -4 \log 2 \; |\log q_0|^2 - 2C_1 \; |\log q_0| + B(q_0)$$

where B is a bounded function of q_0, and

$$C_1 = 2(\log 2)^2 - 4 \int_0^1 \frac{\mathrm{d}x}{x} \; \log \left(\frac{1+2x}{1+x} \right)$$

Proof: By (17), it suffi ces to show that

$$I(q_0) = 2 \log 2 \; |\log q_0|^2 + C_1 \; |\log q_0| + \tilde{B}(q_0)$$

with bounded $\tilde{B}_0$. We rewrite the argument ε of Φ_{q_0} as $\varepsilon = x'(2y' + y) - xy'$. We fi rst bound the contribution of $y' \le q_0$. To do this, bound

$$|\Phi_{q_0}(\varepsilon)| \le \frac{1}{q_0^2 + \varepsilon^2}$$

Use that this bound is decreasing in ε and that $\varepsilon \ge x'y$. Therefore

$$\begin{aligned}
4 \int_0^1 \mathrm{d}y \int_0^{q_0} \mathrm{d}y' \int_0^1 \mathrm{d}x' \int_0^{x'} \mathrm{d}x \; \Phi_{q_0} \left((2x' - x)y' + yx' \right) \\
\le 4 \int_0^1 \mathrm{d}y \int_0^{q_0} \mathrm{d}y' \int_0^1 \mathrm{d}x' \int_0^{x'} \mathrm{d}x \; \frac{1}{q_0^2 + (x'y)^2} \\
= 4q_0 \int_0^1 \mathrm{d}x' \int_0^1 \mathrm{d}y \; \frac{x'}{q_0^2 + (x'y)^2} \\
= 4q_0 \int_0^1 \mathrm{d}x' \; \frac{1}{q_0} \arctan \frac{x'}{q_0} \\
\le 2\pi.
\end{aligned}$$

Thus it suffices to calculate the asymptotic behaviour of

$$\tilde{I}(q_0) = 4 \int_0^1 \mathrm{d}y \int_{q_0}^1 \mathrm{d}y' \int_0^1 \mathrm{d}x' \int_0^{x'} \mathrm{d}x \; \Phi_{q_0}\left(x'(2y'+y) - xy'\right)$$

for small $q_0 > 0$. By (15)

$$\Phi_{q_0}(\varepsilon) = -\frac{\partial}{\partial \varepsilon} \, \frac{\varepsilon}{q_0^2 + \varepsilon^2}. \tag{19}$$

Because $\frac{\partial \varepsilon}{\partial x} = -y'$,

$$\int_0^{x'} \mathrm{d}x \; \Phi_{q_0}(\varepsilon(x)) = \frac{1}{y'} \left[\frac{\varepsilon}{\varepsilon^2 + q_0^2} \right]_{(2y'+y)x'}^{(y'+y)x'}.$$

The integral over x' can now be done, using

$$\int_0^1 \mathrm{d}x' \; \frac{\alpha x'}{(\alpha x')^2 + q_0^2} = \frac{1}{2\alpha} \log\left(1 + \frac{\alpha^2}{q_0^2}\right).$$

Thus

$$\tilde{I}(q_0) = \tilde{I}_1 - \tilde{I}_2$$

with

$$\begin{aligned} \tilde{I}_j &= 4 \int_{q_0}^1 \frac{\mathrm{d}y'}{y'} \int_0^1 \mathrm{d}y \; \frac{1}{2(jy'+y)} \log\left(1 + \frac{(jy'+y)^2}{q_0^2}\right) \\ &= 2 \int_1^{\frac{1}{q_0}} \frac{\mathrm{d}\eta}{\eta} \int_{j\eta}^{\frac{1}{q_0}+j\eta} \frac{\mathrm{d}\xi}{\xi} \; \log\left(1+\xi^2\right) \end{aligned}$$

where we have made the change of variables $\xi = \frac{jy'+y}{q_0}$, $d\xi = \frac{1}{q_0} dy$ followed by the change of variables $\eta = \frac{y'}{q_0}$, $d\eta = \frac{1}{q_0} dy'$. Thus

$$\begin{aligned} \tilde{I}(q_0) &= \int_1^{q_0^{-1}} \frac{\mathrm{d}\eta}{\eta} \left(J_{[\eta, q_0^{-1}+\eta]} - J_{[2\eta, q_0^{-1}+2\eta]} \right) \\ &= \int_1^{q_0^{-1}} \frac{\mathrm{d}\eta}{\eta} \left(J_{[\eta, 2\eta]} - J_{[q_0^{-1}+\eta, q_0^{-1}+2\eta]} \right) \end{aligned}$$

with

$$J_A = 2 \int_A \frac{\mathrm{d}\xi}{\xi} \; \log\left(1+\xi^2\right) \geq 0 \text{ for } A \subset [0, \infty)$$

We have

$$J_{[a,b]} = 2\int_a^b \frac{\mathrm{d}\xi}{\xi}\ \log\left(1+\xi^2\right) = \int_{a^2}^{b^2} \frac{\mathrm{d}t}{t}\ \log(1+t). \tag{20}$$

In our case both integration intervals for J are subsets of $[1,\infty)$, so we can expand the logarithm, to get

$$\begin{aligned} J_{[a,b]} &= \int_{a^2}^{b^2} \frac{\mathrm{d}t}{t}\left[\log(t) + \log\left(1+\frac{1}{t}\right)\right] \\ &= \frac{1}{2}\left[(\log b^2)^2 - (\log a^2)^2\right] - \sum_{n\geq 1} \frac{(-1)^n}{n^2}\ (a^{-2n} - b^{-2n}) \\ &= 2\left[(\log b)^2 - (\log a)^2\right] - \sum_{n\geq 1} \frac{(-1)^n}{n^2}\ (a^{-2n} - b^{-2n}) \\ &= 2\log(ab)\ \log\frac{b}{a} - \sum_{n\geq 1} \frac{(-1)^n}{n^2}\ (a^{-2n} - b^{-2n}) \end{aligned} \tag{21}$$

The final integral over η gives, for the first term,

$$\begin{aligned} \int_1^{q_0^{-1}} \frac{\mathrm{d}\eta}{\eta}\ J_{[\eta,2\eta]} &= 2\log 2\ |\log q_0|^2 + 2(\log 2)^2\ |\log q_0| \\ &\quad -\frac{1}{2}\sum_{n\geq 1}\frac{(-1)^n}{n^3}(1-4^{-n})(1-q_0^{2n}) \end{aligned}$$

with the last term analytic, and hence bounded, for $|q_0| < 1$. The second term gives two contributions:

$$\int_1^{q_0^{-1}} \frac{\mathrm{d}\eta}{\eta}\ J_{[q_0^{-1}+\eta,q_0^{-1}+2\eta]} = W - R$$

with (here $M = q_0^{-1}$)

$$R = \int_1^M \frac{\mathrm{d}\eta}{\eta}\ X(\eta), \qquad X(\eta) = \sum_{n\geq 1}\frac{(-1)^n}{n^2}\ \left[(M+\eta)^{-2n} - (M+2\eta)^{-2n}\right]$$

and

$$W = 2\int_1^M \frac{\mathrm{d}\eta}{\eta}\ \left[(\log(M+\alpha\eta))^2\right]_{\alpha=1}^{\alpha=2}.$$

For $|q_0| \le 1$ the series for X converges absolutely and gives

$$X = \sum_{n\ge1} \frac{(-1)^n}{n^2}\, q_0^{2n} \left[(1+\eta q_0)^{-2n} - (1+2\eta q_0)^{-2n}\right]$$

so $|X| \le \sum_{n\ge1} q_0^{2n}/n^2 \le 2q_0^2$, hence for $|q_0| \le 1$,

$$R \le 2q_0^2 |\ln q_0| \le q_0.$$

In W, scaling back to $x = q_0\eta$ gives

$$\begin{aligned} W &= 2\int_{q_0}^1 \frac{\mathrm{d}x}{x} \left[\left(\log q_0^{-1} + \log(1+\alpha x)\right)^2\right]_{\alpha=1}^{\alpha=2} \\ &= 2\int_{q_0}^1 \frac{\mathrm{d}x}{x} \left[2\log q_0^{-1}\, \log(1+\alpha x) + (\log(1+\alpha x))^2\right]_{\alpha=1}^{\alpha=2}. \end{aligned}$$

Because $\alpha x \ge 0$, $0 \le \log(1+\alpha x) \le \alpha x$, so W is bounded by a constant times $\log q_0^{-1}$. The integral of the same function from 0 to q_0 is of order $q_0|\log q_0|$, therefore

$$W = 4\log q_0^{-1} \int_0^1 \frac{\mathrm{d}x}{x}\, \log\left(\frac{1+2x}{1+x}\right) + \tilde{B}(q_0)$$

where $\tilde{B}$ is a bounded function. ■

5.3 First spatial derivatives

In our model case, we have $E_2 = xy$ and $E_3 = x'y'$. Moreover we take $\frac{\mathbf{q}}{2\pi} = (\xi, \eta)$ so that, fixing k_1 by momentum conservation, $E_1 = (\xi + x - x')(\eta + y - y')$. It is clear from (11) that the spatial derivatives also act on the Fermi function $f_\beta(E_1)$, so we get two terms. The derivative of the Fermi function $f_\beta(x)$ is minus the approximate delta $\delta_\beta(x) = \beta/(4\cosh^2(\beta x/2))$ so that

$$-\partial_i \Sigma_2(q_0, q) = \left\langle (\partial_i e)\left[-\delta_\beta(E_1)\frac{f_2 - f_3}{\mathrm{i}q_0 + \varepsilon} + \frac{(f_1 + b_{23})(f_2 - f_3)}{(\mathrm{i}q_0 + \varepsilon)^2}\right]\right\rangle_q \tag{22}$$

where $\partial_i e$ has argument $q + k_2 - k_3$, $f_i = f_\beta(E_i)$, $b_{23} = b_\beta(E_2 - E_3)$ and, as before, $\varepsilon = E_2 - E_3 - E_1$. At $q = 0$

$$-\frac{\partial}{\partial \xi}\Sigma_2(q_0, 0) = S_1(\beta, q_0) + S_2(\beta, q_0)$$

with

$$\begin{aligned}
S_1(\beta,q_0) &= \int_{[-1,1]^4} \mathrm{d}x\mathrm{d}y\mathrm{d}x'\mathrm{d}y' \;\; (y-y')\, \frac{f_\beta(xy)-f_\beta(x'y')}{\mathrm{i}q_0+\varepsilon} \\
&\qquad\qquad (-\delta_\beta)((x-x')(y-y')) \\
S_2(\beta,q_0) &= \int_{[-1,1]^4} \mathrm{d}x\mathrm{d}y\mathrm{d}x'\mathrm{d}y' \;\; (y-y')\, \frac{f_\beta(xy)-f_\beta(x'y')}{(\mathrm{i}q_0+\varepsilon)^2} \\
&\qquad\qquad [f_\beta((x-x')(y-y'))+b_\beta(xy-x'y')]
\end{aligned}$$

Here $\epsilon = xy - x'y' - (x-x')(y-y') = xy' + x'y - 2x'y'$.

Now consider S_1 and apply the reflection $(x,y,x',y') \to (-x,-y,-x',-y')$ to the integration variables. The domain of integration is invariant. The only noninvariant factor is $y-y'$ and it changes its sign. Thus S_1 vanishes. By the same argument, S_2 vanishes as well. By symmetry, the same holds for the η–derivative. Thus

$$\nabla\Sigma_2(q_0,0) = 0.$$

5.4 The second spatial derivatives

Let $\frac{\mathbf{q}}{2\pi} = (\xi,\eta)$. The real part of $\Sigma_2(\xi,\eta)$ is a correction to $e(\xi,\eta) = \xi\eta$. Since $\partial_\xi\partial_\eta e(\xi,\eta) = 1$, we calculate the correction that Σ_2 gives to that quantity.

Lemma 5.2 *For small* $q_0 \neq 0$

$$\lim_{\beta\to\infty} \mathrm{Re}\, \frac{\partial^2}{\partial\xi\partial\eta}\Sigma_2(q_0,0) = (2+4\log 2)\;(\log|q_0|)^2 + O(|\log|q_0||) \tag{23}$$

Proof: By symmetry it suffices to consider $q_0 > 0$. In general, the second order spatial derivatives are

$$-\partial_i\partial_j\Sigma_2(q_0,q) = Z_1^{(i,j)} + Z_2^{(i,j)} + Z_3^{(i,j)}$$

where

$$\begin{aligned}
Z_1^{(i,j)} &= \left\langle \left((\partial_i\partial_j e) + 2\frac{(\partial_i e)(\partial_j e)}{\mathrm{i}q_0+\varepsilon} \right) \frac{(f_1+b_{23})(f_2-f_3)}{(\mathrm{i}q_0+\varepsilon)^2} \right\rangle_q \\
Z_2^{(i,j)} &= \left\langle \left((\partial_i e)(\partial_j e)\;(-\delta'_\beta)(E_1) + (\partial_i\partial_j e)(-\delta_\beta)(E_1) \right) \frac{f_2-f_3}{\mathrm{i}q_0+\varepsilon} \right\rangle_q \\
Z_3^{(i,j)} &= \left\langle 2(\partial_i e)(\partial_j e)(-\delta_\beta)(E_1)\; \frac{f_2-f_3}{(\mathrm{i}q_0+\varepsilon)^2} \right\rangle_q
\end{aligned}$$

Here $e = e(k_2 - k_3 + q) = E_1$ and $\varepsilon = E_2 - E_3 - E_1$. Denote

$$\zeta(q_0) = -\frac{\partial^2}{\partial\xi\partial\eta}\Sigma_2(q_0, 0, 0)$$

In the xy case, the two integration momenta are denoted by (x, y) and (x', y'), and $E_2 = xy$, $E_3 = x'y'$, and $E_1 = e(k_2 - k_3 + q) = (\xi + x - x')(\eta + y - y')$. Since $\partial_\xi e \mid_{\xi=\eta=0}= (y - y')$, $\partial_\eta e \mid_{\xi=\eta=0}= (x - x')$, and $\partial^2_{\xi\eta} e = 1$,

$$\zeta = \zeta_1 + \zeta_2 + \zeta_3$$

with

$$\begin{aligned}
\zeta_1 &= \int_{[-1,1]^4} \mathrm{d}^4 X \ \left(1 + 2\tfrac{(x-x')(y-y')}{\mathrm{i}q_0+\varepsilon(x,y,x',y')}\right) \tfrac{(f_1+b_{23})(f_2-f_3)}{(\mathrm{i}q_0+\varepsilon(x,y,x',y'))^2} \\
\zeta_2 &= \int_{[-1,1]^4} \mathrm{d}^4 X \ \left(-E_1\delta'_\beta(E_1) - \delta_\beta(E_1)\right) \ \tfrac{f_2-f_3}{\mathrm{i}q_0+\varepsilon(x,y,x',y')} \\
\zeta_3 &= \int_{[-1,1]^4} \mathrm{d}^4 X \ 2(-E_1)\delta_\beta(E_1) \ \tfrac{f_2-f_3}{(\mathrm{i}q_0+\varepsilon(x,y,x',y'))^2}
\end{aligned}$$

Here $X = (x, y, x', y')$ and $\mathrm{d}^4 X = \mathrm{d}x \ \mathrm{d}y \ \mathrm{d}x' \ \mathrm{d}y'$, and

$$\varepsilon(x, y, x', y') = xy - x'y' - E_1 = xy - x'y' - (x - x')(y - y').$$

Using the decomposition given in Appendix B,

$$\zeta_1 \underset{\beta\to\infty}{\longrightarrow} -2\int_{[0,1]^4} \mathrm{dxdydx'dy'} \sum_{j=1}^{4} \left(1 + 2\frac{F_j}{\mathrm{i}q_0 + \varepsilon_j}\right) \frac{1}{(\mathrm{i}q_0 + \varepsilon_j)^2} \ \mathbb{1}(\rho_j)$$

(The limit can be taken under the integral. Decomposing according to the signs of $x, x' \ldots$, only $j \in \{1, 2, 3, 4, 9, 10, 11, 12\}$ can contribute. Using the symmetry (31), one obtains the above. The minus sign arises from the combination of Fermi functions f_β, as discussed around (16).) We write $\zeta_1 = -(\zeta_{11} + \zeta_{12})$. The term involving the 1 is

$$\zeta_{11} = 2\int_{[0,1]^4} \mathrm{dxdydx'dy'} \sum_{i=1}^{4} \frac{1}{(\mathrm{i}q_0 + \varepsilon_i)^2} \ \mathbb{1}(\rho_i)$$

By (32) and (33),

$$\zeta_{11} = 8\int_{[0,1]^4} \mathrm{dxdydx'dy'} \ \mathrm{Re} \ \frac{1}{(\mathrm{i}q_0 + \varepsilon_1)^2} \ \mathbb{1}(\rho_1) \ .$$

This is the same (up to a sign) as (17), (18), hence $\zeta_{11} = 4\log 2\ (\log|q_0|)^2 + O(|\log|q_0||)$.

The term ζ_{12} involving the F_j gives another contribution,

$$\zeta_{12} = 2\int_{[0,1]^4} \mathrm{dxdydx'dy'}\ \sum_{j=1}^{4} 2\frac{F_j}{(\mathrm{i}q_0+\varepsilon_j)^3}\ \mathbb{1}(\rho_j)$$

The summand for $j=2$ gives the same integral as that for $j=1$ because the integrand is related by the exchange $\mathrm{x} \leftrightarrow \mathrm{y}$ and $\mathrm{x'} \leftrightarrow \mathrm{y'}$. Ditto for $j=4$ and $j=3$. Thus

$$\zeta_{12} = 8\int_{[0,1]^4} \mathrm{dxdydx'dy'}\ \sum_{j=1,3} \frac{F_j}{(\mathrm{i}q_0+\varepsilon_j)^3}\ \mathbb{1}(\rho_j)$$

Because $F_1 = -F_3$ and $\varepsilon_1 = -\varepsilon_3$, and because $\rho_1 \Leftrightarrow \rho_3$,

$$\sum_{j=1,3} \frac{F_j}{(\mathrm{i}q_0+\varepsilon_j)^3}\ \mathbb{1}(\rho_j)\ =\ \mathbb{1}(\rho_1)\quad F_1\ 2\ \mathrm{Re}\ \frac{1}{(\mathrm{i}q_0+\varepsilon_1)^3}.$$

Thus

$$\zeta_{12} = 16\ \mathrm{Re}\ \int_0^1 \mathrm{dy}\int_0^1 \mathrm{dy'}\int_0^1 \mathrm{dx'}\int_0^{\mathrm{x'}} \mathrm{dx}\ \frac{(\mathrm{x}-\mathrm{x'})(\mathrm{y}+\mathrm{y'})}{(\mathrm{i}q_0+\mathrm{x'}(\mathrm{y}+\mathrm{y'})+\mathrm{y'}(\mathrm{x'}-\mathrm{x}))^3}$$

Let $b = \mathrm{x'}(\mathrm{y}+\mathrm{y'})$, change integration variables from x to $u = \mathrm{x'}-\mathrm{x} \in [0,\mathrm{x'}]$, and use

$$\int_0^{\mathrm{x'}} \mathrm{d}u\ \frac{u}{(\mathrm{i}q_0+b+\mathrm{y'}u)^3} = \frac{(\mathrm{x'})^2}{2(\mathrm{i}q_0+b+\mathrm{x'y'})^2(\mathrm{i}q_0+b)}$$

Renaming to $z = \mathrm{x'}$, we have

$$\zeta_{12} = -8\int_0^1 \mathrm{dy}\int_0^1 \mathrm{dy'}\int_0^1 \mathrm{d}z\ \mathrm{Re}\ \frac{(\mathrm{y}+\mathrm{y'})\ z^2}{(\mathrm{i}q_0+z(\mathrm{y}+\mathrm{y'}))\ (\mathrm{i}q_0+z(\mathrm{y}+2\mathrm{y'}))^2}$$

The bound

$$\left|\frac{(\mathrm{y}+\mathrm{y'})\ z^2}{(\mathrm{i}q_0+z(\mathrm{y}+\mathrm{y'}))\ (\mathrm{i}q_0+z(\mathrm{y}+2\mathrm{y'}))^2}\right| \le \frac{z}{q_0^2+z^2(\mathrm{y}+2\mathrm{y'})^2}$$

for the integrand implies that

$$|\zeta_{12}| \le 4\int_0^1 \mathrm{dy}\int_0^1 \mathrm{dy'}\frac{1}{(\mathrm{y}+2\mathrm{y'})^2}\ \ln\left(1+\frac{(\mathrm{y}+2\mathrm{y'})^2}{q_0^2}\right)$$

which shows that $|\zeta_{12}| \leq$ const $[\log|q_0|]^2$ for small $|q_0|$.

We now calculate the coefficient of the $\log^2$. First rewrite

$$\frac{(\mathrm{y}+\mathrm{y}')\,z^2}{(\mathrm{i}q_0+z(\mathrm{y}+\mathrm{y}'))\,(\mathrm{i}q_0+z(\mathrm{y}+2\mathrm{y}'))^2} = \frac{1}{(\mathrm{y}+2\mathrm{y}')^2}\,\frac{z^2}{(z+\mathrm{i}A)(z+\mathrm{i}B)^2}$$

with

$$A = \frac{q_0}{\mathrm{y}+\mathrm{y}'} \qquad B = \frac{q_0}{\mathrm{y}+2\mathrm{y}'}.$$

Partial fractions give

$$\frac{z^2}{(z+\mathrm{i}A)(z+\mathrm{i}B)^2} = \frac{\alpha}{z+\mathrm{i}A} + \frac{\beta}{z+\mathrm{i}B} + \mathrm{i}\,\frac{b_2}{(z+\mathrm{i}B)^2}$$

with

$$\begin{aligned}\alpha &= \frac{A^2}{(B-A)^2} = \left(\frac{\mathrm{y}+2\mathrm{y}'}{\mathrm{y}'}\right)^2 \\ \beta &= \frac{B(B-2A)}{(B-A)^2} = 1-\left(\frac{\mathrm{y}+2\mathrm{y}'}{\mathrm{y}'}\right)^2 \\ b_2 &= \frac{B^2}{A-B} = B\,\frac{\mathrm{y}+\mathrm{y}'}{\mathrm{y}'}.\end{aligned}$$

α, β and b_2 are real, so we need

$$\mathrm{Re}\,\int_0^1 \frac{\mathrm{d}z}{z+\mathrm{i}A} = \frac{1}{2}\ln(1+A^{-2})$$

and

$$\mathrm{Re}\ \mathrm{i}b_2\int_0^1 \frac{\mathrm{d}z}{(z+\mathrm{i}B)^2} = \frac{b_2}{B(1+B^2)}.$$

With this, we have

$$\begin{aligned}\zeta_{12} \;=\; -4\int_0^1 \mathrm{d}\mathrm{y}\int_0^1 \mathrm{d}\mathrm{y}'\,\frac{1}{(\mathrm{y}+2\mathrm{y}')^2} \\ \left[\alpha\ln(1+A^{-2}) + \beta\ln(1+B^{-2}) + \frac{2b_2}{B(1+B^2)}\right].\end{aligned}$$

Collecting terms and renaming $\mathrm{y}' = \eta$ gives

$$\zeta_{12} = -4 \int_0^1 \mathrm{dy} \int_0^1 \mathrm{d}\eta \ \Big[\frac{1}{\eta^2} \ \ln \frac{q_0^2 + (\mathrm{y}+\eta)^2}{q_0^2 + (\mathrm{y}+2\eta)^2} + 2\frac{\mathrm{y}+\eta}{\eta} \ \frac{1}{q_0^2 + (\mathrm{y}+2\eta)^2} + \frac{1}{(\mathrm{y}+2\eta)^2} \ \ln \left(1 + \frac{(\mathrm{y}+2\eta)^2}{q_0^2} \right) \Big]$$

Although the first summands individually contain nonintegrable singularities at $\eta = 0$, these singularities cancel in the sum. A convenient way to implement this is to use that

$$\frac{1}{\eta^2} \ \ln \frac{q_0^2 + (\mathrm{y}+\eta)^2}{q_0^2 + (\mathrm{y}+2\eta)^2} \quad + \quad 2\frac{\mathrm{y}}{\eta} \ \frac{1}{q_0^2 + (\mathrm{y}+2\eta)^2} = \frac{\partial}{\partial \eta} \left[-\frac{1}{\eta} \ \ln \frac{q_0^2 + (\mathrm{y}+\eta)^2}{q_0^2 + (\mathrm{y}+2\eta)^2} \right] + \frac{1}{\eta} \left[\frac{2(\mathrm{y}+\eta)}{q_0^2 + (\mathrm{y}+\eta)^2} - \frac{2(\mathrm{y}+4\eta)}{q_0^2 + (\mathrm{y}+2\eta)^2} \right] . \quad (24)$$

Moreover

$$\frac{1}{(\mathrm{y}+2\eta)^2} \ \ln \left(1 + \frac{(\mathrm{y}+2\eta)^2}{q_0^2} \right) = \frac{1}{2} \ \frac{\partial}{\partial \eta} \left[-\frac{1}{y+2\eta} \ \ln \left(1 + \frac{(\mathrm{y}+2\eta)^2}{q_0^2} \right) \right] + \frac{2}{q_0^2 + (\mathrm{y}+2\eta)^2}$$

Thus

$$\zeta_{12} = -4 \int_0^1 \mathrm{dy} \left[-\frac{1}{\eta} \ \ln \frac{q_0^2 + (\mathrm{y}+\eta)^2}{q_0^2 + (\mathrm{y}+2\eta)^2} - \frac{1}{2} \frac{1}{y+2\eta} \ \ln \left(1 + \frac{(\mathrm{y}+2\eta)^2}{q_0^2} \right) \right]_{\eta=0}^{\eta=1} -4 \int_0^1 \mathrm{dy} \int_0^1 \mathrm{d}\eta \ \frac{2}{\eta} \left[\frac{\mathrm{y}+\eta}{q_0^2 + (\mathrm{y}+\eta)^2} - \frac{\mathrm{y}+2\eta}{q_0^2 + (\mathrm{y}+2\eta)^2} \right]$$

Evaluation at $\eta = 1$ gives one bounded term and one term of order $\log |q_0|$. The terms at $\eta = 0$ give

$$-4 \int_0^1 \mathrm{dy} \left[\frac{2\mathrm{y}}{q_0^2 + \mathrm{y}^2} - \frac{1}{2\mathrm{y}} \ln \left(1 + \frac{\mathrm{y}^2}{q_0^2} \right) \right]$$

The first summand integrates to $O(\ln|q_0|)$. By (21), the second term gives

$$2(\ln|q_0|)^2 + \text{ bounded}$$

The remaining integrals are

$$\begin{aligned}
&-8\int_0^1 \mathrm{dy} \int_0^1 \mathrm{d}\eta \ \left[\frac{1}{q_0^2+(\mathrm{y}+\eta)^2} - \frac{2}{q_0^2+(\mathrm{y}+2\eta)^2}\right] \\
&= 8\int_0^1 \mathrm{dy} \int_1^2 \mathrm{d}\eta \ \frac{1}{q_0^2+(\mathrm{y}+\eta)^2} \le 8
\end{aligned}$$

since the integrand is bounded by 1, and

$$-4\int_0^1 \mathrm{dy} \int_0^1 \mathrm{d}\eta \, \frac{2}{\eta}\left[\frac{\mathrm{y}}{q_0^2+(\mathrm{y}+\eta)^2} - \frac{\mathrm{y}}{q_0^2+(\mathrm{y}+2\eta)^2}\right]$$

The integrand is bounded by a constant times $\frac{1}{q_0^2+\mathrm{y}^2+\eta^2}$ so the integral is of order $\log|q_0|$. Thus $\zeta_{12} = 2(\log|q_0|)^2+$ bounded terms, so that $\zeta_1 = -(\zeta_{11}+\zeta_{12}) = -(4\log 2+2)(\log|q_0|)^2+$ less singular terms.

We now show that ζ_2 and ζ_3 vanish in the limit $\beta\to\infty$. Consider ζ_2 first. Let $G_\beta(E_1) = -E_1\delta'_\beta(E_1) - \delta_\beta(E_1)$. (Note that $G_\beta(E) = \beta G_1(\beta E)$.) At $\xi=\eta=0$, $E_1 = (x-x')(y-y')$ is invariant under the exchange $(x,y)\leftrightarrow(x',y')$, so

$$\zeta_2 = \int_{[-1,1]^4} \mathrm{d}^4X \ G_\beta(E_1) f_\beta(xy) \left(\frac{1}{\mathrm{i}q_0+\varepsilon(x,y,x',y')} - \frac{1}{\mathrm{i}q_0+\varepsilon(x',y',x,y)}\right)$$

Because $\varepsilon(x,y,x',y') = xy'+x'y-2x'y'$,

$$\varepsilon(x',y',x,y) - \varepsilon(x,y,x',y') = 2(x'y'-xy) = 2[x(y'-y)+(x'-x)y']$$

Hence

$$\zeta_2 = 2\int_{[-1,1]^4} \mathrm{d}^4X \ G_\beta(E_1) \ f_\beta(xy) \ \frac{x(y'-y)+(x'-x)y'}{(\mathrm{i}q_0+\varepsilon(x,y,x',y'))\ (\mathrm{i}q_0+\varepsilon(x',y',x,y))}$$

Consider first

$$T_1 = 2\int_{[-1,1]^4} \mathrm{d}^4X \ G_\beta(E_1) \ f_\beta(xy) \ \frac{x(y'-y)}{(\mathrm{i}q_0+\varepsilon(x,y,x',y'))\ (\mathrm{i}q_0+\varepsilon(x',y',x,y))}$$

Because $[-1,1]^4$ and the integrand are invariant under the reflection R which maps (x,y,x',y') to $(-x,-y,-x',-y')$ and

$$R\{(x,y,x',y') \mid y \geq y'\} = \{(x,y,x',y') \mid y \leq y'\},$$

we can put in a factor $2\,\mathbb{1}(y>y')$. Changing variables from x' to E_1, so that

$$x' = x - \frac{E_1}{y-y'}\ ,$$

gives

$$\begin{aligned} T_1 &= 4\int_{[-1,1]^3} \mathrm{d}x\mathrm{d}y\mathrm{d}y'\ \ \mathbb{1}(y-y'>0)\ \ (-x)\ f_\beta(xy) \\ &\qquad \int_{(x-1)(y-y')}^{(x+1)(y-y')} \mathrm{d}E_1\ \frac{G_\beta(E_1)}{(\mathrm{i}q_0+\varepsilon(x,y,x',y'))\ (\mathrm{i}q_0+\varepsilon(x',y',x,y))} \end{aligned}$$

Change variables one last time, from E_1 to $u=\beta E_1$, to get

$$\begin{aligned} T_1 &= 4\int_{[-1,1]^3} \mathrm{d}x\mathrm{d}y\mathrm{d}y'\ \int_{\mathbb{R}} \mathrm{d}u \\ &\qquad \mathbb{1}(y-y'>0)\ \ \mathbb{1}(\beta(x-1)(y-y') \leq u \leq \beta(x+1)(y-y')) \\ &\qquad (-x) f_\beta(xy)\ \frac{G_1(u)}{(\mathrm{i}q_0+\varepsilon(x,y,x',y'))\ (\mathrm{i}q_0+\varepsilon(x',y',x,y))} \end{aligned}$$

where $x' = x - \frac{u}{\beta(y-y')} \to x$ as $\beta\to\infty$. The integrand is bounded in magnitude by the L^1 function

$$(x,y,y',u) \mapsto G_1(u)\frac{1}{q_0^2}$$

and it converges almost everywhere (namely, for $x \neq \pm 1$, $xy \neq 0$) to

$$\mathbb{1}(y-y'>0)\ \ x\ \Theta(-xy)\frac{1}{q_0^2+x^2(y-y')^2}[(-u)\delta_1'(u)-\delta_1(u)].$$

By dominated convergence, the limit $\beta\to\infty$ can be taken under the integral. The limiting range of integration for u is $\mathbb{R}$. By the fundamental theorem of calculus

$$\int_{\mathbb{R}} \mathrm{d}u\ G_1(u) = \int_{\mathbb{R}} \mathrm{d}u\ \tfrac{d}{du}[-u\delta_1(u)] = 0,$$

so T_1 vanishes as $\beta \to \infty$. The calculation of the limit $\beta \to \infty$ of

$$T_2 = 2 \int_{[-1,1]^4} \mathrm{d}^4 X \; G_\beta(E_1) \; f_\beta(xy) \; \frac{(x'-x)y'}{(\mathrm{i}q_0 + \varepsilon(x,y,x',y')) \; (\mathrm{i}q_0 + \varepsilon(x',y',x,y))}$$

is similar and gives 0 as well. Thus $\zeta_2 \to 0$ as $\beta \to \infty$. By the same arguments

$$\zeta_3 = \int_{[-1,1]^4} \mathrm{d}^4 X \; 2(-E_1)\delta_\beta(E_1) \; f_\beta(xy) \frac{2\mathrm{i}q_0(\tilde{\varepsilon} - \varepsilon) + (\tilde{\varepsilon} - \varepsilon)(\tilde{\varepsilon} + \varepsilon)}{(\mathrm{i}q_0 + \varepsilon)^2 \; (\mathrm{i}q_0 + \tilde{\varepsilon})^2}$$

where $\varepsilon = \varepsilon(x,y,x',y')$ and $\tilde{\varepsilon} = \varepsilon(x',y',x,y)$. After the same limiting argument as before, the u–integral is now

$$\int_{\mathbb{R}} u\delta_1(u) \; \mathrm{d}u = 0$$

because the integrand is odd. Thus $\zeta_3 \to 0$ as $\beta \to \infty$, too. ■

Lemma 5.3 *The real part of the second derivative of the self–energy with respect to ξ grows at most logarithmically as $q_0 \to 0$: there are constants A and B such that for all $|q_0| < 1$*

$$\left| \mathit{Re} \; \frac{\partial^2}{\partial \xi^2} \Sigma_2(q_0, 0) \right| \leq A + B \log \frac{1}{|q_0|}$$

The same holds by symmetry for the second derivative with respect to η.

Proof: We need to bound

$$\partial_\xi^2 \left\langle \frac{(f_\beta(E_1) + b_\beta(E_2 - E_3)) \; (f_\beta(E_2) - f_\beta(E_3))}{\mathrm{i}q_0 + \varepsilon} \right\rangle$$

at $\xi = \eta = 0$ where all ξ–dependence is in $E_1 = (\xi + x - x')(\eta + y - y')$ and in $\varepsilon = E_2 - E_3 - E_1 = xy' + x'y - 2x'y'$. We proceed as for the mixed derivative $\partial^2/\partial\xi\partial\eta$, but now some terms are different because $\partial_\xi^2 E_1 = 0$. We obtain

$$\frac{\partial^2 \Sigma_2}{\partial \xi^2}(q_0, 0, 0) = \mathbb{X}_1 + \mathbb{X}_2 + \mathbb{X}_3$$

with

$$\mathbb{X}_1 = 2\left\langle (y-y')^2 \, \frac{(f_\beta(E_1)+b_\beta(E_2-E_3))\,(f_\beta(E_2)-f_\beta(E_3))}{(\mathrm{i}q_0+\varepsilon)^3}\right\rangle$$

$$\mathbb{X}_2 = \left\langle \frac{\delta'_\beta(E_1)\,(y-y')^2}{\mathrm{i}q_0+\varepsilon}\,(f_\beta(E_2)-f_\beta(E_3))\right\rangle$$

$$\mathbb{X}_3 = 2\left\langle \frac{-\delta_\beta(E_1)\,(y-y')^2}{(\mathrm{i}q_0+\varepsilon)^2}\,(f_\beta(E_2)-f_\beta(E_3))\right\rangle$$

We first calculate the zero–temperature limit of $\mathbb{X}_1$. Using the notations of Appendix B,

$$\mathbb{X}_1^{(0)} = \lim_{\beta\to\infty}\mathbb{X}_1 = -4\int_{[0,1]^4} \mathrm{dxdx'dydy'}\ \sum_{n=1}^{4}\frac{D_n^2}{(\mathrm{i}q_0+\varepsilon_n)^3}\ \mathbb{1}(\rho_n)$$

We use that $D_1^2 = D_3^2$ and $\varepsilon_3 = -\varepsilon_1$, to combine $n=1$ and $n=3$ in one term, and $n=2$ and $n=4$ in another, so that $\mathbb{X}_1^{(0)} = \mathbb{X}_{1,1}^{(0)} + \mathbb{X}_{1,2}^{(0)}$ with

$$\mathbb{X}_{1,1}^{(0)} = -8\mathrm{i}\int_0^1 \mathrm{dy}\int_0^1 \mathrm{dy'}\ (\mathrm{y'}+\mathrm{y})^2\ \int_0^1 \mathrm{dx'}\int_0^{\mathrm{x'}} \mathrm{dx}\ \mathrm{Im}\ \frac{1}{(\mathrm{i}q_0+\mathrm{x'y}+\mathrm{y'}(2\mathrm{x'}-\mathrm{x}))^3}$$

$$\mathbb{X}_{1,2}^{(0)} = -8\mathrm{i}\int_0^1 \mathrm{dy'}\int_0^{\mathrm{y'}} \mathrm{dy}\ (\mathrm{y'}-\mathrm{y})^2\ \int_0^1 \mathrm{dx'}\int_0^1 \mathrm{dx}\ \mathrm{Im}\ \frac{1}{(\mathrm{i}q_0+\mathrm{xy'}+\mathrm{x'}(2\mathrm{y'}-\mathrm{y}))^3}$$

Thus $\mathbb{X}_1$ does not contribute to Re $\partial_\xi^2\Sigma_2$.

We now show that $\mathbb{X}_3$ is imaginary in the limit $\beta\to\infty$, because after a rewriting of terms, the difference $f_\beta(E_2)-f_\beta(E_3)$ effectively implies taking the imaginary part. Because $(y-y')^2\delta_\beta(E_1)$ is invariant under $(x,y)\leftrightarrow(x',y')$ (and denoting $\tilde\varepsilon(x,y,x',y') = \varepsilon(x',y',x,y)$)

$$\begin{aligned}\mathbb{X}_3 &= -2\int \mathrm{d}^4X\ (y-y')^2\ \delta_\beta(E_1)\ f_\beta(xy)\ \left[\frac{1}{(\mathrm{i}q_0+\varepsilon)^2}-\frac{1}{(\mathrm{i}q_0+\tilde\varepsilon)^2}\right]\\ &= -4\int \mathrm{d}y\int \mathrm{d}y'\ \mathbb{1}(y-y'>0)\ \int \mathrm{d}x\ f_\beta(xy)\ (y-y')\\ &\qquad \int \mathrm{d}x'\ (y-y')\delta_\beta(E_1)\left[\frac{1}{(\mathrm{i}q_0+\varepsilon)^2}-\frac{1}{(\mathrm{i}q_0+\tilde\varepsilon)^2}\right]\end{aligned}$$

For the second equality, we used invariance under the reflection $(x,y,x',y')\to(-x,-y,-x',-y')$. The convergence argument used in the analysis of the T_1 contribution to $\partial^2_{\xi\eta}\Sigma_2$ can be summarized in the following Lemma.

Lemma 5.4 *Let $\beta_0 \geq 0$ and $F : [\beta_0, \infty) \times [-1,1]^4 \to \mathbb{C}$ be bounded, and*

$$\lim_{\beta\to\infty} F(\beta, x, y, x - \tfrac{u}{\beta(y-y')}, y') = f(x,y,y')$$

a.e. in (x,y,u,y'). Let $E_1 = (x-x')(y-y')$. Then

$$\begin{aligned}
&\lim_{\beta\to\infty} \int F(\beta, X)\, \delta_\beta(E_1)\, (y-y')\ \mathbb{1}(y>y')\ \mathrm{d}^4X \\
=\ & \int \mathrm{d}y \int \mathrm{d}y'\, \mathbb{1}(y>y') \int \mathrm{d}x\ f(x,y,y')
\end{aligned}$$

Applying Lemma 5.4, and using that $\varepsilon(x,y,x,y') = x(y-y') = -\tilde\varepsilon(x,y,x'y')$, we get

$$\lim_{\beta\to\infty} \mathbb{X}_3 = -4 \int_{-1}^{1} \mathrm{d}y \int_{-1}^{y} \mathrm{d}y'\ (y-y') \int_{-1}^{1} \mathrm{d}x\ \Theta_{\frac12}(-xy)\, \mathcal{I}(x,y,y')$$

where $\Theta_{\frac12}(x) = \lim\limits_{\beta\to\infty} f_\beta(-x)$ is the Heaviside function, except that $\Theta_{\frac12}(0) = 1/2$,

$$\begin{aligned}
\mathcal{I}(x,y,y') &= \left[\frac{1}{(\mathrm{i}q_0 + x(y-y'))^2} - \frac{1}{(\mathrm{i}q_0 - x(y-y'))^2}\right] \\
&= 2\mathrm{i}\ \mathrm{Im}\ \frac{1}{(\mathrm{i}q_0 + x(y-y'))^2}.
\end{aligned}$$

Thus, in the limit $\beta\to\infty$, $\mathbb{X}_3$ does not contribute to the real part of $\partial_\xi^2 \Sigma_2$ either.

It remains to bound $\mathbb{X}_2$. Here we integrate by parts, to remove the derivative from the approximate delta function, and then take $\beta \to \infty$.

We first do the standard rewriting

$$\begin{aligned}
\mathbb{X}_2 = 2 \int \mathrm{d}^4X\ \mathbb{1}(y-y'>0)\ (y-y')^2 \\
f_\beta(xy)\ \delta'_\beta((x-x')(y-y'))\ \left[\frac{1}{\mathrm{i}q_0+\varepsilon} - \frac{1}{\mathrm{i}q_0+\tilde\varepsilon}\right]
\end{aligned}$$

We integrate by parts in x, using that $(y-y')\delta'_\beta((x-x')(y-y')) = \frac{\partial}{\partial x}\delta_\beta((x-x')(y-y'))$. This gives the boundary term

$$\begin{aligned}
B_{1/\beta} = 2 \sum_{x=\pm1} x \int \mathrm{d}y \int \mathrm{d}y'\ \mathbb{1}(y-y'>0) \int \mathrm{d}x'\ f_\beta(xy) \\
(y-y')\delta_\beta((x-x')(y-y'))\ \left[\frac{1}{\mathrm{i}q_0+\varepsilon} - \frac{1}{\mathrm{i}q_0+\tilde\varepsilon}\right]
\end{aligned}$$

and two integral terms, namely

$$I_1 = 2\int \mathrm{d}y \int \mathrm{d}y' \, \mathbb{1}(y-y'>0) \int \mathrm{d}x \int \mathrm{d}x' \; y\delta_\beta(xy)$$
$$(y-y')\delta_\beta((x-x')(y-y')) \left[\frac{1}{\mathrm{i}q_0+\varepsilon} - \frac{1}{\mathrm{i}q_0+\tilde{\varepsilon}}\right]$$

and

$$I_2 = 2\int \mathrm{d}y \int \mathrm{d}y' \, \mathbb{1}(y-y'>0) \int \mathrm{d}x \int \mathrm{d}x' \; f_\beta(xy)$$
$$(y-y')\delta_\beta((x-x')(y-y')) \left[\frac{y'}{(\mathrm{i}q_0+\varepsilon)^2} - \frac{y'-2y}{(\mathrm{i}q_0+\tilde{\varepsilon})^2}\right]$$

An obvious variant of Lemma 5.4, where x is summed over ± 1 instead of integrated, applies to the boundary term and gives

$$B_0 = \lim_{\beta\to\infty} B_{1/\beta} = 2\sum_{x=\pm1} x \int_{-1}^{1} \mathrm{d}y \int_{-1}^{y} \mathrm{d}y' \; \Theta_{\frac{1}{2}}(-xy)$$
$$\left[\frac{1}{\mathrm{i}q_0 + x(y-y')} - \frac{1}{\mathrm{i}q_0 - x(y-y')}\right]$$

Thus B_0 is real. Because $\Theta_{\frac{1}{2}}(y) + \Theta_{\frac{1}{2}}(-y) = 1$, we get

$$B_0 = 4\int_{-1}^{1} \mathrm{d}y \int_{-1}^{y} \mathrm{d}y' \; \frac{y-y'}{q_0^2 + (y-y')^2}$$

With $z = y - y'$, we have

$$\begin{aligned} B_0 &= 4\int_{-1}^{1} \mathrm{d}y \int_{0}^{1+y} \mathrm{d}z \; \frac{z}{q_0^2+z^2} = 4\int_{-1}^{1} \mathrm{d}y \; \frac{1}{2}\ln(q_0^2+z^2)\Big|_0^{1+y} \\ &= 2\int_{-1}^{1} \mathrm{d}y \; \ln\left(1+\frac{(1+y)^2}{q_0^2}\right) = 2\int_0^2 \mathrm{d}\eta \; \ln\left(1+\frac{\eta^2}{q_0^2}\right) \\ &= O(\log|q_0|). \end{aligned}$$

In I_1, we change variables from (x', x) to (u, v) where $u = \beta(x - x')(y - y')$ and $v = \beta x|y|$ and get

$$\begin{aligned} I_1 &= 2\int_{-1}^{1} \mathrm{d}y \int_{-1}^{y} \mathrm{d}y' \,\mathrm{sgn}(y) \int_{-\beta|y|}^{\beta|y|} \mathrm{d}v\ \delta_1(v) \int_{(\frac{v}{|y|}-\beta)(y-y')}^{(\frac{v}{|y|}+\beta)(y-y')} \mathrm{d}u\ \delta_1(u) \\ &\quad \left[\frac{1}{\mathrm{i}q_0 + \varepsilon(\frac{v}{\beta|y|}, y, \frac{v}{\beta|y|} - \frac{u}{\beta(y-y')}, y')} - \frac{1}{\mathrm{i}q_0 + \varepsilon(\frac{v}{\beta|y|} - \frac{u}{\beta(y-y')}, y', \frac{v}{\beta|y|}, y)}\right] \end{aligned}$$

The integral converges in the limit $\beta \to \infty$ by dominated convergence. The last factor vanishes in that limit, so $I_1 \to 0$ as $\beta \to \infty$.

Finally, Lemma 5.4 implies that $I_2^{(0)} = \lim_{\beta\to\infty} I_2$ exists and equals

$$\begin{aligned} I_2^{(0)} &= 2\int_{-1}^{1} \mathrm{d}y \int_{-1}^{y} \mathrm{d}y' \int_{-1}^{1} \mathrm{d}x\ \Theta_{\frac{1}{2}}(-xy) \\ &\quad \left[\frac{y'}{(\mathrm{i}q_0 + x(y-y'))^2} - \frac{y' - 2y}{(\mathrm{i}q_0 - x(y-y'))^2}\right] \end{aligned}$$

In the real part, the terms with y' in the numerator cancel, so that

$$\begin{aligned} \mathrm{Re}\ I_2^{(0)} &= 2\int_{-1}^{1} \mathrm{d}y \int_{-1}^{y} \mathrm{d}y' \int_{-1}^{1} \mathrm{d}x\ \Theta_{\frac{1}{2}}(-xy)\ \mathrm{Re}\ \frac{2y}{(\mathrm{i}q_0 - x(y-y'))^2} \\ &= 2\int_{-1}^{1} \mathrm{d}y \int_{-1}^{1} \mathrm{d}x\ \Theta_{\frac{1}{2}}(-xy)\ \mathrm{Re}\ \int_0^{1+y} \mathrm{d}z \frac{2y}{(\mathrm{i}q_0 - xz)^2} \\ &= -4\int_{-1}^{1} y\ \mathrm{d}y \int_{-1}^{1} \mathrm{d}x\ \Theta_{\frac{1}{2}}(-xy) \frac{1+y}{q_0^2 + x^2(1+y)^2} \\ &= -4\int_{-1}^{1} y\ \mathrm{d}y \int_{-(1+y)}^{1+y} \mathrm{d}v\ \Theta_{\frac{1}{2}}\left(-\frac{vy}{1+y}\right) \frac{1}{q_0^2 + v^2} \\ &= -4\int_{0}^{1} y\ \mathrm{d}y \int_{1-y}^{1+y} \frac{\mathrm{d}v}{q_0^2 + v^2} \end{aligned}$$

The contribution from the region $v \geq 1$ is obviously bounded. The remaining integral is

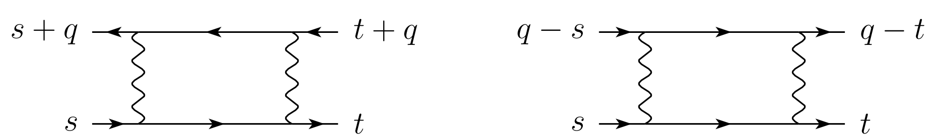
$$\int_0^1 y \, \mathrm{d}y \int_{1-y}^1 \frac{\mathrm{d}v}{q_0^2 + v^2} = \int_0^1 \frac{\mathrm{d}v}{q_0^2 + v^2} \int_{1-v}^1 y \, \mathrm{d}y$$

$$= \int_0^1 \frac{\mathrm{d}v}{q_0^2 + v^2} \left(v - \frac{v^2}{2} \right) = O(|\log |q_0||)$$

■

5.5 One–loop integrals for the xy case

For the discussion in Section 6, it is useful to calculate the lowest order contributions to the four–point function, the so–called bubble integrals. Again we restrict to the xy type singularity. Because the fermionic bubble integrals are not continuous at zero temperature, it is best to calculate them by setting $q_0 = 0$ first, then letting the spatial part q tend to 0, all at a fixed inverse temperature β, and then calculate the asymptotics as $\beta \to \infty$.



particle–hole bubble particle–particle bubble

5.5.1 The particle–hole bubble

Write $x' = x + \xi$, $y' = y + \eta$. The bubble is

$$\begin{aligned} B_{\mathrm{ph}}(q_0, \xi, \eta) &= \int_{[-1,1]^2} \mathrm{d}x\mathrm{d}y \, \frac{1}{\beta} \sum_\omega \frac{1}{\mathrm{i}\omega - xy} \frac{1}{\mathrm{i}(q_0 + \omega) - x'y'} \\ &= \int_{[-1,1]^2} \mathrm{d}x\mathrm{d}y \, \frac{f_\beta(xy) - f_\beta(x'y')}{\mathrm{i}q_0 + xy - x'y'} \\ &= \int_0^1 \mathrm{d}t \int_{[-1,1]^2} \mathrm{d}x\mathrm{d}y \, \frac{xy - x'y'}{\mathrm{i}q_0 + xy - x'y'} (-\delta_\beta)(txy + (1-t)x'y') \end{aligned} \tag{25}$$

Obviously, at $q_0 \neq 0$, $B_{\rm ph}(q_0,\xi,\eta) \to 0$ as $(\xi,\eta)\to 0$. So set $q_0 = 0$, i.e. consider

$$B_{\rm ph}^{(0)} = \lim_{(\xi,\eta)\to 0} B_{\rm ph}(0,\xi,\eta).$$

Lemma 5.5 *The large β asymptotics of $B_{\rm ph}^{(0)}$ is*

$$B_{\rm ph}^{(0)} = -2\ln\beta + 2K + O(\mathrm{e}^{-\beta})$$

where $K = \int_0^\infty \frac{\mathrm{d}u}{2\cosh^2\frac{u}{2}} \ln u$.

Proof: By (25),

$$\begin{aligned} B_{\rm ph}^{(0)} &= \int_{[-1,1]^2} \mathrm{d}x\mathrm{d}y\ (-\delta_\beta)(xy) = 4\int_{[0,1]^2} \mathrm{d}x\mathrm{d}y\ (-\delta_\beta)(xy) \\ &= -4\int_0^1 \frac{\mathrm{d}x}{x} \int_0^{\beta x} \frac{\mathrm{d}u}{4\cosh^2\frac{u}{2}} = -4\int_0^\beta \frac{\mathrm{d}u}{4\cosh^2\frac{u}{2}}\ \ln\frac{\beta}{u} \\ &= -2\ln\beta \int_0^\infty \frac{\mathrm{d}u}{2\cosh^2\frac{u}{2}} + 2K - \int_\beta^\infty \frac{\mathrm{d}u}{\cosh^2\frac{u}{2}}\ \ln\frac{u}{\beta}. \end{aligned}$$

The last integral is exponentially small in β because of the decay of $1/\cosh^2$. ∎

5.5.2 The particle–particle bubble

This time write $x' = x-\xi$, $y' = y-\eta$. The bubble is

$$\begin{aligned} B_{\rm pp}(q_0,\xi,\eta) &= \int_{[-1,1]^2} \mathrm{d}x\mathrm{d}y\ \frac{1}{\beta}\sum_\omega \frac{1}{\mathrm{i}\omega - xy}\frac{1}{\mathrm{i}(q_0-\omega) - x'y'} \\ &= \int_{[-1,1]^2} \mathrm{d}x\mathrm{d}y\ \frac{f_\beta(-xy) - f_\beta(x'y')}{-\mathrm{i}q_0 + xy + x'y'} \\ &\underset{(\xi,\eta)\to 0}{\longrightarrow} \int_{[-1,1]^2} \mathrm{d}x\mathrm{d}y\ \frac{1}{-\mathrm{i}q_0 + 2xy}\ \tanh\left(\frac{\beta}{2}xy\right) \end{aligned}$$

Again, we set $q_0 = 0$ and keep $\beta < \infty$. Then

$$\begin{aligned} B_{\rm pp}^{(0)} &= B_{\rm pp}(0,0,0) \\ &= \int_{[-1,1]^2} \mathrm{d}x\mathrm{d}y\ \frac{1}{2xy}\ \tanh\left(\frac{\beta}{2}xy\right) = 2\int_{[0,1]^2} \mathrm{d}x\mathrm{d}y\ \frac{1}{xy}\ \tanh\left(\frac{\beta}{2}xy\right) \end{aligned}$$

$$
\begin{aligned}
&= 2\int_0^1 \frac{\mathrm{d}x}{x}\int_0^x \frac{\mathrm{d}E}{E}\ \tanh\frac{\beta}{2}E = -2\int_0^1 \mathrm{d}E\ \frac{\ln E}{E}\ \tanh\frac{\beta}{2}E \\
&= \left[-(\ln E)^2\ \tanh\frac{\beta}{2}E\right]_0^1 + \int_0^1 \mathrm{d}E\ (\ln E)^2\ \frac{\beta}{2\cosh^2\frac{\beta}{2}E} \\
&= \int_0^{\beta/2} \mathrm{d}v\ \left(\ln\frac{2v}{\beta}\right)^2 \frac{1}{\cosh^2 v} \\
&= \int_0^{\infty} \mathrm{d}v\ \left(\ln\frac{2v}{\beta}\right)^2 \frac{1}{\cosh^2 v} - \int_{\beta/2}^{\infty} \mathrm{d}v\ \left(\ln\frac{2v}{\beta}\right)^2 \frac{1}{\cosh^2 v}
\end{aligned}
$$

Thus we have

Lemma 5.6

$$
B_{\rm pp}^{(0)} = (\ln\beta)^2 - 2K\ln\beta + K' + O\left(\mathrm{e}^{-\beta}\right)
$$

where $K = \int_0^\infty \frac{\ln(2v)}{\cosh^2 v}dv$ *and* $K' = \int_0^\infty \frac{(\ln(2v))^2}{\cosh^2 v}dv$.

6 Interpretation

Let us discuss these results a bit more informally. The above calculations for the xy case can be summarized as follows. Evidently, the derivatives we were looking at diverge in the limit $q_0 \to 0$ at zero temperature. To proceed, we discuss positive temperatures and replace q_0 by π/β, and thus translate everything into β–dependent quantities. We have proven that to all orders r in λ,

$$
|\nabla\Sigma_r| \leq \text{ const} \tag{26}
$$

(where the constant depends on the order r in λ). In the model computations of the last section, we have seen that to second order in the coupling constant λ

$$
\begin{aligned}
\text{Im } \partial_0\Sigma &\sim -4\ln 2\ (\lambda\ln\beta)^2 \\
\text{Re } \tfrac{\partial^2}{\partial\xi\partial\eta}\Sigma &\sim (2+4\ln 2)\ (\lambda\ln\beta)^2 \\
\text{Re } \tfrac{\partial^2}{\partial\xi^2}\Sigma &\sim O(\lambda^2\ln\beta) \\
B_{\rm ph}(0) &\sim -2\ \lambda\ln\beta \\
B_{\rm pp}(0) &\sim \lambda\ (\ln\beta)^2 - \frac{\alpha}{2}(\lambda\ln\beta)^2
\end{aligned}
$$

We have redefined B_{ph} and B_{pp} to include the appropriate coupling constant dependence. The line for B_{pp} includes second order corrections to the superconducting vertex, where the coefficient given by the loop integral is α. If λ is negative (attractive interaction), the superconducting instability driven by the $\lambda(\ln\beta)^2$ term is always strongest, but if $\lambda > 0$ (repulsive bare interaction), the $\lambda(\ln\beta)^2$ term suppresses the leading order Cooper pair interaction, and the higher order term proportional to α is only of order $(\lambda\ln\beta)^2$. In this case, all terms that can drive instabilities are linear or quadratic in $\lambda\ln\beta$. A first attempt to weigh the relative strength of these divergences is to look at the prefactors. Here it seems that the second derivative gets the largest contribution. The asymmetry between this logarithmic divergence and the boundedness of the gradient is striking.

We now put the results of [2, 3, 4] for the half–filled, $t = 0$, Hubbard model into context. It was proven there that perturbation theory in the coupling constant λ converges in the regime $W_{\beta,\lambda} = \{(\lambda, \beta) : |\lambda| \ll 1,\ |\lambda|(\log\beta)^2 \ll 1\}$. Our analysis, while presently restricted to all–order perturbation theory, is not restricted to $W_{\beta,\lambda}$. Indeed our results are most interesting at temperatures lower than those given (at small fixed $\lambda > 0$) by $W_{\beta,\lambda}$: in the regime $W_{\beta,\lambda}$ all the interesting effects that we summarized at the beginning of this section are still $O(\lambda)$ and hence small. As mentioned in [1], we expect that some of our results can be proven nonperturbatively, using sector techniques, provided the flow of the four–point function can be tracked in enough detail to see the differences between the various attractive and repulsive initial interactions in the bounds.

A nonperturbative treatment of the half–filled, $t = 0$, Hubbard model at temperatures below those permitted by $W_{\beta,\lambda}$ will require great care. The restriction $|\lambda|(\log\beta)^2 \ll 1$ of $W_{\beta,\lambda}$ eliminates the singularity that occurs for $\lambda < 0$ in the sum over particle–particle ladders. The square of the logarithm arises from the Van Hove singularity, as discussed in the Introduction of [1]. For $\lambda > 0$, however, there is no singularity in the flow of the four-point function when $\lambda(\log\beta)^2$ becomes of order one. (The coupling constant for s-wave superconductivity is suppressed rather than enhanced by the flow.) The $O(\lambda^2)$ terms in B_{pp} discussed above can generate singularities in the sum of particle–particle ladders, but this happens only when $\lambda\log\beta$ becomes of order 1. In this regime, all the other effects we have studied here, as well as nesting effects, come into play and compete with each other. In the exactly half-filled, $t = 0$ case, the singularities in the particle–hole channel drive Néel antiferromagnetism. For $t' \neq 0$, antiferromagnetism is weakened because nesting is destroyed. In the exactly half-filled, $t = 0$, $\lambda > 0$ case, Néel antiferromagnetism is believed to be the true ordering. However this is unproven.

Indeed, at the present time, even Fermi liquid behaviour has not been definitively ruled out. The divergence of the second derivative of the selfenergy with respect to the frequency ω was proven by a second–order calculation in [4]. While the property that Σ is C^2 appears in the sufficient condition for Fermi liquid behaviour of [22], and while the second spatial derivative enters in the curvature of the Fermi surface and hence needs to be controlled carefully (unless the Fermi surface is fixed by a symmetry), only a divergence in the first ω–derivative will change the asymptotic behaviour at small frequency and result in a breakdown of Fermi liquid behaviour. But the first ω–derivative still remains small in the regime $W_{\beta,\lambda}$, so that the Z factor stays close to 1 there.

In the following we further discuss the physical significance of our findings. This discussion is not rigorous, but it reveals some interesting possibilities that should be studied further.

We first note that the above results were achieved in renormalized perturbation theory, that is, the above all–order results are in the context of an expansion where a counterterm is used to fix the Fermi surface. (The second order explicit calculations assume only that the first order corrections can be taken into account by a shift in μ, which is true because the interaction is local.) Using a counterterm is the only way to get an all–order expansion that is well–defined in the limit of zero temperature. A scheme where the scale decomposition adapts to the Fermi surface movement was outlined in [18] and developed mathematically in [19] and [20]; but in any such scheme the expansions have to be done iteratively and cannot be cast in the form of a single renormalized expansion, because the singularity moves in every adjustment of the Fermi surface [18]. The meaning of the counterterm was explained in detail in [13, 14, 15, 16]. In brief, the dispersion relation we are using in our propagators is not the bare one, but the renormalized one, whose zero level set is the Fermi surface of the interacting system. Thus, we have in fact made the assumption that

> the Fermi surface of the *interacting* system has the properties **H1–H6**. In particular it contains singular points and these singular points are nondegenerate.

We shall first discuss our results under this assumption and then speculate about how commonly it will be valid. In the case of a nonsingular, strictly convex, curved Fermi surface, there was a similar assumption, which was, however, mathematically justified by our proof of an inversion theorem [16] that gives a bijective relation between the free and interacting dispersion relation.

6.1 Asymmetry and Fermi velocity suppression

First note that the regularized (discrete–time) functional integral for many–body systems has a symmetry that allows one to make an arbitrary *nonzero* rescaling of the field variables. This is based on the behaviour of the measure under $\psi_i \to g_i \psi_i$ and $\bar\psi_i \to \tilde g_i \bar\psi_i$. The result is

$$\int \prod_i \mathrm{d}\bar\psi_i \mathrm{d}\psi_i \; \mathrm{e}^{-\mathcal{A}(\bar\psi,\psi)} = \left(\prod_i g_i \tilde g_i\right)^{-1} \prod_i \mathrm{d}\bar\psi_i \mathrm{d}\psi_i \; \mathrm{e}^{-\mathcal{A}(\tilde g^{-1}\bar\psi, g^{-1}\psi)} \tag{27}$$

Source terms get a similar rescaling. This can, of course, be used to remove a factor Z^{-1} from the quadratic term in the fields, but the factor Z (see (13)) will then reappear in the interaction and source terms. Note that Z depends on momentum. In the limit of very small Z, some terms may get greatly enhanced or suppressed.

The Fermi velocity is defined as $v_F(p) = Z\nabla\Sigma$ on the Fermi surface. If one extrapolates the above formula for $\partial_0\Sigma_2$ to $\beta \to \infty$, the Z factor, defined in (13), becomes zero at the Van Hove points.

There is a crucial difference between the one– and two–dimensional cases. In dimension one, both $\partial_0\Sigma_2$ *and* $\nabla\Sigma_2$ behave like $\lambda^2 \log\beta$, so that, after extracting the field strength renormalization, the Fermi velocity retains its original value. In our two–dimensional situation, however, $\nabla\Sigma$ remains bounded, and thus the Fermi velocity gets suppressed in a neighbourhood of the Van Hove singularity because it contains a factor of Z. Because the time derivative term in the action is $Z^{-1}\bar\psi \mathrm{i} k_0 \psi$ and k_0 is an odd multiple of the temperature $T = \beta^{-1}$, one can also interpret $Z(k)^{-1}T = T(k)$ heuristically as a “momentum–dependent temperature” that varies over the Fermi surface and that increases as one approaches the Van Hove points (“hot spots”). This behaviour is illustrated in Figure 1.

6.2 Inversion problem

There is, however, also a more basic problem that is exhibited by these results. It arises when one starts questioning the assumption that the *interacting* Fermi surface contains singularities. The second derivative of the self–energy in spatial directions is divergent at zero temperature. Thus even the lowest nontrivial correction may change the structure of e significantly. This is related to the inversion problem, which was solved for strictly convex Fermi surfaces in [16].

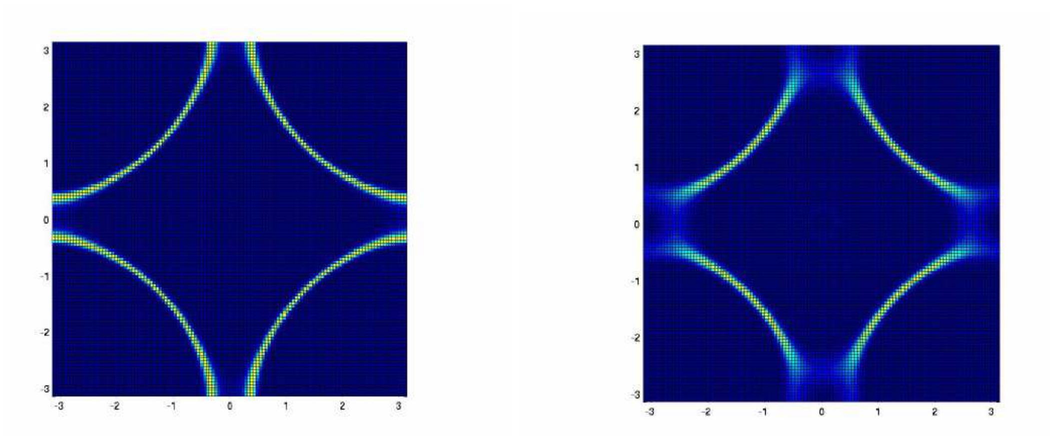


Figure 1: Comparison of the derivative of the Fermi function, where on the right, the inverse temperature β is scaled by an angle–dependent factor that vanishes near the points $(\pi, 0)$ and $(0, \pi)$. The dark regions correspond to small derivatives, the bright ones to large derivatives.

In all of the following discussion, we concentrate on two dimensions and assume that by adjusting μ, we can arrange things such that the interacting Fermi surface still contains a point where the gradient of the dispersion relation vanishes.

In the general theory of expansions for many–fermion systems, the most singular terms are those created by the movement of the Fermi surface. Counterterms need to be used to make perturbation theory well-defined, or an adaptive scale decomposition has to be chosen. To show that the model with counterterms corresponds to a bare model in some desired class, one needs to prove an inversion theorem, which requires regularity estimates. To get an idea, it is instructive to consider a neighbourhood of a regular point on the Fermi surface. By an affine transformation to coordinates (u, v) in momentum space, i.e. shifting the origin to that point and rotating so that the tangent plane to the Fermi surface is given by $u = 0$, the function e can be transformed to a function $\tilde{e}(u, v) = u + \frac{\kappa}{2}v^2 + \ldots$, where κ denotes the curvature and $\ldots$ the higher order terms. (By an additional change of coordinates, the error terms can be made to vanish, but this is not important here.) Let $\Sigma_0(p)$ be the self–energy at frequency zero (or $\pm\pi/\beta$). Then

$$(e + \Sigma_0)^{\sim}(u, v) = u + \frac{\kappa}{2}v^2 + \lambda\sigma(u, v).$$

In order for the correction $\lambda\sigma$ not to overwhelm the zeroth order contribution e, we need $\partial_u \sigma$ to be bounded and $\partial_v^2 \sigma$ to be bounded. Then the correction is small when λ is small. Note that different regularity is needed in normal and tangential directions. Indeed, even in the absence of Van Hove singularities, the function σ is not twice differentiable in u at $T = 0$, but it is C^2 in v if e has the same property [14].

Thus, the mere divergence of a derivative does not tell us about a potential problem with renormalization; the relevant question is whether the correction is larger than the zeroth order term. In the strictly convex, curved case, there is no problem. However, in two dimensions, the normal form for the Van Hove singularity, $\tilde{e}(uv) = uv$ gets changed significantly because

$$\tilde{\zeta}_2 = \frac{\partial^2}{\partial u \partial v} \left[e(u,v) + \tilde{\sigma}(u,v) \right] \big|_{u=v=0} \tag{28}$$

diverges at zero temperature. To leading order in q_0, u and v, our result for the inverse full propagator to second order takes the form $\zeta_2 \, \mathrm{i}q_0 - \tilde{\zeta}_2 \, uv$ with

$$\begin{aligned} \zeta_2 &= 1 + \vartheta_2 (\lambda \ln \beta)^2 \\ \tilde{\zeta}_2 &= 1 + \tilde{\vartheta}_2 (\lambda \ln \beta)^2 . \end{aligned}$$

Because $\tilde{\zeta}_2$ diverges in the zero–temperature limit, it is not even in second order consistent to assume that the type of $(u,v) = (0,0)$ as a critical point of e and $E = e + \tilde{\sigma}$ is the same. One can attempt to fix this problem by rescaling the fields by $\tilde{\zeta}^{1/2}$. Then the $\mathrm{i}q_0$ term gets rescaled similarly because ζ depends on the same combination of λ and $\ln \beta$ as $\tilde{\zeta}$. It should, however, be noted that, as in the above–discussed suppression of the Fermi velocity, a corresponding rescaling of all interaction and source terms also occurs. In particular, σ, which gives the Fermi surface shift, itself gets replaced by $\sigma \tilde{\zeta}^{-1}$. Because σ remains bounded (only its derivatives diverge), this would imply that the Fermi surface shift gets scaled down in the rescaling transformation, indicating a "pinning" of the Fermi surface at the Van Hove points.

In our case, $\vartheta_2 = 4 \ln 2$ and $\tilde{\vartheta}_2 = 2 + 4 \ln 2$. With these numerical values, $\tilde{\zeta}_2$ is larger than ζ_2, so $\tilde{\zeta}_2$ appears as the natural factor for the rescaling. In the Hubbard model case, the transformations leading to the normal form depend on the parameter θ in (1), so that one can expect ζ and $\tilde{\zeta}$ to depend on θ. The study of these dependencies, as well as the interplay between the two critical points that contribute, is left to future work.

One point of criticism of the Van Hove scenario has always been that it appears nongeneric because the logarithmic singularity gets weakened very fast as one moves away from the Van Hove densities. If the above speculations can be substantiated by careful studies, the singular Fermi surface scenario may turn out to be much more natural than one would naively assume. Moreover, there is a natural way how extended Van Hove singularities may arise by interaction effects.

A Interval Lemma

The following standard result is included for the convenience of the reader.

Lemma A.1 *Let $\epsilon,\ \eta$ be strictly positive real numbers and k be a strictly positive integer. Let $I \subset \mathbb{R}$ be an interval (not necessarily compact) and f a C^k function on I obeying*

$$|f^{(k)}(x)| \geq \eta \qquad \text{for all } x \in I$$

Then

$$\mathrm{Vol}\{\, x \in I \mid |f(x)| \leq \epsilon \,\} \leq 2^{k+1} (\tfrac{\epsilon}{\eta})^{1/k} \ .$$

Proof: Denote $\alpha = (\frac{\epsilon}{\eta})^{1/k}$. In terms α, we must show

$$|f^{(k)}(x)| \geq \tfrac{\epsilon}{\alpha^k} \quad \text{for all } x \in I \quad \Longrightarrow \quad \mathrm{Vol}\{\, x \in I \mid |f(x)| \leq \epsilon \,\} \leq 2^{k+1}\alpha$$

Define c_k inductively by $c_1 = 2$ and $c_k = 2 + 2c_{k-1}$. Because $b_k = 2^{-k} c_k$ obeys $b_1 = 1$ and $b_k = 2^{-k+1} + b_{k-1}$ we have $b_k \leq 2$ and hence $c_k \leq 2^{k+1}$. We shall prove

$$|f^{(k)}(x)| \geq \tfrac{\epsilon}{\alpha^k} \quad \text{for all } x \in I \quad \Longrightarrow \quad \mathrm{Vol}\{\, x \in I \mid |f(x)| \leq \epsilon \,\} \leq c_k\alpha$$

by induction on k.

Suppose that $k = 1$ and let x and y be any two points in $\{\, x \in I \mid |f(x)| \leq \epsilon \,\}$. Then

$$|x - y| = \tfrac{|x-y|}{|f(x)-f(y)|} |f(x) - f(y)| = \tfrac{|f(x)-f(y)|}{|f'(\zeta)|} \leq \tfrac{2\epsilon}{|f'(\zeta)|}$$

for some $\zeta \in I$. As $|f'(\zeta)| \geq \frac{\epsilon}{\alpha}$ we have $|x-y| \leq 2\alpha$. Thus $\{\, x \in I \mid |f(x)| \leq \epsilon \,\}$ is contained in an interval of length at most 2α as desired.

Now suppose that the induction hypothesis is satisfied for $k-1$ and that $|f^{(k)}(x)| \geq \frac{\epsilon}{\alpha^k}$ on I. As in the last paragraph the set $\{\, x \in I \mid |f^{(k-1)}(x)| \leq \frac{\epsilon}{\alpha^{k-1}} \,\}$ is contained in a subinterval I_0 of I of length at most 2α. Then I is the union of

I_0 and at most two other intervals I_+, I_- on which $|f^{(k-1)}(x)| \geq \frac{\epsilon}{\alpha^{k-1}}$. By the inductive hypothesis

$$\begin{aligned} \mathrm{Vol}\{\, x \in I \mid |f(x)| \leq \epsilon \,\} &\leq \mathrm{Vol}(I_0) + \sum_{i=\pm} \mathrm{Vol}\{\, x \in I_i \mid |f(x)| \leq \epsilon \,\} \\ &\leq 2\alpha + 2c_{k-1}\alpha = c_k\alpha \end{aligned}$$

■

B Signs etc.

The restrictions (16) are summarized in the following table

$E_2 = xy$	$E_3 = x'y'$	$E_1 = (x-x')(y-y')$	s_f
$+$	$-$	$-$	(-1)
$-$	$+$	$+$	(-1)

(29)

In both cases, the product of indicator functions resulting from the limit of Fermi functions is -1.

Let R be the reflection at zero,

$$R(x, y, x', y') = (-x, -y, -x', -y'). \tag{30}$$

The function

$$\varepsilon = \varepsilon(x, y, x', y') = xy' + x'y - 2x'y'$$

satisfi es

$$\varepsilon(x, y, x', y') = \varepsilon(-x, -y, -x', -y').$$

The function $D(x, y, x', y') = y - y'$ satisfi es

$$D(R(x, y, x', y')) = -D(x, y, x', y')$$

The function $F(x, y, x', y') = (x - x')(y - y')$ is invariant under R.

In the following we list all cases of signs for x, x', y and y', together with ε, D, F written as functions of

$$\mathrm{x} = |x|, \mathrm{y} = |y|, \mathrm{x}' = |x'|, \mathrm{y}' = |y'|.$$

to be able to restrict the integrals to $[0, 1]$ whenever this is convenient (by transforming to $\mathrm{x}, \ldots, \mathrm{y}'$ as integration variables), and to exhibit some important sign changes. In the last column, we list the condition ρ_n obtained from the restriction on the sign of $(x-x')(y-y')$ in (29).

n	$x\,y\,x'\,y'$	$\varepsilon_n = xy' + x'y - 2x'y'$	$D_n = y - y'$	F_n	ρ_n
1	$+++-$	$\varepsilon_1 = \mathrm{x'y} + (2\mathrm{x'} - \mathrm{x})\mathrm{y'}$	$D_1 = \mathrm{y} + \mathrm{y'}$	$(\mathrm{x} - \mathrm{x'})(\mathrm{y} + \mathrm{y'})$	$\mathrm{x} < \mathrm{x'}$
2	$++-+$	$\varepsilon_2 = \mathrm{xy'} + (2\mathrm{y'} - \mathrm{y})\mathrm{x'}$	$D_2 = \mathrm{y} - \mathrm{y'}$	$(\mathrm{x} + \mathrm{x'})(\mathrm{y} - \mathrm{y'})$	$\mathrm{y} < \mathrm{y'}$
3	$+-++$	$\varepsilon_3 = \mathrm{xy'} - (2\mathrm{y'} + \mathrm{y})\mathrm{x'}$	$D_3 = -(\mathrm{y} + \mathrm{y'})$	$-(\mathrm{x} - \mathrm{x'})(\mathrm{y} + \mathrm{y'})$	$\mathrm{x} < \mathrm{x'}$
4	$+---$	$\varepsilon_4 = \mathrm{x'y} - (2\mathrm{x'} + \mathrm{x})\mathrm{y'}$	$D_4 = -(\mathrm{y} - \mathrm{y'})$	$-(\mathrm{x} + \mathrm{x'})(\mathrm{y} - \mathrm{y'})$	$\mathrm{y} < \mathrm{y'}$
5	$++++$	$\varepsilon_5 = \mathrm{xy'} - \mathrm{x'}(2\mathrm{y'} - \mathrm{y})$			
6	$++--$	$\varepsilon_6 = -(\mathrm{xy'} + \mathrm{x'}(2\mathrm{y'} + \mathrm{y}))$			
7	$+-+-$	$\varepsilon_7 = -(\mathrm{xy'} - \mathrm{x'}(2\mathrm{y'} - \mathrm{y}))$			
8	$+--+$	$\varepsilon_8 = \mathrm{xy'} + \mathrm{x'}(2\mathrm{y'} + \mathrm{y})$			
9	$---+$	ε_1	$-D_1$		ρ_1
10	$--+-$	ε_2	$-D_2$		ρ_2
11	$-+--$	ε_3	$-D_3$		ρ_3
12	$-+++$	ε_4	$-D_4$		ρ_4
13	$----$	ε_5			
14	$--++$	ε_6			
15	$-+-+$	ε_7			
16	$-++-$	ε_8			

Cases 1-4 and 9-12 obey the restrictions (29). Cases 5-8 and 13-16 do not because there, the signs of xy and $x'y'$ are the same. They are used to discuss some terms at finite β. Since the first two restrictions are not satisfied, the column for the last restriction, ρ, is left empty in these cases. Case $n+8$ is obtained from n by the reflection R. Thus

$$\varepsilon_{n+8} = \varepsilon_n, \quad D_{n+8} = -D_n, \quad F_{n+8} = F_n, \quad \rho_{n+8} = \rho_n \tag{31}$$

Moreover,

$$\varepsilon_1 = -\varepsilon_3 \text{ and } \varepsilon_2 = -\varepsilon_4 \tag{32}$$

and

$$\varepsilon_2(\mathrm{y},\mathrm{x},\mathrm{y'},\mathrm{x'}) = \varepsilon_1(\mathrm{x},\mathrm{y},\mathrm{x'},\mathrm{y'}) \text{ and } \rho_2(\mathrm{y},\mathrm{x},\mathrm{y'},\mathrm{x'}) \Leftrightarrow \rho_1(\mathrm{x},\mathrm{y},\mathrm{x'},\mathrm{y'}). \tag{33}$$

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